

# Fractal Structures in Functions Related to Number Theory

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## Credits

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## Benoit B. Mandelbrot (1924–2010)

- “If we talk about impact inside mathematics, and applications to the sciences, he is one of the most important figures of the last 50 years.” -[Hans-Otto Peitgen](#).
- He brought background into foreground, made exceptions into the rule. His work reorganized how people see things.
- [Example](#). Note in the following photograph a possible fractal structure in the hair (après A. Einstein).

Benoit Mandelbrot



## Some Important Themes:

- Structures having **self-similar** and **self-affine** substructures.
- Structures produced by **multiplicative product processes** on trees; canonical cascade measures, a model for turbulence (“multifractal products”), generalizing a model of **Yaglom (1966)**.
- Measures of fractal behavior on different scales: the **multi-fractal formalism**

## A Typical Paper

B. B. Mandelbot, **Negative Fractal Dimensions and Multifractals**, *Physica A* **163** (1990), 306–315.

- **Abstract:** “A new notion of fractal dimension is defined. When it is positive, it effectively falls back on known definitions. But its motivating virtue is that it can take negative values, which measure usefully the degree of emptiness of empty sets.”
- **Citation list:** 21 references, of which 10 are to the author’s previous papers and talks. **Self-citation dimension:**  $10/21 = 0.47619$  (an empirical estimate).

# Functions Related To Number Theory

We discuss two functions related to number theory with fractal-like behavior.

- **Farey Fractions.** The geologist [Farey \(1816\)](#) noted them in: “On a curious Property of vulgar Fractions.” His observation then proved by [Cauchy \(1816\)](#). But the curious property already noted earlier by [Haros \(1802\)](#).
- **Takagi function** ([Takagi \(1903\)](#)). This particular continuous function on  $[0, 1]$  is everywhere non-differentiable.

# Farey Fractions

The **Farey sequence**  $\mathcal{F}_N$  consists of all rational fractions  $r = \frac{p}{q}$  in  $[0, 1]$ , in lowest terms, having  $\max(p, q) \leq N$ . Write them in increasing order as  $\{r_n : 0 \leq n \leq |\mathcal{F}_N| - 1\}$ .

Thus:

$$\mathcal{F}_1 = \left\{ \frac{0}{1}, \frac{1}{1} \right\}, \quad |\mathcal{F}_1| = 2$$

$$\mathcal{F}_2 = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\}, \quad |\mathcal{F}_2| = 3$$

$$\mathcal{F}_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\}, \quad |\mathcal{F}_3| = 5$$



## Farey Fractions-2

- The Farey sequence  $\mathcal{F}_N$  has cardinality

$$|\mathcal{F}_N| = \frac{6}{\pi^2} N^2 + O(N \log N).$$

- (Farey's curious Property) Neighboring elements  $\frac{a}{b} < \frac{a'}{b'}$  of  $\mathcal{F}_N$  satisfy

$$\det \begin{bmatrix} a & a' \\ b & b' \end{bmatrix} = ab' - ba' = -1.$$

- The Riemann hypothesis is encoded in the following question ...

# How well spaced are the Farey fractions?

- What we know:

**Theorem.** *The ensemble spacing of  $\mathcal{F}_N$  approaches the uniform distribution on  $[0, 1]$  as  $N \rightarrow \infty$ .*

The approach holds in many senses, e.g. the **Kolmogorov-Smirnov** statistic.

- However the individual gaps between neighboring member of the Farey sequence  $\mathcal{F}_N$  are of quite different sizes, varying between  $\frac{1}{N}$  and  $\frac{1}{N^2}$ .
- The **rate of approach** to the uniform distribution is what encodes the **Riemann hypothesis**, by...

# Franel's Theorem

- **Franel's Theorem (1924)** *The Riemann hypothesis is equivalent to: For each  $\epsilon > 0$  and all  $N$  one has*

$$\sum_{n=1}^{|\mathcal{F}_N|} \left( r_n - \frac{n}{|\mathcal{F}_N|} \right)^2 \leq C_\epsilon N^{-1+\epsilon}.$$

- This says, in some sense, the individual discrepancies from uniform distribution are of average size  $\frac{1}{N^{3/2-\epsilon}}$ .
- Generalizations to other discrepancy functions given by Mikolas (1948, 1949), and by Kanemitsu, Yoshimoto and Balasubramanian (1995, 2000).

