Facts and Conjectures about Factorizations of Fibonacci and Lucas Numbers

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Édouard Lucas Memorial Lecture

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Rochester, New York

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Topics

- Will cover some history, starting with Fibonacci.
- The work of Édouard Lucas suggests some new problems that may be approachable in the light of what we now know.
- Caveat: the majority of open problems stated in this talk seem out of the reach of current methods in number theory. (“impossible”)

Table of Contents

1. Leonardo of Pisa (Fibonacci)

- Leonardo Pisano Bigollo (ca 1170–after 1240), son of Guglieimo Bonacci.

- Schooled in Bugia (Béjaïa, Algeria) where his father worked as customs house official of Pisa; Leonardo probably could speak and read Arabic

- Traveled the Mediterranean at times till 1200, visited Constantinople, then mainly in Pisa, received salary/pension in 1240.

Fibonacci Books -1

- *Liber Abbaci (1202, rewritten 1228)*

  [Introduced Hindu-Arablc numerals. Business, interest, changing money.]

- *De Practica Geometrie, 1223*

  [Written at request of Master Dominick. Results of Euclid, some borrowed from a manuscript of Plato of Tivoli, surveying, land measurement, solution of indeterminate equations.]

Fibonacci Books-2

- *Flos, 1225*

  [Solved a challenge problem of Johannes of Palermo, a cubic equation, $x^3 + 2x^2 + 10x - 20 = 0$ approximately, finding $x = 1.22.7.42.33.4.40 \approx 1.3688081075$ in sexagesimal.]

- *Liber Quadratorum, 1225* "The Book of Squares"

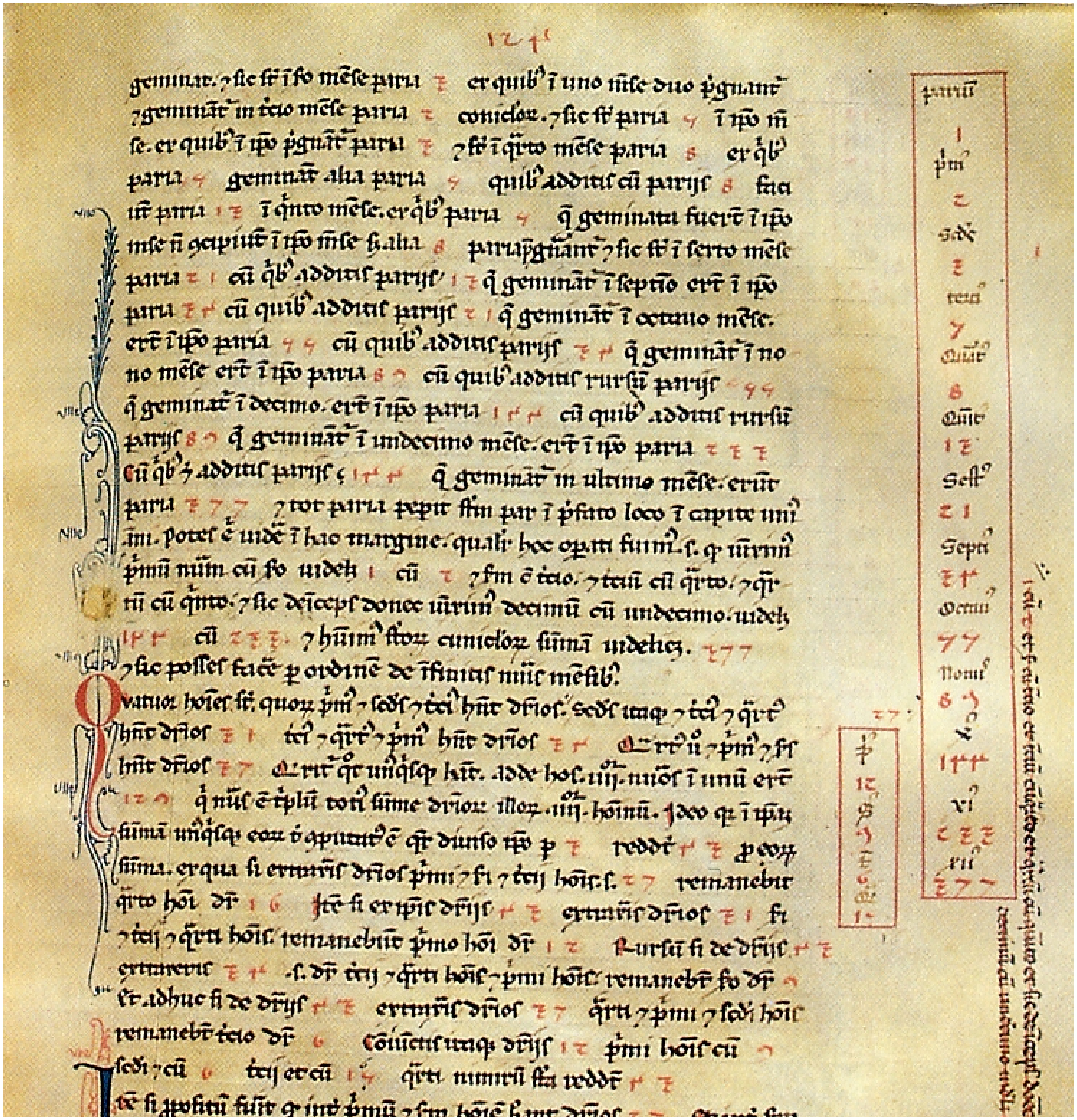
  [Solved another challenge problem of Johannes of Palermo. Determined "congruent numbers" k such that $x^2 + k = y^2$ and $x^2 - k = z^2$ are simultaneously solvable in rationals, particularly $k = 5$. This congruent numbers problem is in Diophantus.]

