A Totally Disconnected Thread: Some Complicated p-Adic Fractals

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Topics Covered

- Part I. Ternary expansions of powers of 2
- Part II. A 3-Adic generalization
- Part III. p-Adic path set fractals
- Part IV. Intersections of translates of 3-adic Cantor sets

Credits-1

- Part I : P. Erdős,
  Some Unconventional Problems in Number Theory,
  Math. Mag. **52** (1979), 67–70.

- Philip J. Davis, The Thread-A Mathematical Yarn,
  Birkhäuser, Basel, 1983. (Second Edition. Harcourt, 1989.)

- "The Thread" follows a quest of the author to find out the first name and its origins of the Russian mathematician and number theorist: P. L. Chebyshev (1821–1894),

  [This quest was done before Google (published in 14 B.G.). Now a mouse click does it.]

Credits-2

- Part II:
  J. C. Lagarias, Ternary Expansions of Powers of 2,
  J. London Math. Soc. **79** (2009), 562–588.

- Part III:
  W. C. Abram and J. C. Lagarias,
  Path sets and their symbolic dynamics,
  Adv. Applied Math. **56** (2014), 109–134.

  W. C. Abram and J. C. Lagarias,
  p-adic path set fractals,
  J. Fractal Geom. **1** (2014), 45–81.

Credits-3

- Part IV:
  W. C. Abram and J. C. Lagarias, Intersections of Multiplicative Translates of 3-adic Cantor sets, J. Fractal Geom. **1** (2014), 349–390.

  W. C. Abram, A. Bolshakov and J. C. Lagarias, Intersections of Multiplicative Translates of 3-adic Cantor sets II, preprint.

- Work of J.C.Lagarias supported by NSF grants DMS-1101373 and DMS-1401224.
  Work by W. C. Abram supported by an NSF Fellowship and Hillsdale College.

Part I. Erdős Ternary Digit Problem

- Problem. Let $(M)_3$ denote the integer M written in ternary (base 3). How many powers 2^n of 2 omit the digit 2 in their ternary expansion?

Examples	Non-examples
$(2^0)_3 = 1$	$(2^3)_3 = 22$
$(2^2)_3 = 11$	$(2^4)_3 = 121$
$(2^8)_3 = 100111$	$(2^6)_3 = 2101$

- Conjecture. (Erdős 1979) There are no solutions for $n \geq 9$.

Erdős Ternary Digit Problem: Binomial Coefficient Motivation

- *Motivation.* 3 does not divide the binomial coefficient $\binom{2^{k+1}}{2^k}$ if and only if the ternary expansion of 2^k omits the digit 2.

Heuristic for Erdős Ternary Problem

- The ternary expansion $(2^n)_3$ has about

$$\alpha_0 n \quad \text{digits}$$

where

$$\alpha_0 := \log_3 2 = \frac{\log 2}{\log 3} \approx 0.63091$$

- Heuristic. If ternary digits were picked randomly and independently from $\{0, 1, 2\}$, then the probability of avoiding the digit 2 would be $\approx \left(\frac{2}{3}\right)^{\alpha_0 n}$.

- These probabilities decrease exponentially in n, so their sum converges. Thus expect only finitely many n to have expansion $[2^n]_3$ that avoids the digit 2.

Part II. 3-Adic Dynamical System Generalizations of Erdős Ternary Digit Problem

- Approach: View the set $\{1, 2, 4, ...\}$ as a forward orbit of the discrete dynamical system $T : x \mapsto 2x$.

- The forward orbit $\mathcal{O}(x_0)$ of x_0 is

$$\mathcal{O}(x_0) := \{x_0, T(x_0), T^{(2)}(x_0) = T(T(x_0)), \cdots\}$$

  Thus: $\mathcal{O}(1) = \{1, 2, 4, 8, \cdots\}$.

- Changed Problem. Study the forward orbit $\mathcal{O}(\lambda)$ of an arbitrary initial starting value λ. How big can its intersection with the "Cantor set" be?

3-adic Integer Dynamical System-1

- View the integers $\mathbb{Z}$ as contained in the set of 3-adic integers $\mathbb{Z}_3$.