# A New Question: Products of Farey Fractions

(Ongoing work with [Harm Derksen](#)) The **Farey product**  $F(N)$  is the product of all Farey fractions in  $\mathcal{F}_N$ , excluding 0.

- **Question 1.** How does  $F(N)$  grow as  $N \rightarrow \infty$ ?

*Answer:*  $\log F(N) = -\frac{\pi^2}{12}N^2 + O(N \log N)$

- **Question 2.** For a fixed prime  $p$ , how does divisibility by  $p$ , that is, the function  $\text{ord}_p(F(N))$ , behave as  $N$  increases?

*Partial Answer:* It exhibits approximately self-similar fractal behavior (empirically) on logarithmic scale. There is a race between primes  $p$  dividing numerator versus denominator.

## Products of Farey Fractions-2

- **Theorem.** (1) *There is upper bound*

$$|\text{ord}_p(F(N))| = O(N(\log N)^2).$$

- (2) *Infinitely often one has*

$$|\text{ord}_p(F(N))| > \frac{1}{2}N \log N.$$

- **Conjecture 1.**  $|\text{ord}_p(F(N))| = O(N \log N)$ ,
- **Conjecture 2.**  $\text{ord}_p(F(N))$  changes sign infinitely often.

## A Toy Model-Total Farey Sequence

The **total Farey sequence**  $\mathcal{G}_N$  consists of all rational fractions  $r = \frac{p}{q}$  in  $[0, 1]$ , *not necessarily given in lowest terms*, having  $\max(p, q) \leq N$ .

Thus

$$\mathcal{G}_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2} \text{ (counted twice)}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\},$$

Thus

$$|\mathcal{G}_4| = 8 > |\mathcal{F}_4| = 7.$$

# Products of Total Farey Fractions-1

The **total Farey product**  $G(N)$  is the product of all total Farey fractions, excluding 0. Here  $G(N) = \frac{1!2!3!\dots N!}{1^1 2^2 3^3 \dots N^N}$ .

- **Problem 1.** How does  $G(N)$  grow as  $N \rightarrow \infty$ ?

*Answer:*  $\log G(N) = -\frac{1}{2}N^2 + O(N \log N)$

- **Question 2.** For a fixed prime  $p$ , how does  $\text{ord}_p(G(N))$  behave as  $N$  increases?

*Answer:* There is a race between primes  $p$  dividing numerator versus denominator. But now it is analyzable and has provably fractal behavior.

## Products of Total Farey Fractions-2

- **Key Fact.**  $1/G(N)$  is an integer, given by a product of binomial coefficients

