The Takagi Function

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The Beauty and Power of Number Theory,
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Topics Covered

- Part I. Introduction and Some History
- Part II. Number Theory
- Part III. Analysis
- Part IV. Rational Values
- Part V. Level Sets

Credits

- J. C. Lagarias and Z. Maddock , Level Sets of the Takagi Function: Local Level Sets, arXiv:1009.0855

- J. C. Lagarias and Z. Maddock , Level Sets of the Takagi Function: Generic Level Sets, arXiv:1011.3183

- Zachary Maddock was an REU Student in 2007 at Michigan. He is now a grad student at Columbia, studying algebraic geometry with advisor Johan de Jong.

- Work partially supported by NSF grant DMS-0801029.

Part I. Introduction and History

- Definition The distance to nearest integer function (sawtooth function)

$$\ll x \gg = dist(x, \mathbb{Z})$$

- The map $T(x) = 2 \ll x \gg$ is sometimes called the symmetric tent map, when restricted to $[0, 1]$.

The Takagi Function

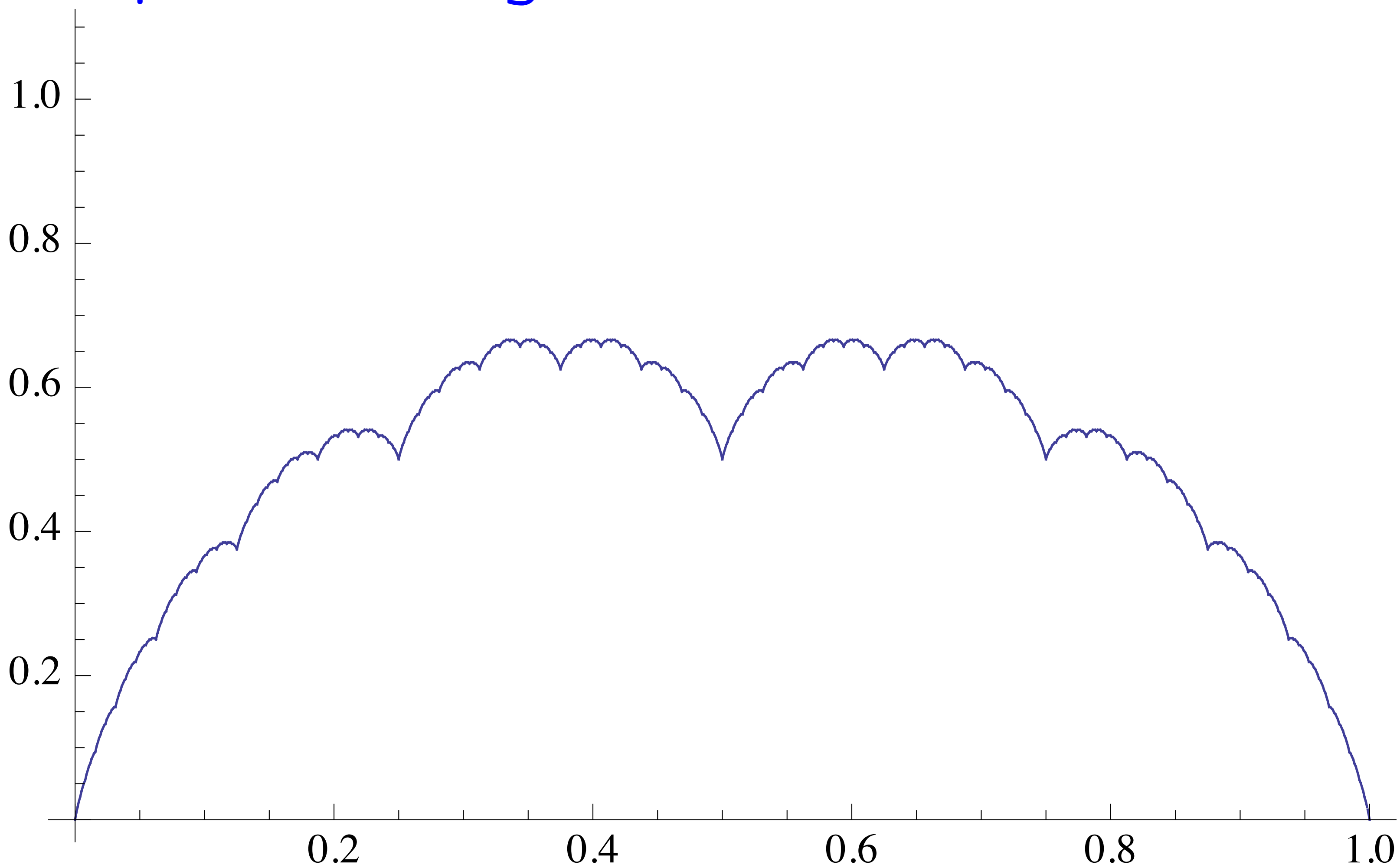
- The Takagi Function $\tau(x) : [0,1] \to [0,1]$ is

$$\tau(x) = \sum_{j=0}^{\infty} \frac{1}{2^j} \ll 2^j x \gg$$

- This function was introduced by Teiji Takagi (1875–1960) in 1903. Takagi is famous for his work in number theory. He proved the fundamental theorem of Class Field Theory (1920, 1922).

- He was sent to Germany 1897-1901. He visited Berlin and Góttingen, saw Hilbert.

Graph of Takagi Function



Main Property: Everywhere Non-differentiability

- Theorem (Takagi (1903) The function $\tau(x)$ is continuous on $[0, 1]$ and has no derivative at each point $x \in [0, 1]$ on either side.

- van der Waerden (1930) discovered the base 10 variant, proved non-differentiability.