- The 3-adic integers $\mathbb{Z}_3$ are the set of all formal expansions

$$\beta = d_0 + d_1 \cdot 3 + d_2 \cdot 3^2 + \ldots$$

  where $d_i \in \{0, 1, 2\}$. Call this the 3-adic expansion of β.

- Set $ord_3(0) := +\infty$ and $ord_3(\beta) := \min\{j : d_j \neq 0\}$.

  The 3-adic size of $\beta \in \mathbb{Q}_3$ is:

$$||\beta||_3 = 3^{-ord_3(\beta)}$$

3-adic Integer Dynamical System-2

- Now view $\{1, 2, 4, 8, ...\}$ as a subset of the 3-adic integers.

- The modified 3-adic Cantor set $\tilde{\Sigma}_{3,\bar{2}}$ is the set of all 3-adic integers whose 3-adic expansion omits the digit 2. The Hausdorff dimension of $\tilde{\Sigma}_{3,\bar{2}}$ is $\log_3 2 \approx 0.630929$.

- We impose the condition: avoid the digit 2 on all 3-adic digits.

- Define for $\lambda \in \mathbb{Z}_3$ the complete intersection set

$$N^*(\lambda; \mathbb{Z}_3) := \{n \geq 1 : \quad \text{the full 3-adic expansion } (\lambda 2^n)_3 \text{ omits the digit 2}\}$$

Complete 3-adic Exceptional Set-2

- The 3-adic exceptional set is

$$\mathcal{E}^*_\infty(\mathbb{Z}_3) \; := \; \{\lambda > 0 : \text{ the complete intersection set } N^*(\lambda; \mathbb{Z}_3) \text{ is infinite.}\}$$

- The set $\mathcal{E}^*_\infty(\mathbb{Z}_3)$ ought to be very small. Conceivably it is just one point $\{0\}$. (If it is larger, then it must be infinite.)

Exceptional Set Conjecture

- *Exceptional Set Conjecture.*
  The 3-adic exceptional set $\mathcal{E}_{\infty}^{*}(\mathbb{Z}_3)$ has
  Hausdorff dimension zero.

- This is our quest: a totally disconnected thread.

- The problem seems approachable because it has nice symbolic dynamics. Hausdorff dimensions of finite intersections can be computed exactly, in principle.

Family of Subproblems

- The Level k exceptional set $\mathcal{E}_k^*(\mathbb{Z}_3)$ has those λ that have at least k distinct powers of 2 with $\lambda 2^k$ in the Cantor set, i.e.

$$\mathcal{E}_k^*(\mathbb{Z}_3) := \{\lambda > 0 : \text{ the set } N^*(\lambda; \mathbb{Z}_3) \geq k.\}$$

- Level k exceptional sets are nested by increasing k:

$$\mathcal{E}_\infty^*(\mathbb{Z}_3) \subset \cdots \subset \mathcal{E}_3^*(\mathbb{Z}_3) \subset \mathcal{E}_2^*(\mathbb{Z}_3) \subset \mathcal{E}_1^*(\mathbb{Z}_3)$$

- Subproblem: Study the Hausdorff dimension of $\mathcal{E}_k^*(\mathbb{Z}_3)$; it gives an upper bound on $dim_H(\mathcal{E}^*(\mathbb{Z}_3))$.

Upper Bounds on Hausdorff Dimension

- Theorem. (Upper Bound Theorem)

  (1). $$dim_H(\mathcal{E}_1^*(\mathbb{Z}_3)) = \alpha_0 \approx 0.63092.$$

  (2). $$dim_H(\mathcal{E}_2^*(\mathbb{Z}_3)) \quad \leq 0.5.$$

- Remark. However there is a lower bound:

$$dim_H(\mathcal{E}_2^*(\mathbb{Z}_3)) \geq \log_3(\frac{1+\sqrt{5}}{2}) \approx 0.438$$

Upper Bounds on Hausdorff Dimension

- Question. Could it be true that

$$\lim_{k\to\infty} dim_H(\mathcal{E}_k^*(\mathbb{Z}_3)) = 0?$$

- If so, this would imply that the complete exceptional set $\mathcal{E}^*(\mathbb{Z}_3)$ has Hausdorff dimension 0.