$$\frac{1}{G(N)} = \prod_{j=0}^N \binom{N}{j}.$$

- **Theorem.**

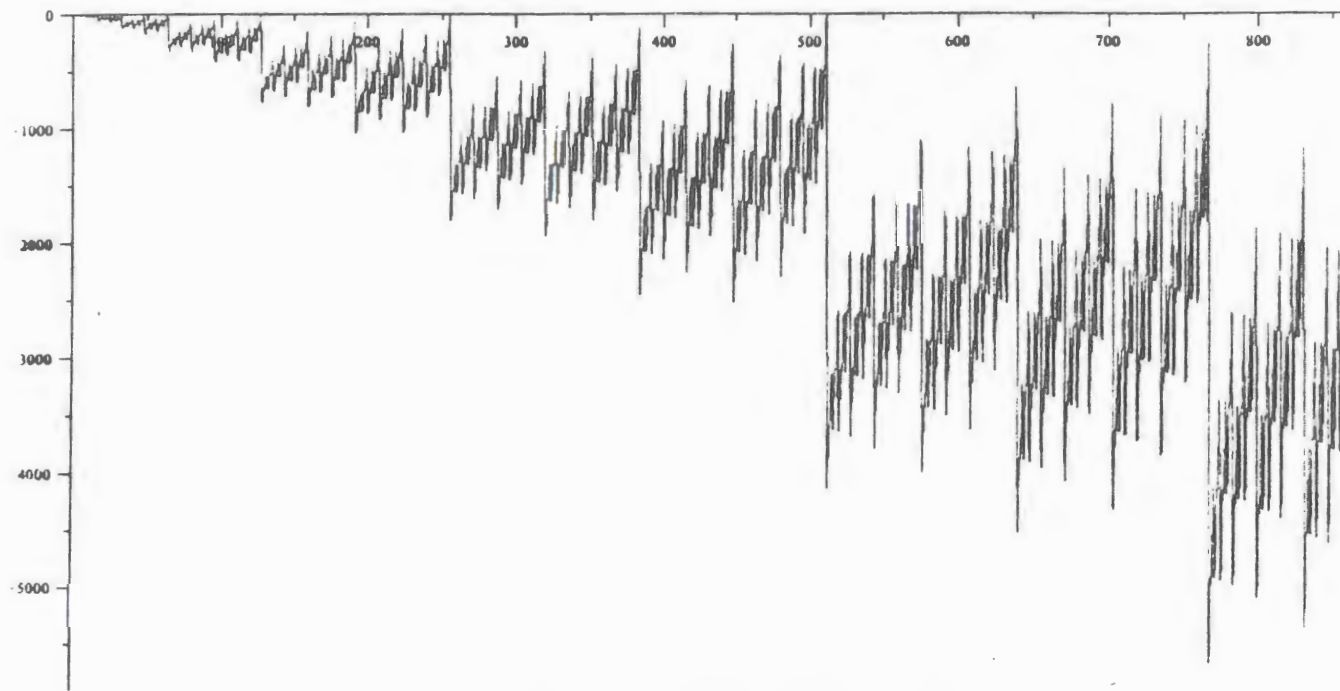
(1) *The size of  $\text{ord}_p(G(N))$  is*