- de Rham (1956) also rediscovered the Takagi function.

History

- The Takagi function $\tau(x)$ has been extensively studied in all sorts of ways, during its 100 year history, often in more general contexts.

- It has some surprising connections with number theory and (less surprising) with probability theory.

- It has showed up as a "toy model" in study of chaotic dynamics, as a fractal, and it has connections with wavelets. For it, many things are explicitly computable.

Generalizations

- For $g(x)$ periodic of period one, and $a, b > 1$, set

$$F_{a,b,g}(x) := \sum_{j=0}^{\infty} \frac{1}{a^j} g(b^j x)$$

- This class includes: Weierstrass nondifferentiable function. Takagi's work may have been motivated by this function.

- Properties of functions depend sensitively on a, b and the function $g(x)$. Sometimes get smooth function on [0, 1] (Hata-Yamaguti (1984))

$$F(x) := \sum_{j=0}^{\infty} \frac{1}{4^j} \ll 2^j x \gg = 2x(1-x).$$

Recursive Construction

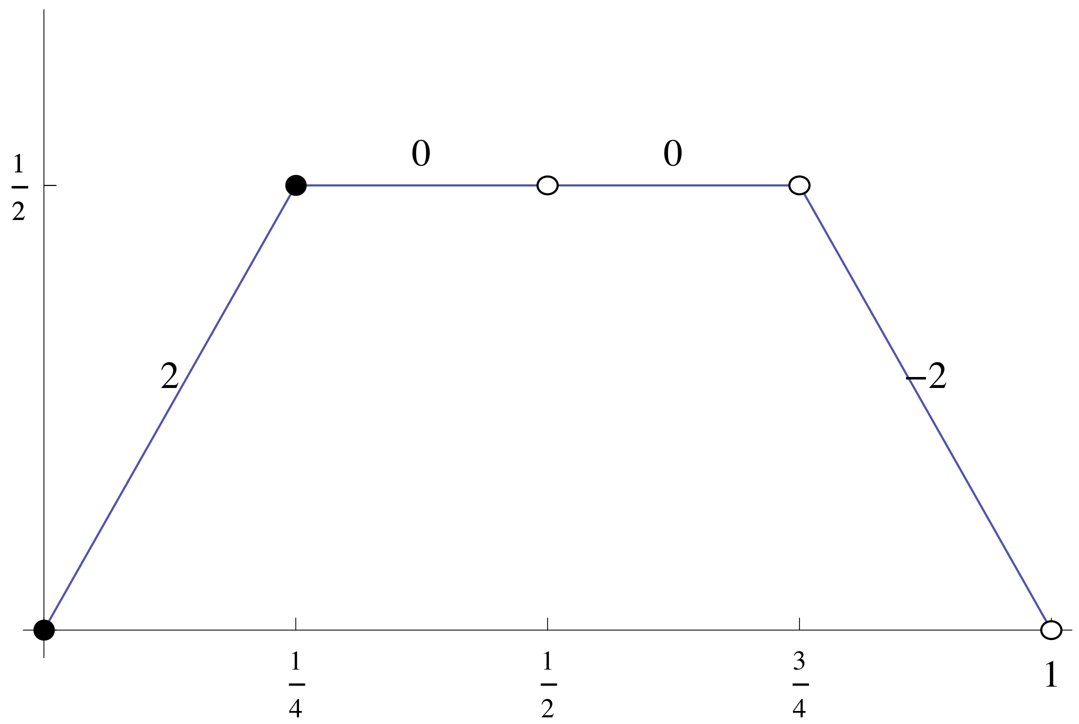
- The n-th approximant function

$$\tau_n(x) := \sum_{j=0}^{n} \frac{1}{2^j} \ll 2^j x \gg$$

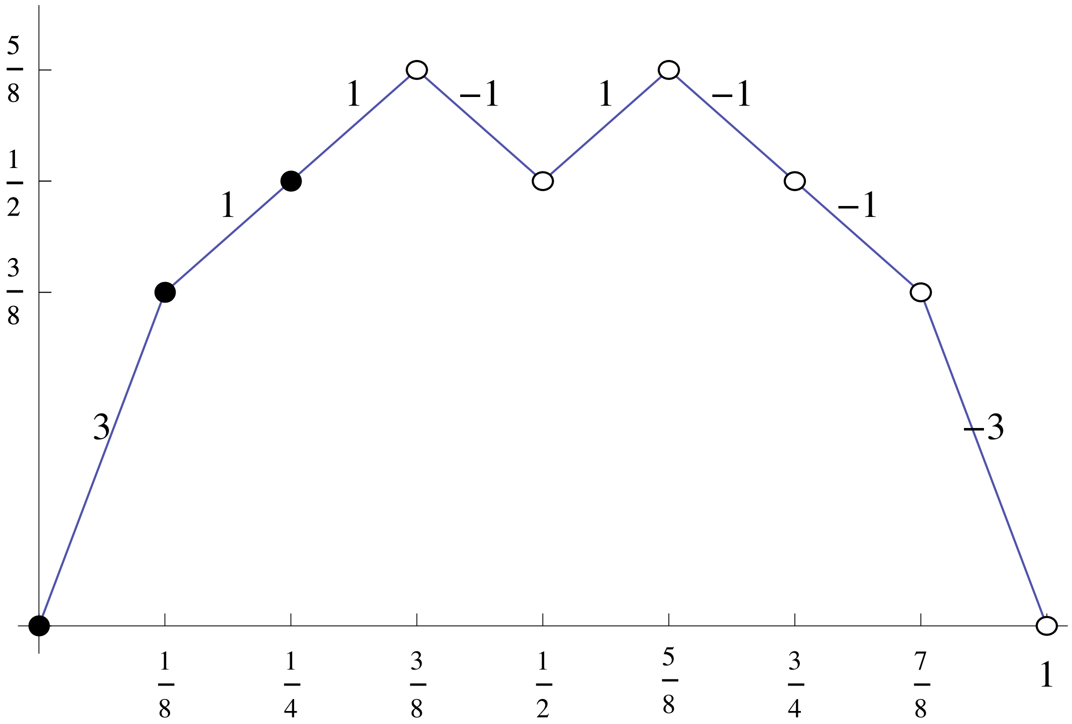
- This is a piecewise linear function, with breaks at the dyadic integers $\frac{k}{2^n}$, $1 \leq k \leq 2^n - 1$.