Upper Bound Theorem: Proof Idea

- The set $\mathcal{E}_k^*(\mathbb{Z}_3)$ is a countable union of closed sets

$$\mathcal{E}_k^*(\mathbb{Z}_3) = \bigcup_{r_1 < r_2 < \ldots < r_k} \mathcal{C}(2^{r_1}, 2^{r_2}, \ldots, 2^{r_k}),$$

  given by

$$\mathcal{C}(2^{r_1}, 2^{r_2}, \ldots, 2^{r_k}) := \{\lambda : \ (2^{r_i}\lambda)_3 \ \text{ omits digit 2}\}.$$

- We have

$$dim_H(\mathcal{E}_k^*(\mathbb{Z}_3)) = \sup\{dim_H\left(\mathcal{C}(2^{r_1}, 2^{r_2}, \ldots, 2^{r_k})\right)\}$$

- Proof for $k = 1, 2$: obtain upper bounds on Hausdorff dimension of all the sets $\mathcal{C}(2^{r_1}, 2^{r_2}, \ldots, 2^{r_k})$.

Part III. Path Sets and p-adic Path Set Fractals

- Definition Consider sets S of all p-adic integers whose p-adic expansions are describable as the set of edge label vectors of any infinite legal path in a finite directed graph) with labeled edges (finite nondeterministic automaton) starting from a fixed origin node.

- Call any such set S a p-adic path set fractal.

- Generalized Problem. Investigate the structure and properties p-adic path set fractals.

Path Sets-1

- Further Abstraction. Keep only the symbolic dynamics and forget the p-adic embedding: regard S as embedded in a symbol space $A^{\mathbb{N}}$ of an alphabet A with N symbols. Call the resulting symbolic object a path set.

- If we allowed only S which are unions of paths starting from any vertex, then the allowable S are a known dynamical object: a one-sided sofic shift.

- But path sets are a more general concept. They are *not closed* under the action of the one-sided shift map.

$$\sigma(a_0 a_1 a_2 a_3 \cdots) = a_1 a_2 a_3 a_4$$

Path Sets-2

- Path sets are closed under several operations.

  1. Finite unions and intersections of path sets are path sets.

  2. A "decimation" operation that saves only symbols in arithmetic progressions takes path sets to path sets

- The topological entropy of a path set is computable from the incidence matrix for a finite directed graph representing the path set (that is in a suitable normal form).

  It is the logarithm to base N of the largest eigenvalue of the incidence matrix.

P-adic path set fractals-1

- p-adic path set fractals are the image of a path set under a map of the symbol space into the p-adic integers. This embedding can be non-trivial because it uses an mapping of the alphabet $A \to \{0, 1, 2, ..., p-1\}$. In particular many symbols in A may get mapped to the same p-adic digit.

- If the alphabet mapping is one-to-one,then the topological entropy of the path set and the Hausdorff dimension of the p-adic path set fractal are proportional, otherwise not.

- The p-adic topology imposes a geometry on the image. The appearance of the image is dependent on the digit assignment map.

p-adic arithmetic on p-adic path set fractals-1

- Theorem. Suppose S_1 and S_2 are p-adic path set fractals. Define the Minkowski sum

$$S_1 + S_2 := \{s_1 + s_2 : \quad s_1 \in S_1 \; s_2 \in S_2\}$$

where the sum is p-adic addition. Then $S_1 + S_2$ is a p-adic path set fractal.

- Theorem. Suppose $\alpha \in \mathbb{Z}_p$ is a rational number $\alpha = \frac{m}{n}$ with $m, n \in \mathbb{Z}$. If S is a p-adic path set fractal then so is the mulitplicative dilation αS, using p-adic multiplication.

p-adic arithmetic on p-adic path set fractals-2

- 1. There are effectively computable algorithms which given an automaton representing S_1 and S_2, reap. α, can compute an automaton representing $S_1 + S_2$, resp. αS_1.

  2. From these automata Hausdorff dimensions can be directly computed.

- The behavior of Hausdorff dimension under Minkowski sum and under intersection of p-adic path set fractals is *complicated and mysterious.* It depends on arithmetic! But the operation of dilation preserves Hausdorff dimension.