Fibonacci-3: Book “Liber Abbaci”

- Exists in 1227 rewritten version, dedicated to Michael Scot (1175-ca 1232) (court astrologer to Emperor Frederick II)

- Of 90 sample problems, over 50 have been found nearly identical in Arabic sources.

- The rabbit problem was preceded by a problem on perfect numbers, followed by an applied problem.



geminat. et sic fiunt in secundo mense paria 3 ex quibus in uno mense duo pregnantur et geminantur in tercio mense paria 2 coniculorum. et sic sunt paria 5 in ipso mense. ex quibus in ipso pregnantur paria 3 et sunt in quarto mense paria 8 ex quibus paria 5 geminant alia paria 5 quibus additis cum pariis 8 faciunt paria 13 in quinto mense. ex quibus paria 5 que geminata fuerunt in ipso mense non concipiunt in ipso mense sed alia 8 paria pregnantur et sic sunt in sexto mense paria 21 cum quibus additis pariis 13 que geminantur in septimo erunt in ipso paria 34 cum quibus additis pariis 21 que geminantur in octavo mense. erunt in ipso paria 55 cum quibus additis pariis 34 que geminantur in nono mense erunt in ipso paria 89 cum quibus additis rursum pariis 55 que geminantur in decimo. erunt in ipso paria 144 cum quibus additis rursum pariis 89 que geminantur in undecimo mense. erunt in ipso paria 233 cum quibus etiam additis pariis 144 que geminantur in ultimo mense. erunt paria 377 et tot paria peperit supradictum par in prefato loco in capite unius anni. potes enim videre in hac margine. qualiter hoc operati fuimus. scilicet quod iunximus primum numerum cum secundo videlicet 1 cum 2 et secundum cum tercio. et tercium cum quarto. et quartum cum quinto. et sic deinceps donec iunximus decimum cum undecimo. videlicet 144 cum 233 et habuimus suprascriptorum cuniculorum summam videlicet 377 et sic posses facere per ordinem de infinitis numeris mensibus.

Quatuor homines sunt, quorum primus, secundus et tercius habent denarios. secundus vero, tercius et quartus habent denarios 27 tercius, quartus et primus habent denarios 31 quartus, primus et secundus habent denarios 37 queritur quot unusquisque habet. adde homines iiii numeros in unum erunt 120 que numerus est triplum totius summe denariorum illorum iiii hominum. ideo quia in ipsa summa unusquisque eorum ter computatus est. quare divisio ipso per 3 reddet 40 pro eorum summa. ex qua si extraxeris denarios primi et secundi et tercii hominis, scilicet 27 remanebit quarto homini denarii 13 item si ex ipsis extraxeris denarios 31 remanebunt secundo homini denarii 9 et rursum si de 40 extraxeris 37 remanebunt tercio homini denarii 3 ...

parium
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primus
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secundus
3
tercius
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quintus
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sextus
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septimus
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octavus
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nonus
89
decimus
144
undecimus
233
duodecimus
377

parium
1
primus
2
Secundus
3
tercius
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Quartus
8
Quintus
13
Sestus
21
Septimus
34
Octauus
55
Nonus
89
Decimus
144
Undecimus
233
Duodecimus
377