- All segments have integer slopes, ranging between $-n$ and $+n$. The maximal slope $+n$ is attained on $[0, \frac{1}{2^n}]$ and the minimal slope $-n$ on $[1 - \frac{1}{2^n}, 1]$.

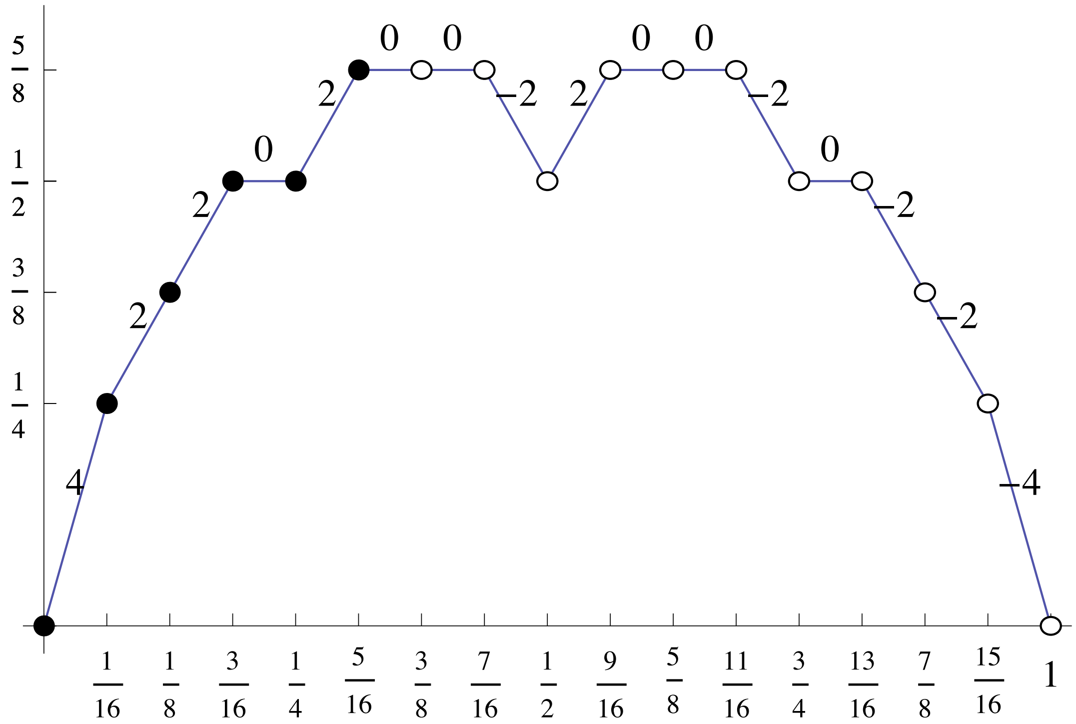
Takagi Approximants-τ_2



Takagi Approximants-τ_3



Takagi Approximants-τ_4



Properties of Approximants

- The n-th approximant

$$\tau_n(x) := \sum_{j=0}^{n} \frac{1}{2^j} \ll 2^j x \gg$$

agrees with $\tau(x)$ at all dyadic rationals $\frac{k}{2^n}$.
These values then freeze, i.e. $\tau_n(\frac{k}{2^n}) = \tau_{n+j}(\frac{k}{2^n})$.

- The approximants are nondecreasing at each step. Thus they approximate Takagi function $\tau(x)$ from below.

Symmetry

- Local symmetry

$$\tau_n(x) = \tau_n(1 - x).$$

- Hence:

$$\tau(x) = \tau(1 - x).$$

Functional Equations

- Fact. The Takagi function, satisfies, for $0 \le x \le 1$, two functional equations:

$$\begin{aligned} \tau(\frac{x}{2}) &= \frac{1}{2}\tau(x) + \frac{1}{2}x \\ \tau(\frac{x+1}{2}) &= \frac{1}{2}\tau(x) + \frac{1}{2}(1-x). \end{aligned}$$

- These are a kind of dilation equation: They relate function values on two different scales.

Takagi Function Formula

- Takagi's Formula (1903): Let $x \in [0,1]$ have the binary expansion

$$x = .b_1 b_2 b_3 ... = \sum_{j=1}^{\infty} \frac{b_j}{2^j}.$$

Then

$$\tau(x) = \sum_{n=1}^{\infty} \frac{l_n(x)}{2^n}.$$

with

$$\begin{aligned} l_n(x) &= b_1 + b_2 + \cdots + b_{n-1} && \text{if bit} \quad b_n = 0. \\ &= (n-1) - (b_1 + b_2 + \cdots + b_{n-1}) && \text{if bit} \quad b_n = 1. \end{aligned}$$

Takagi Function Formula-2

Example. $\frac{1}{3} = .0101\overline{01}...$ (binary expansion)