$$|\text{ord}_p(G(N))| = O(N \log N).$$

(2)  $\text{ord}_p(G(N)) \leq 0$ . *Thus it never changes sign. But:*

$$\text{ord}_p(G(N)) = 0 \quad \textit{infinitely often}.$$

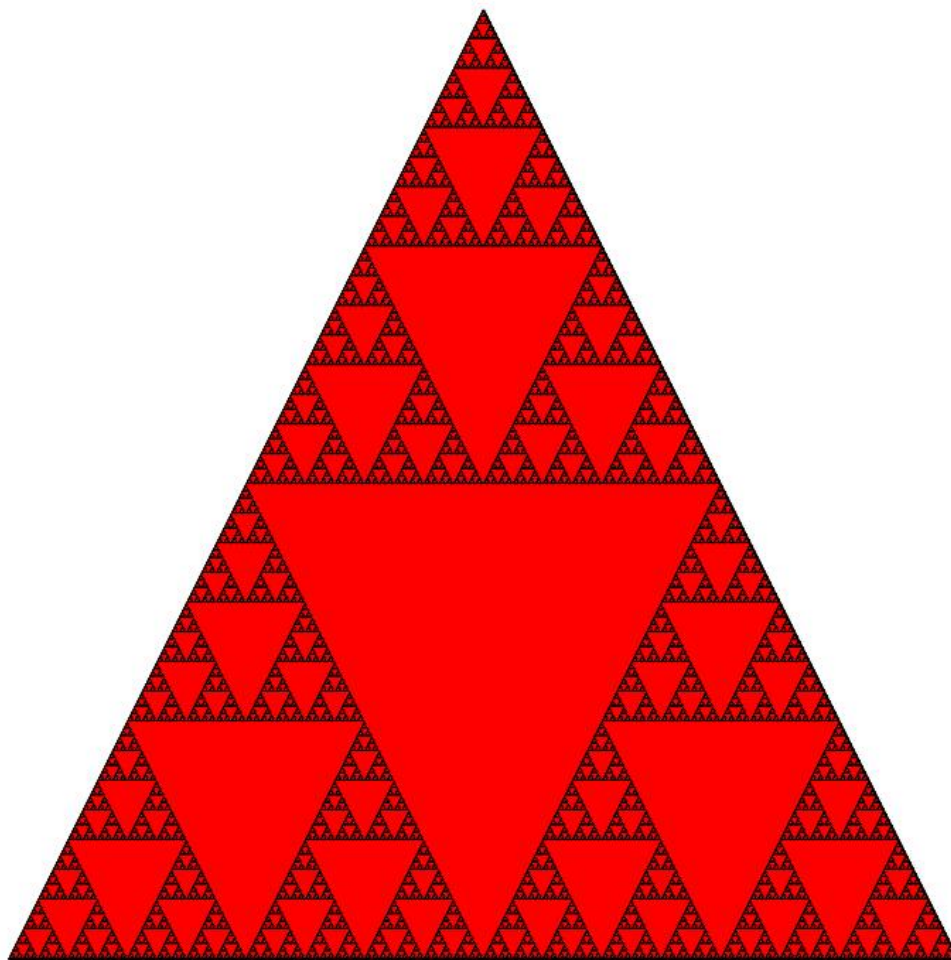




# Total Farey Products-Fractal Behavior

- Binomial coefficients viewed  $(\text{mod } p)$  have self-similar fractal behavior. For example [Pascal's triangle](#) viewed  $(\text{mod } 2)$  produces the [Sierpinski gasket](#).
- [Lucas's theorem](#)(1878) specifies the  $(\text{mod } p)$  behavior of  $\binom{a}{b}$  in terms of the base  $p$  expansions of  $a$  and  $b$ .
- More complicated scaling behavior occurs  $(\text{mod } p^n)$ .
- Obtain a scaling limit in terms of the base  $p$ -expansion of  $N$ . If the top  $d$  digits of  $N$  are fixed, and one averages over the other digits, then get a kind of limit...

## Fractal Behavior: Binomial Coefficients modulo 2



## Farey Products-Fractal Behavior?

- From  $F(N)$  one gets  $G(N)$ , via:

$$G(N) = \prod_{j=1}^N F(\lfloor \frac{N}{j} \rfloor).$$

- Therefore, by Möbius inversion,

$$F(N) = \prod_{j=1}^N G(\lfloor \frac{N}{j} \rfloor)^{\mu(j)}$$

- Results about  $G(N)$  permit some analysis of  $F(N)$ .

## Another Function: The Takagi Function

The Takagi function was constructed by [Teiji Takagi \(1903\)](#) as an example of continuous nowhere differentiable function on unit interval.

Let  $\ll x \gg$  be the distance of  $x$  to the nearest integer (a tent function). The function is:

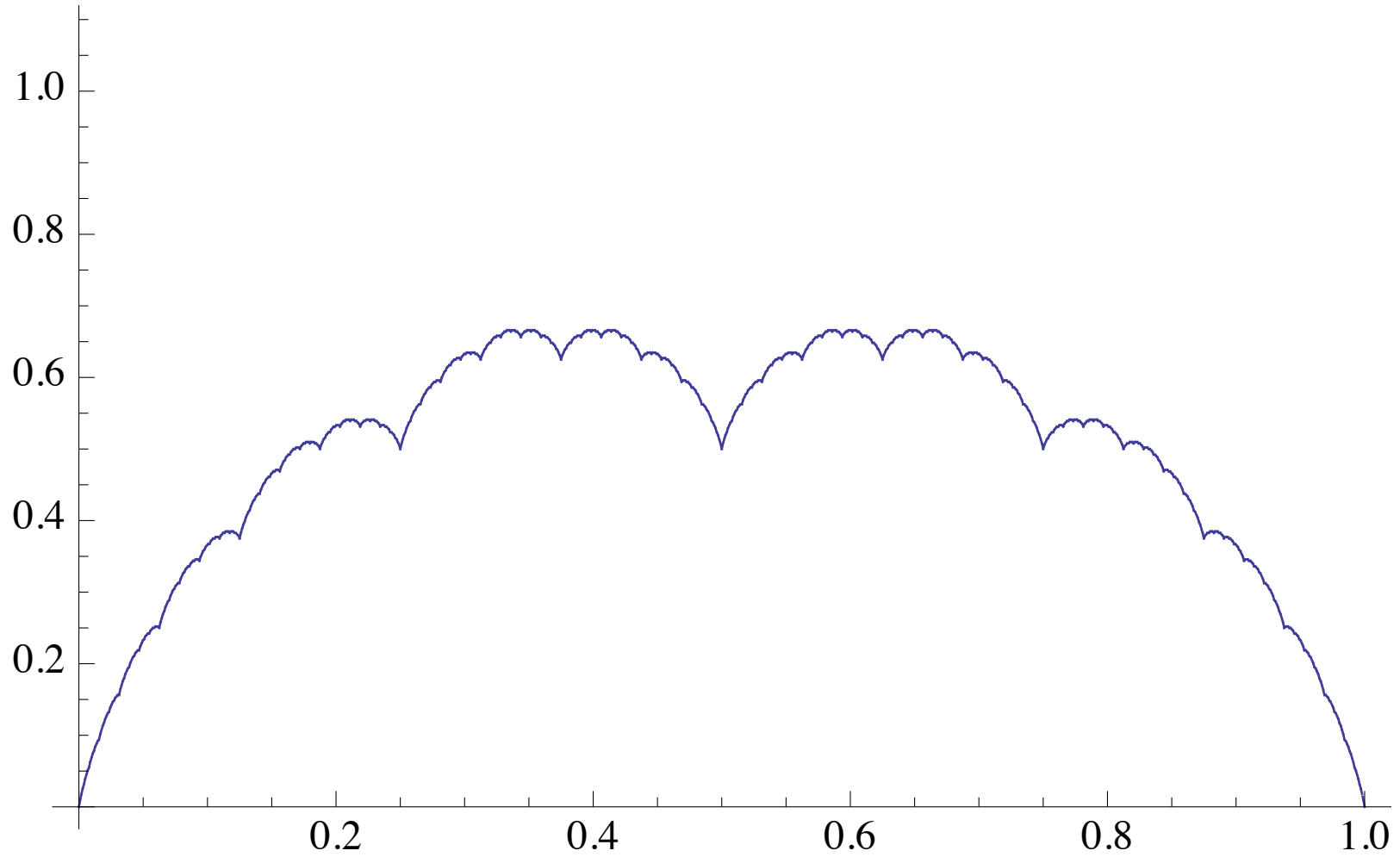
$$\tau(x) := \sum_{n=0}^{\infty} \frac{\ll 2^n x \gg}{2^n}$$

Takagi may have been motivated by [Weierstrass](#) nondifferentiable function (1870's).

## Teiji Takagi (1875–1960)

- **Teiji Takagi** grew up in a rural area, was sent away to school. He read math texts in English, since no texts were available in Japanese. He was sent to Germany in 1897-1901, studied first in Berlin, then moved to Göttingen to study with Hilbert.
- In isolation, he established the main theorems of class field theory (around 1920). This made him famous as a number theorist.
- He founded the modern Japanese mathematics school, writing many textbooks for schools at all levels.

# Graph of Takagi Function



## Main Property: Everywhere Non-differentiability

- **Theorem (Takagi (1903))** The function  $\tau(x)$  is continuous on  $[0, 1]$  and has no finite derivative at each point  $x \in [0, 1]$  on either side.
- Base 10 variant function discovered by [van der Waerden \(1930\)](#), Takagi function rediscovered by [de Rham \(1956\)](#).