« *Quot paria coniculorum in uno anno ex uno pario germinentur.*

Qvidam posuit unum par cuniculorum in quodam loco, qui erat undique pariete circundatus, ut sciret, quot ex eo paria germinarentur in uno anno : cum natura eorum sit per singulum mensem aliud par germinare ; et in secundo mense ab eorum natiuitate germinant. Quia suprascriptum par in primo mense germinat, duplicabis ipsum, erunt paria duo in uno mense. Ex quibus unum, scilicet primum, in secundo mense geminat ; et sic sunt in secundo mense paria 3 ; ex quibus in uno mense duo pregnantur ; et, gemitiantur in tercio mense paria 2 coniculorum ; et sic sunt paria 5 in ipso mense ; ex quibus in ipso pregnantur paria 3 ; et sunt in quarto mense paria 8 ; ex quibus paria 5 geminant alia paria 5 : quibus additis cum parijs 8, faciunt paria 13 in quinto mense ; ex quibus paria 5, que geminata fuerunt in ipso mense, non concipiunt in ipso mense ; sed alia 8 paria pregnantur ; et sic sunt in sexto mense paria 21 ; cum quibus additis parijs 13, que geminantur in septimo, erunt in ipso paria 34 ; cum quibus additis parijs 21, que geminantur in octauo mense, erunt in ipso paria 55 ; cum quibus additis parijs 34, que geminantur in nono mense, erunt in ipso paria 89 ; cum quibus additis rursum parijs 55, que geminantur in decimo, erunt in ipso paria 144 ; cum quibus additis rursum parijs 89, que geminantur in undecimo mense, erunt in ipso paria 233. Cum quibus etiam additis parijs 144, que geminantur in ultimo mense, erunt paria 377 ; et tot paria peperit suprascriptum par in prefato loco in capite unius anni. Potes enim uidere in hac margine, qualiter hoc operati fuimus, scilicet quod iunximus primum numerum cum secundo, uidelicet 1 cum 2 ; et secundum cum tercio ; et tercium cum quarto ; et quartum cum quinto, et sic deinceps, donec iunximus decimum cum undecimo, uidelicet 144 cum 233; et habuimus suprascriptorum cuniculorum summam, uidelicet 377 ; et sic posses facere per ordinem de infinitis numeris mensibus ».

Michael Scot (1175–1232)

- Born in Scotland, studied at cathedral school in Durham, also Paris. Spoke many languages, including Latin, Greek, Hebrew, eventually Arabic.

- Wandering scholar. In Toledo, Spain, learned Arabic. Translated some manuscripts of Aristotle from Arabic.

- Court astrologer to Emperor Frederick II(of Palermo) (1194–1250), patron of science and arts.

- Second version of Fibonacci's *Liber Abaci* (1227) dedicated to him.

Michael Scot-2

- Wrote manuscripts on astrology, alchemy, psychology and occult, some to answer questions of Emperor Frederick

- Books: *Super auctorem spherae*, *De sole et luna*, *De chiromantia*, etc.

- Regarded as magician after death. Was consigned to the eighth circle of Hell in Dante's *Inferno* (canto xx. 115–117). [This circle reserved for sorcerers, astrologers and false prophets.]

Popularizer: Fra Luca Pacioli (1445–1517)

- Born in Sansepolcro (Tuscany), educated in vernacular, and in artist's studio of Piera Della Francesca in Sansepolcro.

- Later a Franciscan Friar, first full-time math professor at several universities.

- Book: *Summa de arithmetic, geometria, proportioni et proportionalitá*, Printed Venice 1494. First detailed book of mathematics. First written treatment of double entry accounting.

- Pacioli praised Leonardo Pisano, and borrowed from him.

Fra Luca Pacioli-2

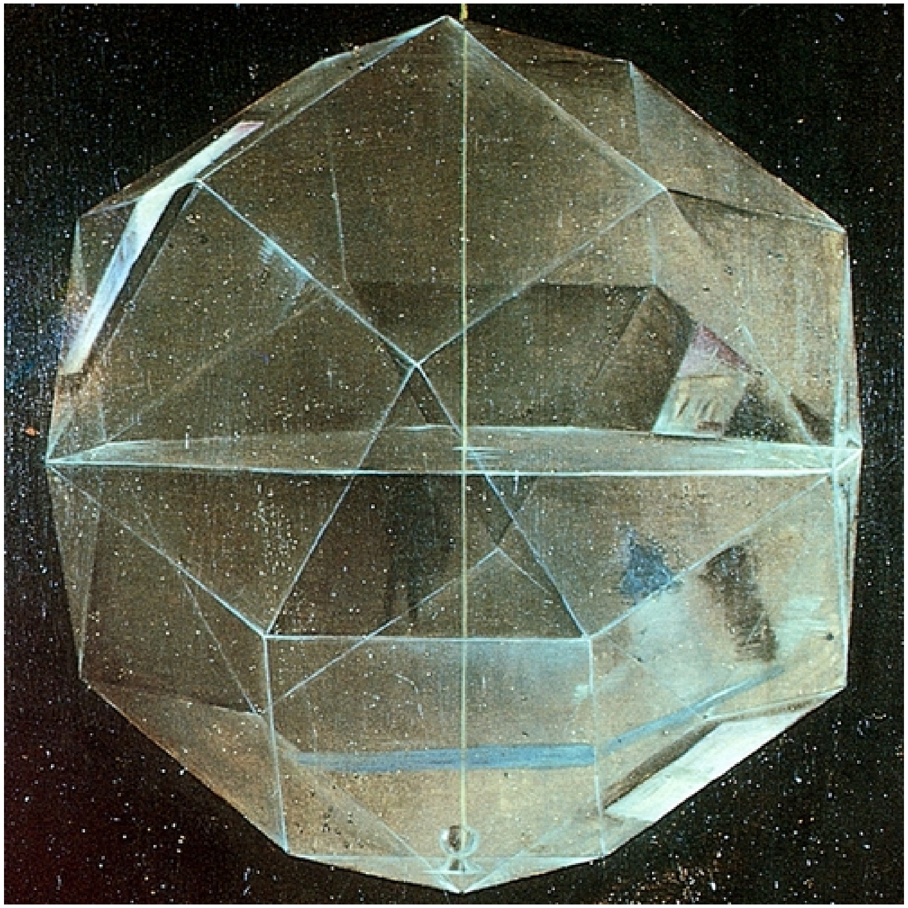
- Pacioli tutored Leonardo da Vinci on mathematics in Florence, 1496–1499. Wrote *De Divina Proportione*, 1496–1498, printed 1509. Mathematics of golden ratio and applications to architecture. This book was illustrated by Leonardo with pictures of hollow polyhedra.