We have

$$\ll 2 \cdot \frac{1}{3} \gg = \ll \frac{2}{3} \gg = \frac{1}{3}, \quad \ll 4 \cdot \frac{1}{3} \gg = \frac{1}{3}, ...$$

so by definition of the Takagi function

$$\tau(\frac{1}{3}) = \frac{\frac{1}{3}}{1} + \frac{\frac{1}{3}}{2} + \frac{\frac{1}{3}}{4} + \frac{\frac{1}{3}}{8} + \cdots = \frac{2}{3}.$$

Alternatively, the Takagi formula gives

$$\tau(\frac{1}{3}) = \frac{0}{2} + \frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{2}{32} + \frac{3}{64} ... = \frac{2}{3}.$$

Takagi Function Formula-3

Example. $\frac{1}{5} = .0011\overline{0011}...$ (binary expansion)

We have

$$\ll 2 \cdot \frac{1}{5} \gg = \frac{2}{5}, \quad \ll 4 \cdot \frac{1}{5} \gg = \frac{1}{5}, \quad \ll 8 \cdot \frac{1}{5} \gg = \frac{2}{5}, ...$$

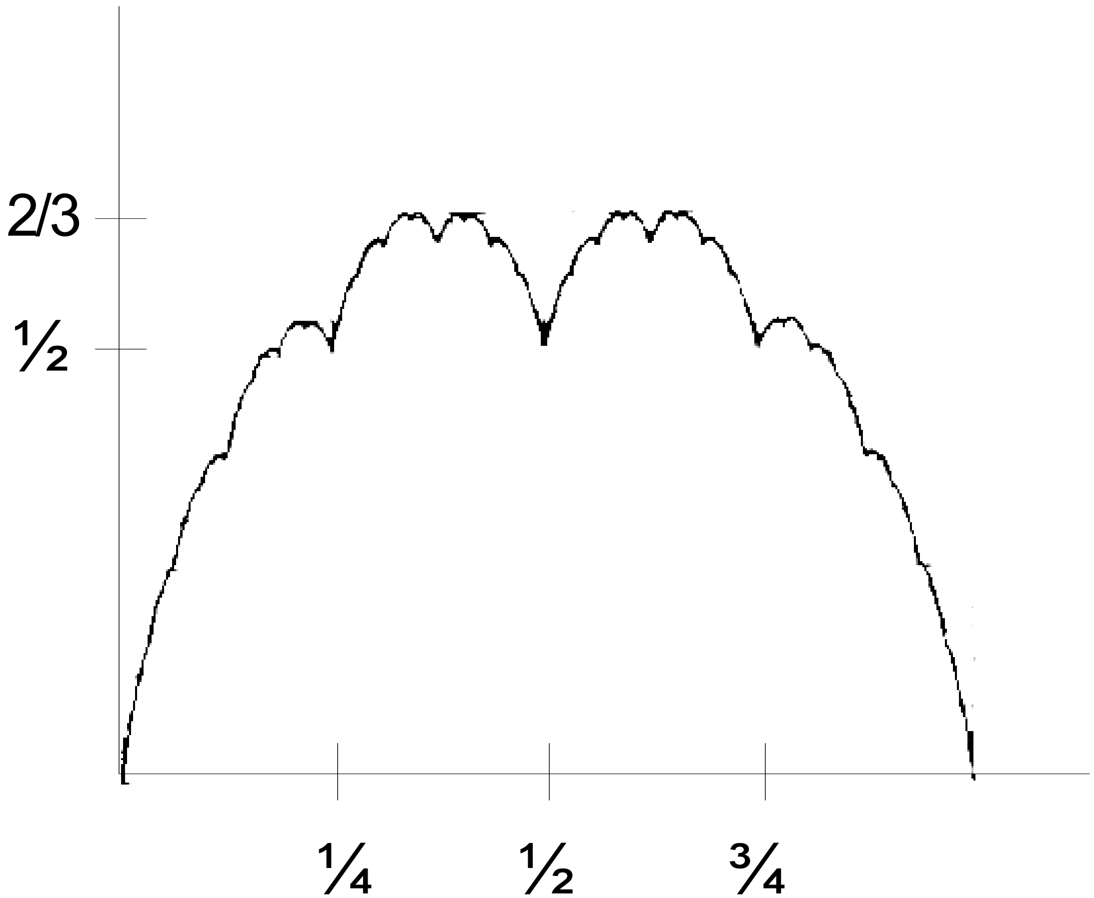
so by definition of the Takagi function

$$\tau(\frac{1}{5}) = \frac{\frac{1}{5}}{1} + \frac{\frac{2}{5}}{2} + \frac{\frac{1}{5}}{4} + \frac{\frac{2}{5}}{8} + \cdots = \frac{8}{15}.$$

Alternatively, the Takagi formula gives

$$\tau(\frac{1}{5}) = \frac{0}{2} + \frac{0}{4} + \frac{0}{8} + \frac{2}{16} + \frac{2}{32} + \frac{2}{64} + \frac{2}{128} + \frac{4}{256} + ... = \frac{8}{15}.$$

Graph of Takagi Function: Review



Fourier Series

Theorem. The Takagi function $\tau(x)$ is periodic with period 1. It is is an even function. So it has a Fourier series expansion

$$\tau(x) := c_0 + \sum_{n=1}^{\infty} c_n \cos(2\pi n x)$$

with Fourier coefficients

$$c_n = 2\int_0^1 \tau(x)\cos(2\pi n x)dx = 2\int_0^1 \tau(x)e^{2\pi i n x}dx$$

These are:

$$c_0 = \int_0^1 \tau(x)dx = \frac{1}{2},$$

and, for $n \geq 1$, writing $n = 2^m(2k+1)$,

$$c_n = \frac{2^m}{(n\pi)^2}.$$

Part II. Number Theory: Counting Binary Digits

- Consider the integers $1, 2, 3, \ldots$ represented in binary notation. Let $S_2(N)$ denote the sum of the binary digits of $0, 1, \ldots, N-1$, i.e. $S_2(N)$ counts the total number of $1's$ in these expansions.