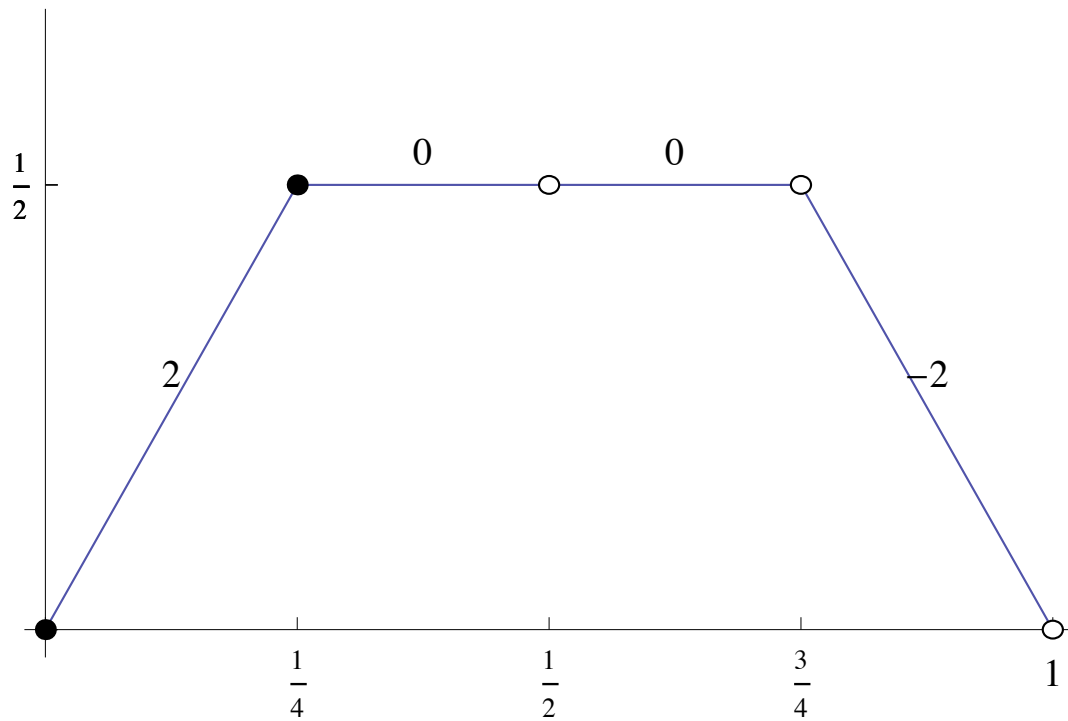
# Recursive Construction

- The  $n$ -th approximant

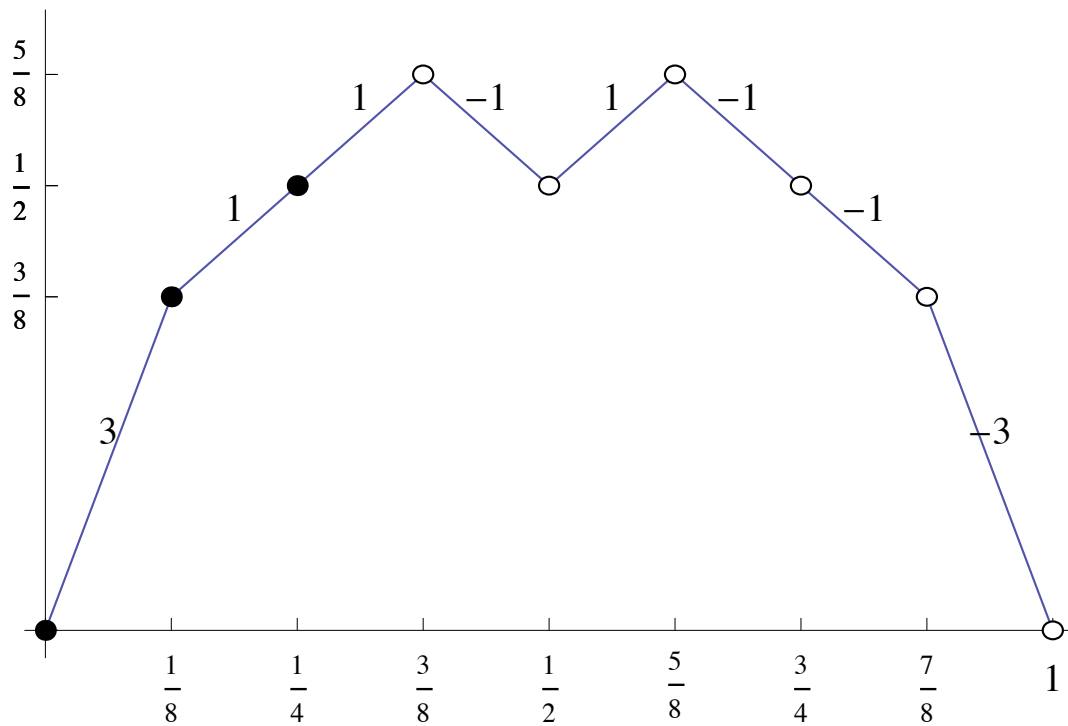
$$\tau_n(x) := \sum_{j=0}^n \frac{1}{2^j} \ll 2^j x \gg$$

- This is a **piecewise linear function**, with breaks at the dyadic integers  $\frac{k}{2^n}$ ,  $1 \leq k \leq 2^n - 1$ .
- All segments have **integer slopes**, in range between  $-n$  and  $+n$ . The maximal slope  $+n$  is attained in  $[0, \frac{1}{2^n}]$  and the minimal slope  $-n$  in  $[1 - \frac{1}{2^n}, 1]$ .