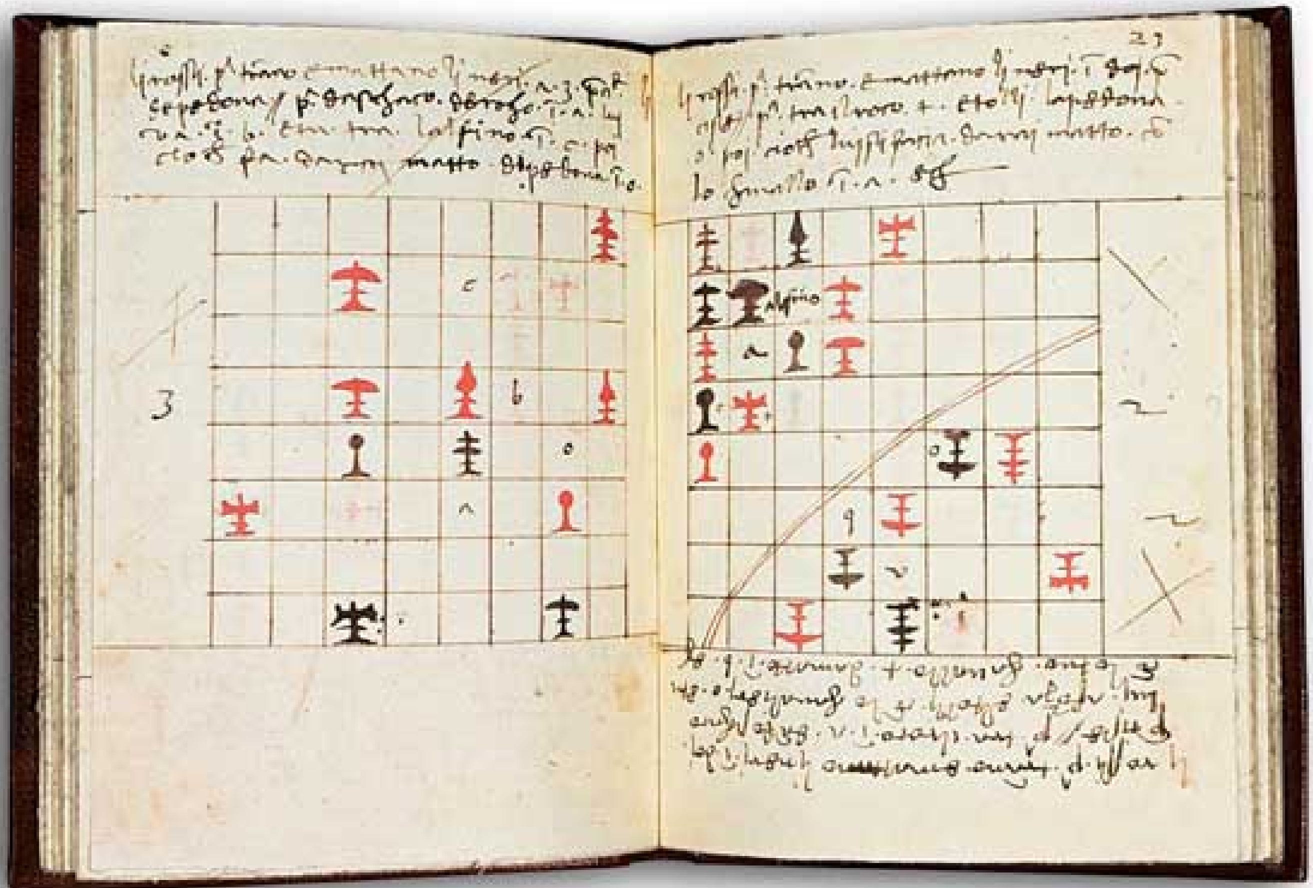
- Pacioli made Latin translation of Euclid's *Elements*, published 1509.