$N =$	1	2	3	4	5	6	7	8	9
	1	10	11	100	101	110	111	1000	1001
$S_2(N) =$	1	2	4	5	7	9	12	13	15

- The function arises in analysis of algorithms for searching: Knuth, Art of Computer Programming, Volume 4 (2011).

Counting Binary Digits-2

- Bellman and Shapiro (1940) showed $S_2(N) \sim \frac{1}{2} N \log_2 N$.

- Mirsky (1949) improved this: $S_2(N) = \frac{1}{2} N \log_2 N + O(N)$.

- Trollope (1968) improved this:

$$S_2(N) = \frac{1}{2} N \log_2 N + N\, E_2(N),$$

  where $E_2(N)$ is a bounded oscillatory function.
  He gave an exact combinatorial formula for $E_2(N)$ involving the Takagi function.

Counting Binary Digits-3

- Delange (1975) gave an elegant improvement of Trollope's result...

- Theorem. (Delange 1975) There is a continuous function $F(x)$ of period 1 such that, for all integer $N \geq 1$,

$$S_2(N) = \frac{1}{2} N \log_2 N + N\, F(\log_2 N),$$

in which:

$$F(x) = \frac{1}{2}(1 - \{x\}) - 2^{-\{x\}} \tau(2^{\{x\}-1})$$

where $\tau(x)$ is the Takagi function, and $\{x\} := x - [x]$.

Counting Binary Digits-4

- The function $F(x) \leq 0$, with $F(0) = 0$.

- Delange found that $F(x)$ has an explicit Fourier expansion whose coefficients involve the values of the Riemann zeta function on the line $Re(s) = 0$, at $\zeta(\frac{2k\pi i}{\log 2})$, $k \in \mathbb{Z}$.

Counting Binary Digits-5

- Flajolet, Grabner, Kirchenhofer, Prodinger and Tichy (1994) gave a direct proof of Delange's theorem using Dirichlet series and Mellin transforms.

- Identity 1. Let $e_2(n)$ sum the binary digits in n. Then
$$\sum_{n=1}^{\infty} \frac{e_2(n)}{n^s} = 2^{-s}(1-2^{-s})^{-1}\zeta(s).$$

Counting Binary Digits-6

- Identity 2: Special case of Perron's Formula. Let

$$H(x) := \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{\zeta(s)}{2^s-1} x^s \frac{ds}{s(s-1)}.$$

  Then for integer N have an exact formula

$$H(N) = \frac{1}{N} S_2(N) - \frac{N-1}{2}.$$

- Proof. Shift the contour to $Re(s) = -\frac{1}{4}$. Pick up contributions of a double pole at $s = 0$ and simple poles at $s = \frac{2\pi ik}{\log 2}$, $k \in \mathbb{Z}, k \neq 0$. Miracle occurs: The shifted contour integral vanishes for all integer values $x = N$. (It is a kind of step function, and does not vanish identically.)

Part III. Analysis: Fluctuation Properties

- The Takagi function oscillates rapidly. It is an analysis problem to understand the size of its fluctuations on various scales.

- These problems have been completely answered, as follows...

Fluctuation Properties: Single Fixed Scale

- The maximal oscillations at scale h are of order: $h \log_2 \frac{1}{h}$.

- Proposition. For all $0 < h < 1$ the Takagi function satisfies

$$|\tau(x+h) - \tau(x)| \leq 2\, h \, \log_2 \frac{1}{h}.$$

- This bound is sharp within a multiplicative factor of 2. Kôno (1987) showed that as $h \to 0$ the constant goes to 1.

Maximal Asymptotic Fluctuation Size

- The asymptotic maximal fluctuations at scale $h \to 0$ are of order: $h\sqrt{2\log_2 \frac{1}{h} \log\log\log_2 \frac{1}{h}}$ in the following sense.

- Theorem (Kôno 1987) Let $\sigma_l(h) = \sqrt{\log_2 \frac{1}{h}}$. Then for all $x \in (0,1)$,

$$\limsup_{h \to 0^+} \frac{\tau(x+h) - \tau(x)}{h\,\sigma_l(h)\sqrt{2\log\log \sigma_l(h)}} = 1,$$

and

$$\liminf_{h \to 0^+} \frac{\tau(x+h) - \tau(x)}{h\,\sigma_l(h)\sqrt{2\log\log \sigma_l(h)}} = -1.$$

Average Scaled Fluctuation Size

- Average Fluctuation size at scale h is Gaussian, proportional to $h\sqrt{\log_2 \frac{1}{h}}$.

- Theorem (Gamkrelidze 1990) Let $\sigma_l(h) = \sqrt{\log_2 \frac{1}{h}}$. Then for each real y,

$$\lim_{h \to 0^+} Meas\left\{x : \frac{\tau(x+h) - \tau(x)}{h\,\sigma_l(h)} \le y\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}t^2} dt.$$

- Kôno's result on maximum asymptotic fluctuation size is analogous to the law of the iterated logarithm.

Part IV. Rational Values

- Easy Fact.

  (1) The Takagi function maps dyadic rational numbers $\frac{k}{2^n}$ to dyadic rational numbers $\tau(\frac{k}{2^n}) = \frac{k'}{2^{n'}}$, where $n' \leq n$.

  (2) The Takagi function maps rational numbers $r = \frac{p}{q}$ to rational numbers $\tau(r) = \frac{p'}{q'}$. Here the denominator of $\tau(r)$ may sometimes be larger than that of r.

- Next formulate four (hard?) unsolved problems...