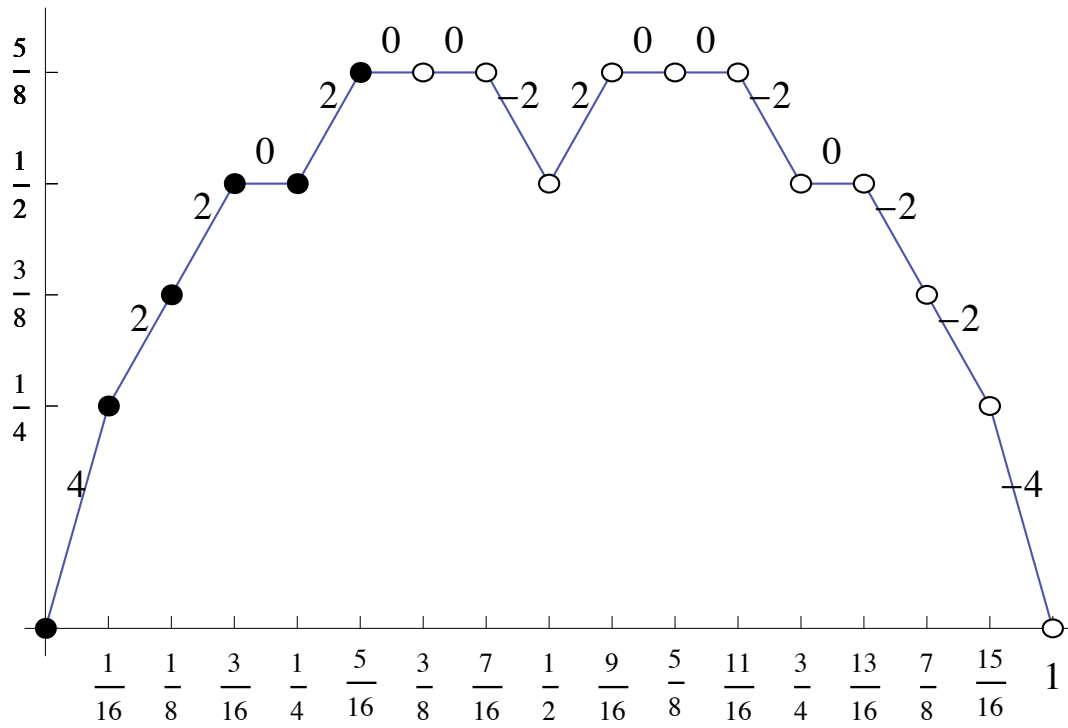
# Takagi Approximants- $\tau_2$



# Takagi Approximants- $\tau_3$



# Takagi Approximants- $\tau_4$



# Properties of Approximants

- The  $n$ -th approximant

$$\tau_n(x) := \sum_{j=0}^n \frac{1}{2^j} \ll 2^j x \gg$$

agrees with  $\tau(x)$  at all dyadic rationals  $\frac{k}{2^n}$ .

These values then **freeze**, i.e.  $\tau_n(\frac{k}{2^n}) = \tau_{n+j}(\frac{k}{2^n})$ .

- The approximants **are nondecreasing** at each step. They approximate Takagi function  $\tau(x)$  from below.

# Functional Equations

- **Fact.** The Takagi function, satisfies, for  $0 \leq x \leq 1$ , two functional equations:

$$\tau\left(\frac{x}{2}\right) = \frac{1}{2}\tau(x) + \frac{1}{2}x$$

$$\tau\left(\frac{x+1}{2}\right) = \frac{1}{2}\tau(x) + \frac{1}{2}(1-x).$$

- These are a kind of **dilation equation**, relating function on two different scales.