- Wrote manuscript, *Die Viribus Quantitatis*, collecting results on mathematics and magic, juggling, chess and card tricks, eat fire, etc. Manuscript rediscovered in 20th century, never printed.


EVCLIDES.

Luca Pacioli polyhedron





2. Édouard Lucas (1842–1891)

- Son of a laborer. Won admission to *École polytechnique* and *École Normale.* Graduated 1864, then worked as assistant astronomer under E. Leverrier at Paris Observatory.

- Artillery officer in Franco-Prussian war (1870/1871). Afterwards became teacher of higher mathematics at several schools: Lycée Moulins, Lycée Paris-Charlemagne, Lycée St.-Louis.

Édouard Lucas-2

- His most well known mathematical work is on recurrence sequences (1876–1878). Motivated by questions in primality testing and factoring, concerning primality of Mersenne and Fermat numbers.

- Culminating work a memoir on Recurrence Sequences- motivated by analogy with simply periodic functions (Amer. J. Math. 1878).

. Édouard Lucas: Books

- Survey paper in 1877 on developments from work of Fibonacci, advertising his results (122 pages)

- Book on Number Theory (1891). This book includes a lot combinatorial mathematics, probability theory, symbolic calculus.

  [Eric Temple Bell had a copy. He buried it during the San Francisco earthquake and dug up the partially burned copy afterwards; it is in Cal Tech library.]

- Recreational Mathematics (four volumes) published after his death. He invented the Tower of Hanoi problem.

RECHERCHES

SUR PLUSIEURS OUVRAGES

DE LÉONARD DE PISE

ET

SUR DIVERSES QUESTIONS D'ARITHMÉTIQUE SUPÉRIEURE

PAR M. ÉDOUARD LUCAS

PROFESSEUR DE MATHÉMATIQUES AU LYCÉE CHARLEMAGNE À PARIS

EXTRAIT DU *BULLETTINO DI BIBLIOGRAFIA E DI STORIA DELLE SCIENZE MATEMATICHE E FISICHE*
TOMO X. — MARZO, APRILE E MAGGIO 1877.

ROME
IMPRIMERIE DES SCIENCES MATHÉMATIQUES ET PHYSIQUES
Via Lata, Num° 3.
1877

Background: Perfect Numbers

A number is perfect if it is the sum of its proper divisors. For example $6 = 1 + 2 + 3$ is perfect.

Theorem (Euclid, Book IX, Prop. 36)
If $2^n - 1$ is a prime, then

$$N = 2^{n-1}(2^n - 1)$$

is a perfect number.

This result led to:

"Good" Unsolved Problem. Find all the prime numbers of the form $2^n - 1$.

"Bad" Unsolved Problem. Are there any odd perfect numbers?

Perfect Numbers-2

- Prime numbers $M_n = 2^n - 1$ are called Mersenne primes after Fr. Marin Mersenne (1588-1648).

- If $M_n = 2^n - 1$ is prime, then $n = p$ must also be prime.

- Fr. Mersenne (1644) asserted that $n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257$ gave $2^n - 1$ primes. In his day the list was verified up to $n = 19$.

- Mersenne missed $n = 61$ and $n = 87$ and he incorrectly included $n = 67$ and $n = 257$. But 127 was a good guess.

Perfect Numbers-3

Three results of Euler.

- Theorem (Euler (1732), unpublished)
  If an even number N is perfect, then it has Euclid's form

  $$N = 2^{n-1}(2^n - 1),$$

  with $M_n = 2^n - 1$ a prime.

- Theorem (Euler(1771), E461) *The Mersenne number $M_{31} = 2^{31} - 1$ is prime.*

- Theorem (Euler (1732), E26, E283) *The Fermat number $2^{2^5} + 1$ is not prime.*

Recurrence Sequences-1

- Lucas considered primality testing from the beginning. He knew the conjectures of primality of certain numbers of Mersenne $M_n = 2^n - 1$ and of Fermat $Fr_n = 2^n + 1$.

- Starting from $n = 0$, we have

$$M_n = 0, 1, 3, 7, 15, 31, 63, 127, 255, 511, ...$$

$$Fr_n = 2, 3, 5, 9, 17, 33, 65, 129, 257, 513, ...$$

- He noted: M_n and Fr_n obey the *same* second-order linear recurrence: $X_n = 3X_{n-1} - 2X_{n-2}$, with different initial conditions: $M_0 = 0, M_1 = 1$, reap. $Fr_0 = 2, Fr_1 = 3$.