Rational Values: Pre-Image Problems

- Problem 1. Determine whether a rational r' has some rational preimage r with $\tau(r) = r'$.

- Problem 2. Determine which rationals r' have an uncountable level set $L(r')$.

  This (unsolved) problem was raised by Donald Knuth in: The Art of Mathematical Programming Volume 4 (Fascicle 3, Problem 83 in 7.2.1.3 (2004)) He says: "WARNING: This problem can be addictive."

Rational Values: Iteration Problems

- Problem 3. Determine the behavior of $\tau(x)$ under iteration, on domain of dyadic rational numbers.

  For dyadic rationals the denominators are nonincreasing, so all iterates go into periodic orbits. Figuring out orbit structure could be an challenging problem.

- Problem 4. Same, on larger domain of all rational numbers.

  Here the denominators can increase or decrease at each iteration. This feature resembles: the $3x + 1$ problem.

Part V. Level Sets of the Takagi Function

- Definition. The level set $L(y) = \{x : \quad \tau(x) = y\}$.

- Problem. How large are the level sets of the Takagi function?

- Quantitative Problem. Determine exact count if finite; Determine Hausdorff dimension if infinite.

- Answer depends on sampling method: Could choose random x-values (abscissas) or random y-values (ordinates)

Aside: Hausdorff Dimension

- Hausdorff dimension is a measure of size of a point set in a metric space. ("Fractional dimension").

- Fact. For any subset S of real line: $0 \leq \text{dim}_H(S) \leq 1$.

- Fact. All countable sets S have Hausdorff dimension 0, so any set of positive Hausdorff dimension is uncountable.

- Fact. The Cantor set has Hausdorff dimension $\frac{\log 2}{\log 3} \approx 0.630$.

Size of Level Sets: Cardinality

- Fact. There exist levels y such that $L(y)$ is finite, countable, or uncountable.

- $L(\frac{1}{5})$ is finite, containing two elements.
  Knuth (2004) showed that $L(\frac{1}{5}) = \{\frac{3459}{87040}, \frac{83581}{87040}\}$.

- $L(\frac{1}{2})$ is countably infinite.

- $L(\frac{2}{3})$ is uncountably infinite.
  Baba (1984) observed this holds...because...

Size of Level Sets: Hausdorff Dimension

- Theorem (Baba 1984) The set $L(\frac{2}{3})$ has Hausdorff dimension $\frac{1}{2}$.

- This result followed up by...

- Theorem (Maddock 2010) All level sets $L(y)$ have Hausdorff dimension at most 0.699.

- Conjecture (Maddock 2010) All level sets $L(y)$ have Hausdorff dimension at most $\frac{1}{2}$.

Local Level Sets-1

- Approach to understand level sets: break them into local level sets, which are easier to understand.

- The local level set containing x is described completely in terms of the binary expansion of $x = \sum_{n \geq 1} b_n 2^{-n}$.

Deficient Digit Function-1

- Definition. The deficient digit function $D_n(x)$ counts the excess of 0's over 1's in the first n digits of the binary expansion of x.

- Example. $x = .00111001...$

n	1	2	3	4	5	6	7	8
b_n	0	0	1	1	1	0	0	1
$D_n(x)$	1	2	1	0	-1	0	-1	0

- Defn. The breakpoints are positions where $D_n(x) = 0$. In example these are positions 4, 6, and 8 ...

Deficient Digit Function-2

- Deficient digit function formula:

$$D_n(x) = n - 2(b_1 + b_2 + \cdots + b_n)$$

- Congruence: $D_n(x) \equiv n \pmod{2}$

- Bounds: $\quad -n \leq D_n(x) \leq n$

- Key Fact. The values $\{D_1(x), D_2(x), D_3(x), ...\}$ for a random x follow a simple random walk that takes equal steps of size ± 1.

Local Level Sets-2

- Given x, look at all the breakpoint values
$$0 = c_0 < c_1 < c_2 < \dots$$
where $D_{c_j}(x) = 0$, i.e. values n where the random walk returns to the origin. Call this set the breakpoint set $Z(x)$.

- The binary expansion of x is broken into blocks of digits with position $c_j < n \le c_{j+1}$. The flip operation exchanges digits 0 and 1 inside a block.

- Definition. The local level set L_x^{loc} consists of all numbers $x' \sim x$ by a (finite or infinite) set of flip operations. All numbers in L_x^{loc} have the same breakpoint set $Z(x) = Z(x')$.

Propeties of Local Level Sets

- Property 1. L_x^{loc} is a closed set.

- Property 2. L_x^{loc} is either a finite set of cardinality $2^{Z(x)}$, if there are finitely many blocks in $Z(x)$, or is a Cantor set if there are infinitely many blocks in $Z(x)$.

- Property 3. Each level set partitions into a disjoint union of local level sets.

Level Sets-Abscissa Viewpoint

- Problem. Draw a random point x uniformly in $[0, 1]$. How large is the level set $L(\tau(x))$?

- Partial Answer. At least as large as the local level set L_x^{loc}.

- Theorem A. For a randomly drawn point x, with probability one the local level set L_x^{loc} is an uncountable (Cantor) set , of Hausdorff dimension 0.

Proof of Theorem A

(1) With probability one, the set of breakpoints $Z(x)$ is infinite: A one-dimensional random walk $D_n(x)$ returns to the origin infinitely often almost surely. This makes L_x^{loc} a Cantor set.

(2) With probability one, the expected time for a one-dimensional random walk $D_n(x)$ to return to the origin is infinite. This “implies” that most local level sets have Hausdorff dimension 0.

Expected Number of Local Level Sets: Ordinate View

- The number of local level sets on a level can be an arbitrarily large integer value and also can be countably infinite.