# Takagi Function in Number Theory

- Let  $e_2(n)$  sum the binary digits in  $n$ . Then

$$\sum_{n=1}^{\infty} \frac{e_2(n)}{n^s} = 2^{-s}(1 - 2^{-s})^{-1}\zeta(s),$$

where  $\zeta(s)$  is the Riemann zeta function.

- Let

$$S_2(N) := \sum_{n=1}^N e_2(n)$$

sum all the binary digits in the binary expansions of the first  $N$  integers.

## Takagi Function in Number Theory-2

Trollope (1968) showed that

$$S_2(N) = \frac{1}{2}N \log_2 N + NE_2(N),$$

where  $E_2(N)$  is an oscillatory function, given by an exact formula involving the Takagi function.

Delange (1975) showed there is a continuous function  $F(x)$  of period 1 such that for all positive integers  $N$ ,

$$E_2(N) = F(\log_2 N),$$

with

$$F(x) = \frac{1}{2}(1 - x) - 2^{\{x\}} \tau(2^{\{x\}-1}),$$

where  $\{x\} = x - \lfloor x \rfloor$  and  $\tau(x)$  is the Takagi function.



# Level Sets of the Takagi Function

- **Definition.** The **level set**  $L(y) = \{x : \tau(x) = y\}$ .  
(Here  $0 \leq y \leq \frac{2}{3}$ . Also:  $\tau(x)$  is rational if  $x$  is rational.)
- **Problem.** How large are the level sets of the Takagi function?
- Various results on this obtained in two papers with [Z. Maddock](#), arXiv:1009.0855, arXiv:1011.3183.

## Size of Level Sets: Cardinality

There exist levels  $y$  such that  $L(y)$  is finite, countable, or uncountable:

- $L(\frac{1}{5})$  is finite, containing two elements.  
Knuth (2005) showed that  $L(\frac{1}{5}) = \{\frac{3459}{87040}, \frac{83581}{87040}\}$ .
- $L(\frac{1}{2})$  is countably infinite.
- $L(\frac{2}{3})$  is uncountably infinite.  
Baba (1984) showed it has Hausdorff dimension  $1/2$ .

## Level Sets-Ordinate view

- We can compute the expected size of a level set  $L(y)$  for a random (ordinate) level  $y$ ...
- **Theorem.**
  - (1) ([Buczolich \(2008\)](#)) The expected size of a level set  $L(y)$  for  $y$  drawn at random is **finite**.
  - (2) ([L-Maddock \(2010\)](#)) The expected number of elements in a level set  $L(y)$  for  $y$  drawn at random is **infinite**.
- Extensive further analysis of finite level sets has been given by [Pieter Allaart](#) in [arXiv:1102.1616](#), [arXiv:1107.0712](#)

## Level Sets-Abscissa view

- We can compute the expected size of a level set  $L(\tau(x))$  for a random (abscissa) value  $x$ ...
- **Theorem.** If a value  $x \in [0, 1]$  is drawn at random, then with probability one the level set  $L(\tau(x))$  is uncountably infinite.
- **Conjecture.** A random  $L(\tau(x))$  drawn this way almost surely has Hausdorff dimension 0.

# Multifractal Spectrum for Level Sets of the Takagi Function?

- **Theorem.** The set *Big* of levels  $y$  such that the level set  $L(y)$  has positive Hausdorff dimension, is itself a set of full Hausdorff dimension 1.
- **Conjecture.** Let  $S(\alpha)$  be the set of levels  $y$  such that the Hausdorff dimension of the level set  $L(y)$  exceeds  $\alpha$ , and let  $f(\alpha)$  be the Hausdorff dimension of  $S(\alpha)$ . Then the function  $f(\alpha)$  exhibits the properties of a multi-fractal spectrum. Namely  $f(\alpha)$  is a convex function of  $\alpha$  on  $[0, 1/2]$  with  $f(0) = 1$ , and  $f(\frac{1}{2}) = 0$ .

# Takagi Function Surveys

- The Takagi function has one hundred years of history and results. See the survey papers:
- P. Allaart and K. Kawamura, *The Takagi Function: A Survey*, arXiv:1110.1691
- J. Lagarias, *The Takagi Function and its Properties*, arXiv:1112:4205.
- Work of J. L. partially supported by grants DMS-0801029 and DMS-1101373.

The End

Thank you for your attention!