Recurrence Sequences-2

- Lucas also noted the strong divisibility property for Mersenne numbers ($n \geq 1$)

$$\text{gcd}(M_m, M_n) = M_{\text{gcd}(m,n)}.$$

- Lucas noted the analogy with trigonometric function identities (singly periodic functions)

$$M_{2n} = M_n Fr_n$$

analogous to

$$\sin(2x) = 2 \sin x \cos x.$$

Enter Fibonacci Numbers.

- Lucas (1876) originally called the Fibonacci numbers F_n the series of Lamé. He denoted them u_n.

- Lamé (1870) counted the number of steps in the Euclidean algorithm to compute greatest common divisor. He found that $gcd(F_n, F_{n-1})$ is the worst case.

- Lucas (1877) introduced the associated numbers $v_n := F_{2n}/F_n$. These numbers are now called the Lucas numbers. They played an important role in his original primality test for certain Mersenne numbers. The currently known test is named: Lucas-Lehmer test.

Fibonacci and Lucas Numbers

- The Fibonacci numbers F_n satisfy $F_n = F_{n-1} + F_{n-2}$, initial conditions $F_0 = 0, F_1 = 1$, giving

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55 \cdots$$

- The Lucas numbers L_n satisfy $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2, L_1 = 1$, giving

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \cdots$$

- They are cousins:

$$F_{2n} = F_n L_n.$$

Divisibility properties of Fibonacci Numbers-1

- "Fundamental Theorem." *The Fibonacci numbers* F_n *satisfy*

$$\gcd(F_m, F_n) = F_{\gcd(m,n)}$$

In particular if m *divides* n *then* F_m *divides* F_n*.*

- The first property is called: a strong divisibility sequence.

Divisibility of Fibonacci Numbers-2

- “Law of Apparition.”

  (1) If a prime p has the form $5n + 1$ or $5n + 4$, then p divides the Fibonacci number F_{p-1}.

  (2) If a prime p has the form $5n + 2$ or $5n + 3$, then p divides F_{p+1}.

- “Law of Repetition.”

  If an odd prime power p^k exactly divides F_n then p^{k+1} exactly divides F_{pn}.

  Exceptional Case $p = 2$. Here $F_3 = 2$ but $F_6 = 2^3$, the power jumps by 2 rather than 1.

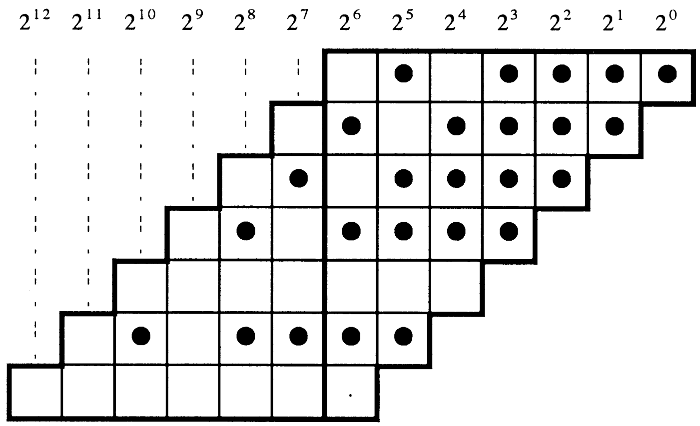
Application: Mersenne Prime testing-1

- Lucas (1876) introduced a sufficient condition to test primality of those Mersenne numbers $M_n = 2^n - 1$ with $n \equiv 3 \pmod 4$. He proved such an M_n is prime if M_n divides the Lucas number L_{2^n}. One computes the right side $\pmod{M_n}$ using the identity, valid for $k \geq 1$,

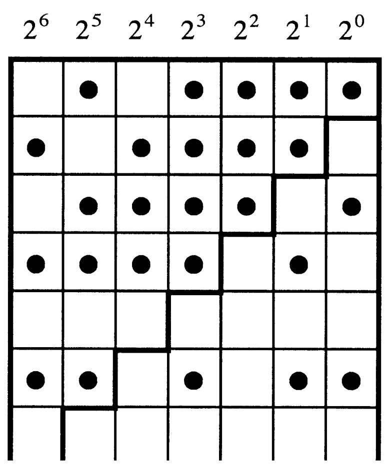
$$L_{2^{k+1}} = (L_{2^k})^2 - 2.$$

  The initial value is $L_2 = 3$.

- Lucas (1877) applied this test to give a new proof of primality for $M_{31} = 2^{31} - 1$. He used calculations in binary arithmetic, stating that one could get pretty fast with it by practice.



Nota. Les gros traits indiquent les bords des deux parties de l'échiquier. Des pions ayant été disposés ainsi que l'indique la figure précédente sur l'échiquier pour la multiplication, il faut faire l'addition, et la division par $2^7 - 1$; on commence alors, par placer l'échiquier dans la seconde position indiquée ci-après sans déranger les pions ; on a ainsi supprimé des multiples de $2^7 - 1$.