Part IV. Intersections of Translates of 3-adic Cantor sets

- New Problem. For positive integers $r_1 < r_2 < \cdots < r_k$ set

$$\mathcal{C}(2^{r_1}, 2^{r_2}, ..., 2^{r_k}) := \{\lambda : (2^{r_i}\lambda)_3 \text{ omits the digit 2}\}$$

  Determine the Hausdorff dimension of $\mathcal{C}(2^{r_1}, 2^{r_2}, ..., 2^{r_k})$.

- More generally, allow arbitrary positive integers $N_1, N_2, ..., N_k$. Determine the Hausdorff dimension of:

$$\mathcal{C}(N_1, N_2, \cdots, N_k) := \{\lambda : \text{all } (N_i\lambda)_3 \text{ omit the digit 2}\}$$

Discovery and Experimentation

- The Hausdorff dimension of sets $\mathcal{C}(N_1, N_2, ..., N_k)$ can in principle be determined exactly. (Structure of these sets describable by finite automata.)

- Mainly discuss special case $\mathcal{C}(1, N)$, for simplicity.

- This special case already has a complicated and intricate structure!

Basic Structure of the answer-1

- The 3-adic expansions of members of sets $\mathcal{C}(N_1, N_2, ..., N_k)$ are describable dynamically as having the symbolic dynamics of a sofic shift, given as the set of allowable infinite paths in a suitable labelled graph (finite automaton).

- The sequence of allowable paths is characterized by the topological entropy of the dynamical system. This is the growth rate ρ of the number of allowed label sequences of length n. It is the maximal (Perron-Frobenius) eigenvalue ρ of the weight matrix of the labelled graph, a non-negative integer matrix. (Adler-Konheim-McAndrew (1965))

Basic Structure of the answer-2

- The Hausdorff dimension of the associated "fractal set" $\mathcal{C}(N_1, ..., N_k)$ is given as the base 3 logarithm of the topological entropy of the dynamical system.

- This is $\log_3 \rho$ where ρ is the Perron-Frobenius eigenvalue of the symbol weight matrix of the labelled graph.

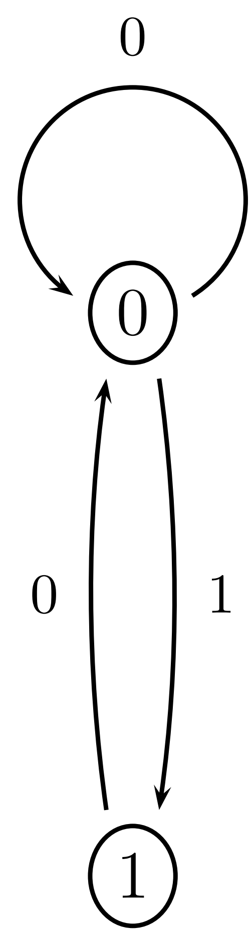
- Remark. These sets are 3-adic analogs of "self-similar fractals" in sense of Hutchinson (1981), as extended in Mauldin-Williams (1985). Such a set is a fixed point of a system of set-valued functional equations.

Basic Structure of the answer-3

- If some $N_j \equiv 2 \pmod 3$ occurs, then Hausdorff dimension $\mathcal{C}(N_1, N_2, ..., N_k)$ will be 0.

- If one replaces N_j with $3^k N_j$ then the Hausdorff dimension does not change.

- Can therefore reduce to case: All $N_j \equiv 1 \pmod 3$.