- We are able to estimate the number of local level sets when the ordinate y is picked at random:

- Theorem B. The expected number of local level sets for an (ordinate) y drawn uniformly from $[0, \frac{2}{3}]$ is exactly 3/2.

Level Sets-Ordinate View

- We can compute the expected size of a level set $L(y)$ for a random (ordinate) level y...

- Theorem C.
  (1) The expected size of a level set $L(y)$ for y drawn at random from $[0, \frac{2}{3}]$ is finite.

  (2) However, the expected number of elements in a level set $L(y)$ for y drawn at random from $[0, \frac{2}{3}]$ is infinite.

- Result (1) first proved by Buczolich(2008).

Local Level Sets: Size Paradox?

- Ordinate View: Level sets $L(y)$ are finite with probability 1.

- Abscissa View: Level sets $L(\tau(x))$ are uncountably infinite with probability 1.

- Reconciliation Mechanism: x-values preferentially select level sets that are "large".

Approach to Results

- Idea is to study the left hand endpoints of local level sets...

- Definition. The deficient digit set Ω^L is the set of left-hand endpoints of all local level sets.

- Fact. The set Ω^L consists of all real numbers x whose binary expansions have at least as many 0's as 1's after n steps. That is, all $D_n(x) \geq 0$.
  (The random walk stays nonnegative!)

Approach to Results-cont'd.

- Key point. Ω^L keeps track of all local level sets. It is a closed set obtained by removing a countable set of open intervals from $[0, 1]$. It has has a Cantor set structure.

- Theorem. Ω^L has measure 0, but has full Hausdorff dimension 1.

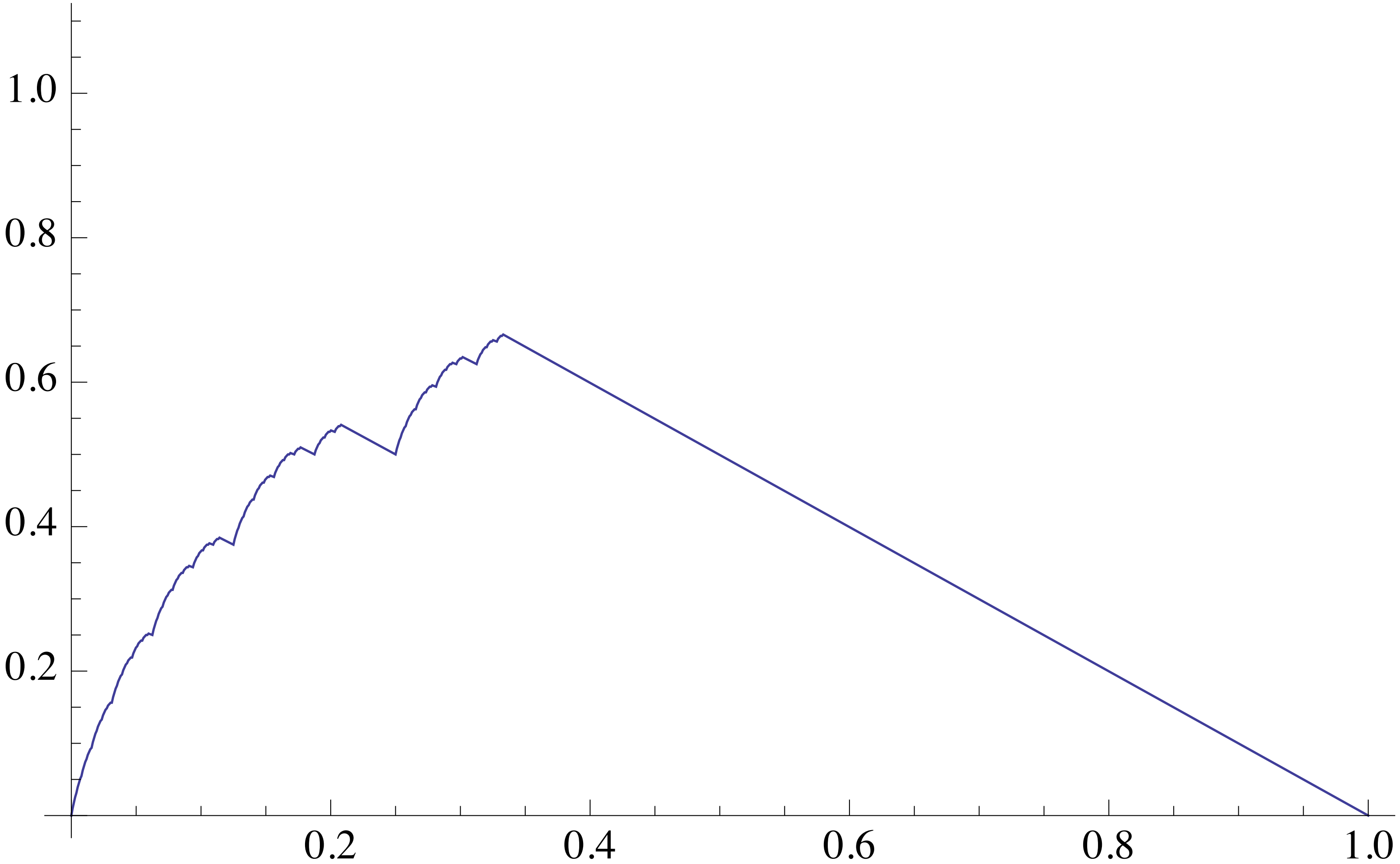
Flattened Takagi Function

- Restrict the Takagi function to Ω^L. On every open interval that was removed to construct Ω^L, linearly interpolate this function between the two endpoints.

- Call the resulting function $\tau^L(x)$ the flattened Takagi function.