Application: Mersenne Prime testing -2

- Lucas [(1877), Sect. 14–16] announced that he had carried out a binary calculation to prove that $M_{127} = 2^{127} - 1$ is prime. This calculation would take hundreds of hours, and he only did it once. The calculations were never written down. Did he make a mistake? The method works in principle, and we now know the answer is right.

- Lucas (1878) went on to develop a modified test to handle M_n with $n \equiv 1 \pmod 4$. He considered more general recursions $G_n = UG_{n-1} + VG_{n-2}$, but ended up with the same recursion $Y_{k+1} = Y_k^2 - 2$. (Now one can start with $Y_1 = 4$ in the modern Lucas-Lehmer test, covering all cases.)

Fibonacci and Lucas Factorization Tables

- Factoring Fibonacci numbers began with Lucas, who completely factored F_n for $n \leq 60$.

- Cunningham factoring project, named after:
  Lt.-Col. (R. E.) A. J. Cunningham, *Factors of numbers*, *Nature* **39** (1889), 559–560.

  Continuing on for 35 years to:
  Lt.-Col. (R. E.) A. J. Cunningham and A. J. Woodall, *Factorizations of* $y^n \pm 1$, Francis Hodgson, London 1925.

- Factoring of Fibonacci and Lucas numbers is a spinoff of the Cunningham project.

Factoring Fibonacci Numbers

- Factoring Fibonacci and Lucas numbers has been carried out on a large scale. J. Brillhart, P. L. Montgomery, R. D. Silverman, (Math. Comp. 1988), and much since. Web pages of current records are maintained by Blair Kelly.

- Fibonacci numbers F_n have been completely factored for $n \leq 1000$, and partially factored for $n \leq 10000$. Fibonacci primes have been determined up to $n \leq 50000$ and have been searched somewhat further, to at least $n = 200000$, without rigorous proofs of primality.

- Lucas numbers L_n and primes determined similarly.

TABLEAU DES FACTEURS PREMIERS DE LA SERIE RECURENTE DE LEONARD DE PISE

n	u_n	Div. impropres	Div. propres	n	u_n	Diviseurs impropres	Diviseurs propres
1	1	—	1.	31	13 46269	—	557 × 2417.
2	1	—	1.	32	21 78309	3 × 7 × 47.	2207.
3	2	—	2.	33	35 24578	2 × 89.	19801.
4	3	—	3.	34	57 02887	1597.	3571.
5	5	—	5.	35	92 27465	5 × 13.	1 41961.
6	8	2^3.	—	36	149 30352	$2^4 \times 3^3 \times 17 \times 19$.	107.
7	13	—	13.	37	241 57817	—	73 × 149 × 2221.
8	21	3.	7.	38	390 88169	37 × 113.	9349.
9	34	2.	17.	39	632 45986	2 × 233.	1 35721.
10	55	5.	11.	40	1023 34155	3 × 5 × 7 × 11 × 41.	2161.
11	89	—	89.	41	1655 80141	—	2789 × 59369.
12	144	$2^4 \times 3^2$	—	42	2679 14296	$2^3 \times 13 \times 29 \times 421$.	211.
13	233	—	233.	43	4334 94437	—	4334 94437.
14	377	13.	29.	44	7014 08733	3 × 89 × 199.	43 × 307.
15	610	2 × 5.	61.	45	11349 03170	2 × 5 × 17 × 61.	1 09441.
16	987	3 × 7.	47.	46	18363 11903	28657.	139 × 461.
17	1597	—	1597.	47	29712 15073	—	29712 15073.
18	2584	$2^3 \times 17$.	19.	48	48075 26976	$2^6 \times 3^2 \times 7 \times 23 \times 47$.	1103.
19	4181	—	37 × 113.	49	77787 42049	13.	97 × 61 68709.
20	6765	3 × 5 × 11.	41.	50	1 25862 69025	$5^2 \times 11 \times 3001$.	101 × 151.
21	10946	2 × 13.	421.	51	2 03650 11074	2 × 1597.	63 76021
22	17711	89.	199.	52	3 29512 80099	3 × 233 × 521.	90481.
23	28657	—	28657.	53	5 33162 91173	—	953 × 559 45741.
24	46368	$2^4 \times 3^2 \times 7$.	23.	54	8 62675 71272	$2^3 \times 17 \times 19 \times 53 \times 109$.	5779.
25	75025	5^2.	3001.	55	13 95838 62445	5 × 89.	661 × 4 74541.
26	1 21393	233.	521.	56	22 58514 33717	$3 \times 7^2 \times 13 \times 29 \times 281$.	14503.
27	1 96418	2 × 17.	53 × 109.	57	36 54352 96162	2 × 37 × 113.	43 71901.
28	3 17811	3 × 13 × 29.	281.	58	59 12867 29879	5 14229.	59 × 19489.
29	5 14229	—	5 14229.	59	95 67220 26041	—	353 × 27102 60697.
30	8 32040	$2^3 \times 5 \times 11 \times 61$.	31.	60	154 80087 55920	$2^4 \times 3^2 \times 5 \times 11 \times 31 \times 41 \times 61$.	2521.