Graph: $N = 2^2 = 4$

Associated Matrix $N = 4$

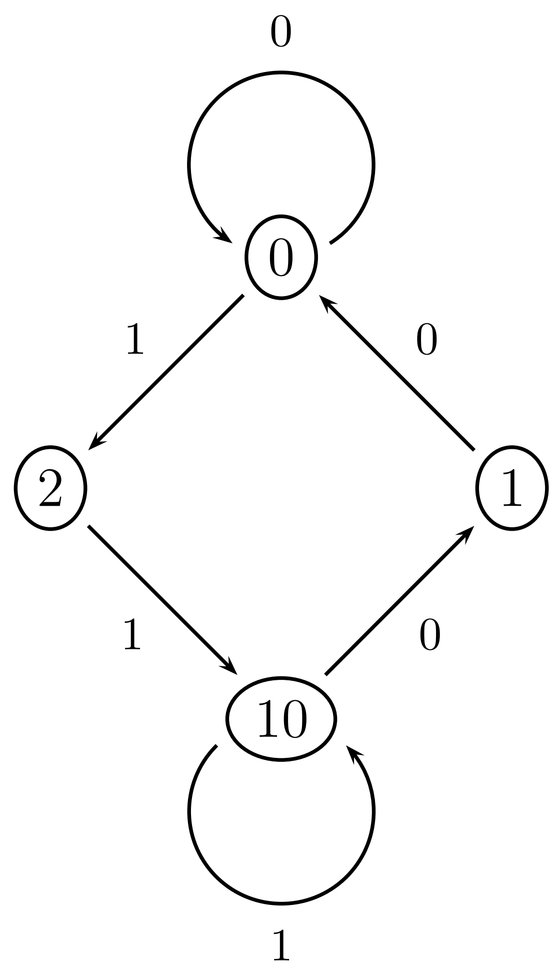
- Weight matrix is:

$$\begin{array}{c c} & \begin{array}{cc} \text{state 0} & \text{state 1} \end{array} \\ \begin{array}{c} \text{state 0} \\ \text{state 1} \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

- This is Fibonacci shift. Perron-Frobenius eigenvalue is:

$$\rho = \frac{1+\sqrt{5}}{2} = 1.6180...$$

- Hausdorff Dimension $= \log_3 \rho \approx 0.438$.



Graph: $N = 7 = (21)_3$

Associated Matrix $N = 7$

- Weight matrix is:

$$
\begin{array}{lcccc}
 & \text{state } 0 & \text{state } 2 & \text{state } 10 & \text{state } 1 \\
\text{state } 0 & [\ 1 & 1 & 0 & 0\ ] \\
\text{state } 2 & [\ 0 & 0 & 1 & 0\ ] \\
\text{state } 10 & [\ 0 & 0 & 1 & 1\ ] \\
\text{state } 1 & [\ 1 & 0 & 0 & 0\ ]
\end{array}
$$

- Perron-Frobenius eigenvalue is : $\rho = \frac{1+\sqrt{5}}{2} = 1.6180...$

- Hausdorff Dimension $= \log_3 \rho \approx 0.438.$

Graphs for $N = (10^k 1)_3$

- Theorem. ("Fibonacci Graphs")
  For $N = (10^k 1)_3$, (i.e. $N = 3^{k+1} + 1$)

  $$dim_H(\mathcal{C}(1, N)) := dim_H(\Sigma_{3,\bar{2}} \cap \frac{1}{N}\Sigma_{3,\bar{2}}) = \log_3(\frac{1+\sqrt{5}}{2}) \approx 0.438$$

- Remark. The finite graph associated to $N = 3^{k+1} + 1$ has 2^k states! The symbolic dynamics depend on k!

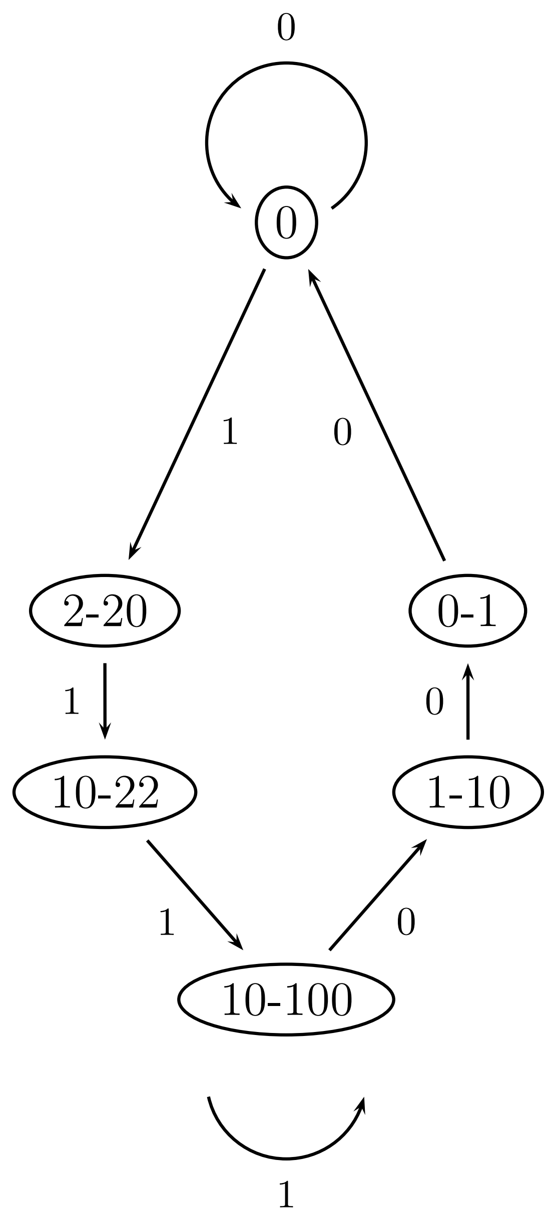
- The eigenvector for the maximal eigenvalue (Perron-Frobenius eigenvalue) of the adjacency matrix of this graph is explicitly describable. It has a self-similar structure, and has all entries in $\mathbb{Q}(\sqrt{5})$.