- Amazing Fact. (Or Trivial Fact.) All the linear interpolations have slope -1.

Graph of Flattened Takagi Function



Flattened Takagi Function-2

- Claim. The flattened Takagi function has much less oscillation than the Takagi function. Namely...

- Theorem F.
  (1) The flattened Takagi function $\tau^L(x)$ is a function of bounded variation. That is, it is the sum of an increasing function (means: nondecreasing) and a decreasing function (means: nonincreasing).
  (This is called: Jordan decomposition of BV function.)

  (2) $\tau^L(x)$has total variation $V_0^1(\tau^L) = 2$.

- This theorem follows from...

Takagi Singular Function

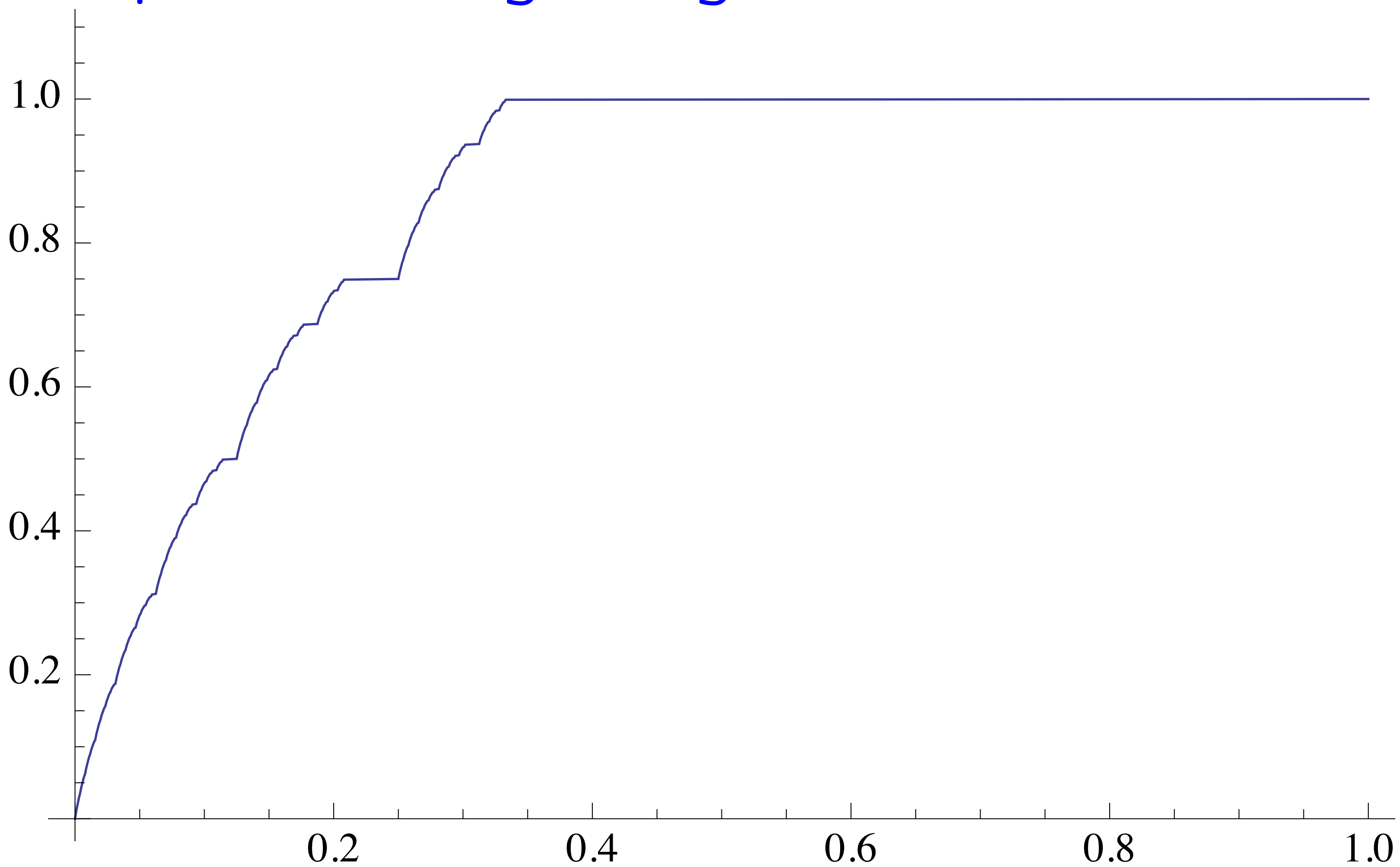
- Theorem D. (1) The flattened Takagi function has a Jordan decomposition

$$\tau^L(x) = \tau^S(x) + (-x),$$

  That is, it is the sum of an upward monotone function $\tau^S(x)$ and a downward monotone function $-x$.

  (2) The function $\tau^L(x)$ is a singular continuous function; it has derivative 0 off the set Ω^L.
  Call it the Takagi singular function.

Graph of Takagi Singular Function



Takagi Singular Function

- The Takagi singular function is the integral of a singular measure:

$$\tau^S(x) = \int_0^x d\mu^S(t)$$

  Call μ^S the Takagi singular measure. It is supported on Ω^L, which has area 0.

- The Takagi singular measure is obviously not translation-invariant. But it satisfies various functional equations coming from those of the Takagi function. It is possible to compute with it. Used to prove results.

Concluding Remarks.

- The Takagi function is a great example of many phenomena in classical analysis and probability theory.

- Found interesting new internal structures: Local level sets and Takagi singular function.

- Raised various open problems:

  (1) Determine the structure of rational levels;

  (2) Study Takagi function as a dynamical system under iteration.

Thank you for your attention!