Lucas (1877) -other work

- Quadratic constraints on Lucas numbers: any L_n divides a number of form either $x^2 \pm 5y^2$, so it has no prime factors of form $20k + 13, 20k + 17$. Similar results for F_{3n}/F_n, etc.

- Gave a symbolic calculus method to generate recurrence series identities. (Resembles umbral calculus)

3. Fibonacci and Lucas Divisibility

- By Lucas' Fundamental theorem, F_n is a strong divisibility sequence.

- By the Lucas Laws of Apparition and Repetition, every prime power (in fact *every integer*) divides infinitely many Fibonacci numbers.

- A much explored topic is that of linear recurrences that give divisibility sequences. These sequences are somewhat complicated but they have been classified.

Divisibility of Lucas Numbers

Lucas numbers have different divisibility properties than Fibonacci numbers.

- Infinitely many primes don't divide any L_n: Lucas excluded $p \equiv 13, 17 \pmod{20}$.

- Even if p_1, p_2 divide some Lucas numbers, $p_1 p_2$ may not.

  Example. 3 divides L_2 and 7 divides L_4 but 21 does not divide any L_n.

Divisibility Sequences

- A sequence u_n is a divisibility sequence if $u_m | u_{mn}$ for all $m, n \geq 1$.

- It is a strong divisibility sequence if

$$gcd(u_m, u_n) = u_{gcd(m,n)}.$$

- The Fibonacci numbers are a strong divisibility sequence. But the Lucas numbers L_n are not even a divisibility sequence. since $L_2 = 3$ does not divide $L_4 = 7$.

Divisibility Sequences-2

- Second and third order recurrence sequences that are divisibility sequences were studied by Marshall Hall(1936), and by Morgan Ward in a series of papers from the 1930's to the 1950's. Two cases: where $u_0 = 0$ and the "degenerate case" where $u_0 \neq 0$. In the latter case there are only finitely many different primes dividing the u_n.

- Linear recurrence sequences that are divisibility sequences were "completely" classified by Bezivin, Pethö and van der Poorten, Amer. J. Math. 1990. Their notion of divisibility is that the quotients u_{nm}/u_m belong to a fixed ring A which is finitely generated over the ring $\mathbb{Z}$.

Almost divisibility sequences

- Let us call a sequence u_n an almost divisibility sequence if there is an integer N such that $u_m | u_{mn}$ whenever $(mn, N) = 1$.

- Similarly call u_n a almost strong almost divisibility sequence if
$$gcd(u_m, u_n) = u_{gcd(m,n)}.$$
whenever $(mn, N) = 1$.

- Theorem. *The Lucas numbers L_m for $m \geq 1$ form an almost strong divisibility sequence with $N = 2$.*

Almost divisibility sequences-2

- The set of Lucas numbers can be partitioned as

$$\Sigma_k := \{L_{2^k m} : \quad m \geq 1 \text{ and } \quad m \text{ odd}\}$$

for $n \geq 1$. The different sets Σ_k are (nearly) pairwise relatively prime, namely: any common factors are 1 or 2.

- The partition above implies that the Lucas sequence L_n violates the "finitely generated" assumption in the work of Bezivin, Pethö and van der Poorten on divisibility sequences. So we arrive at:

  Open Problem. *Classify all linear recurrences giving* **almost divisibility sequences**, (*resp.* **strong almost divisibility sequences**).

Divisibility of Fibonacci Numbers: Open Problem

- p^2-Problem. For each prime p, is it true that there is some Fibonacci number exactly divisible by p?

- "Folklore" Conjecture. *There are infinitely many p such that p^2 divides every Fibonacci number divisible by p.*

- This conjecture is based on a heuristic analogous to the infinitude of solutions to $2^{p-1} \equiv 1 \pmod{p^2}$. Here two solutions are known: $p = 1093$, $p = 3511$.

- Heuristic says that density of such $p \leq x$ is $\log\log x$.

General Recurrence Divisibility

- Theorem.(Lagarias (1985)). *The density of prime divisors of the Lucas sequences is* $2/3$.

  This density result classifies the primes using splitting conditions in number fields and uses the Chebotarev density theorem as an ingredient. The proof started from work of Morgan Ward and Helmut Hasse.

- Large bodies of work have attempted to specify prime divisors of general linear recurrences, General results are limited. Work of Christian Ballot gives information on maximal prime divisors of higher order linear recurrences.

Thank you for your attention!