Graphs for $N = (20^k1)_3$

- Empirical Results. Take $N = 2 \cdot 3^{k+1} + 1 = (20^k1)_3$. For $1 \le k \le 7$, the graphs have increasing numbers of strongly connected components.

- There is an outer component with about k states, whose Hausdorff dimension goes rapidly to 0 as k increases. (This is provable for all $k \ge 1$).

- There is also an strongly connected inner component, which appears to have exponentially many states, and whose Hausdorff dimension monotonically increases for small k, and eventually exceeds that of the outer component.



Graph: $N = 19 = (201)_3$

Graph for $N = 139 = (12011)_3$

- This value N=139 is a value of $N \equiv 1 \pmod 3$ where the associated set has Hausdorff dimension 0.

- The corresponding graph has 5 strongly connected components; each one separately has Perron-Frobenius eigenvalue 1, giving Hausdorff dimension 0!

General Graphs-Some Properties of $\mathcal{C}(1, N)$

- The states in the graph can be labelled with integers k satisfying $0 \le k \le \lfloor \frac{N}{6} \rfloor$ (if entering edge label is 0) and $\lfloor \frac{N}{3} \rfloor \le k \le \lfloor \frac{N}{2} \rfloor$ (if entering edge label is 1).

- The paths in the graph starting from given state k describe the symbolic dynamics of numbers in the intersection of shifted multiplicatively translated 3-adic Cantor sets

$$\mathcal{C}_k := \Sigma_{3,\bar{2}} \cap \frac{1}{N}\left(\Sigma_{3,\bar{2}} + k\right).$$

- The Hausdorff dimension of "shifted intersection set" is the maximal Hausdorff dimension of a strongly connected component of graph reachable from the state k.

Lower Bound for Hausdorff Dimension

- Theorem. (**Lower Bound Theorem**) For any any $k \geq 1$ there exist

$$N_1 < N_2 < \cdots < N_k, \quad \text{all } N_i \equiv 1 \pmod 3$$

such that

$$dim_H(\mathcal{C}(N_1, N_2, ..., N_k)) := dim_H(\bigcap_{i=1}^{k} \frac{1}{N_i}\Sigma_{3,\bar{2}}) \geq 0.35.$$

Thus: the maximal Hausdorff dimension of intersection of translates is uniformly bounded away from zero.

- Proof. Take suitable N_i of the form $3^j + 1$ for various large j. One can show the Hausdorff dimension of intersection remains large (large overlap of symbolic dynamics).

Conclusions: Part IV

(1) The graphs for $\mathcal{C}(1, N)$ exhibit a complicated structure depending on an irregular way on the ternary digits of N. Their Hausdorff dimensions vary irregularly.

(2) Conjecture of Part II is false if generalized from powers of 2 to all $N \equiv 1 \pmod 3$.

Conclusions:

(3) Conjecture of Part II that

$$\lim_{k\to\infty} dim_H(\mathcal{E}_k^*(\mathbb{Z}_3)) = 0$$

could still be true, but...

(4) Lower bound theorem suggests: analyzing the special case where all $N_i = 2^{r_i}$ may not be easy!

Conclusions

- Our quest has failed! (So far)
- Perhaps a different approach using abstract ergodic theory should be tried.

Thank you for your attention!