Products of Farey Fractions

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Numbers, Geometries and Games: A Centenarian of Mathematics

(Steve Butler and Barbara Faires, Organizers)

Topics Covered

- 0. Richard Guy
- 1. Farey Fractions
- 2. Products of Farey Fractions-1
- 3. Interlude: Products of Unreduced Farey Fractions
- 4. Products of Farey Fractions-2

0. Richard K. Guy

Quotations from Richard Guy:

- "*Problems are the lifeblood of any mathematical discipline.*"

  On the other hand:

- " R. K. Guy, **Don't try to solve these problems!**, American Math. Monthly *90* (1983), 35–41.

- Exordium: "*Some of you are already scribbling, in spite of the warning....*"

1. Farey Fractions

• The Farey fractions $\mathcal{F}_n$ of order n are fractions $0 \le \frac{h}{k} \le 1$ with $gcd(h,k) = 1$. Thus

$$\mathcal{F}_4 = \{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\}.$$

The non-zero Farey fractions are

$$\mathcal{F}_4^* := \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\}.$$

• The number $|\mathcal{F}_n^*|$ of nonzero Farey fractions of order n is

$$\Phi(n) := \phi(1) + \phi(2) + \cdots + \phi(n).$$

Here $\phi(n)$ is *Euler totient function*. One has

$$\Phi(n) = \frac{3}{\pi^2} N^2 + O(N \log N).$$

Farey Fractions-2

• The Farey fractions have a limit distribution as $N \to \infty$. They approach the uniform distribution on $[0,1]$.

• **Theorem.** *The distribution of Farey fractions described by sum of (scaled) delta measures at members of* $\mathcal{F}_n$*, weighted by* $\frac{1}{\Phi(n)}$. *Let*

$$\mu_n := \frac{1}{\Phi(n)} \sum_{j=1}^{\Phi(n)} \delta(\rho_j)$$

Then these measures μ_n *converge weakly as* $n \to \infty$ *to the uniform (Lebesgue) measure on* $[0,1]$.

Farey Fractions-3

• The rate at which Farey fractions approach the uniform distribution is related to the Riemann hypothesis!

• **Theorem.** (Franel's Theorem (1924)) *Consider the statistic*

$$S_n = \sum_{j=1}^{\Phi(n)} (\rho_j - \frac{j}{\Phi(n)})^2$$

Then as $n \to \infty$

$$S_n = O(n^{-1+\epsilon})$$

for each $\epsilon > 0$ *if and only if the Riemann hypothesis is true.*

• One knows unconditionally that $S_n \to 0$ as $n \to \infty$. This fact is equivalent to the Prime Number Theorem.

2. Products of Farey Fractions

• **Motivation.** There is a mismatch in scales between *addition* and *multiplication* in the rationals $\mathbb{Q}$, which in some way influences the distribution of prime numbers. To understand this better one might study (new) *arithmetic statistics* that mix addition and multiplication in an interesting way.

• The Farey fractions $\mathcal{F}_n$ encode data that seems "additive". So why not study the **product** of the Farey fractions? (We exclude the Farey fraction $\frac{0}{1}$ in the product!)

• Define the Farey product $F_n := \prod_{j=1}^{\Phi(n)} \rho_j$, where ρ_j runs over the nonzero Farey fractions in increasing order.

Products of Farey Fractions-2

• It turns out convenient to study instead the reciprocal Farey product $\overline{F}_n := 1/F_n$.

• Studying Farey products seems interesting because will be a lot of cancellation in the resulting fractions. There are about $\frac{3}{\pi^2}n^2$ terms in the product, but all numerators and denominators of ρ_j contain only primes $\leq n$, and there are certainly at most n of these. So there **must be** enormous cancellation in product numerator and denominator! How much? And what is left over afterwards?

• (History) This research project was done with REU student Harsh Mehta (now grad student at Univ. South Carolina).

Products of Farey Fractions-3

- **Question.** The products of all (nonzero) Farey fractions

$$F_n := \prod_{\rho_r \in \mathcal{F}_n^*} \rho_r.$$

give a single statistic for each n. Is the Riemann hypothesis encoded in its behavior?

- Amazing answer: Yes!

- **Theorem.** (Mikolás (1952)- rephrased) *Let* $\overline{F}_n = 1/F_n$. *The Riemann hypothesis is equivalent to the assertion that*

$$\log(\overline{F}_n) = \Phi(n) - \frac{1}{2}n + O(n^{1/2+\epsilon}).$$

(Here $\Phi(n) \sim \frac{3}{\pi^2}n^2$ *counts the number of Farey fractions.) The RH is encoded in the size of the remainder term.*

Products of Farey Fractions-4

• For Farey products we can ask some *new questions*: what is the behavior of the **divisibility** of $\overline{F}_n$ by a fixed prime p: What power of p divides $\overline{F}_n$? Call if

$$f_p(n) := \mathrm{ord}_p(\overline{F}_n)$$

This value can be positive or negative, because $\overline{F}_n$ is a rational number.

• **Question.** Could some information about RH be encoded in the individual functions $f_p(n)$ for a single prime p?

• **Approach.** Study this question experimentally by computation for small n and small primes.

• But first–a simpler problem: **unreduced Farey fractions.**

3. Products of Unreduced Farey Fractions

- **Idea.** Study a simpler "toy model", products of unreduced Farey fractions.

- The (nonzero) unreduced Farey fractions $\mathcal{G}_n^*$ of order n are all fractions $0 < \frac{h}{k} \le 1$ with $1 \le h \le k \le n$
(no gcd condition imposed).

$$\mathcal{G}_4^* := \{\frac{\mathbf{1}}{\mathbf{4}}, \frac{\mathbf{1}}{\mathbf{3}}, \frac{1}{2}, \frac{\mathbf{2}}{\mathbf{4}}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}, \frac{\mathbf{2}}{\mathbf{2}}, \frac{\mathbf{3}}{\mathbf{3}}, \frac{\mathbf{4}}{\mathbf{4}}\}.$$

- The number of unreduced Farey fractions is

$$|\mathcal{G}_n^*| = \Phi^*(n) := 1 + 2 + 3 + \cdots + n = \binom{n+1}{2} = \frac{1}{2}n(n+1).$$

Unreduced Farey Products are Binomial Products

• **Fact.** The reciprocal unreduced Farey product $\overline{G}_n := 1/G_n$ is always an integer.
(Harm Derksen and L, MONTHLY problem 11594 (2011))

• **Proposition.** *The reciprocal product $\overline{G}_n$ of unreduced Farey fractions is the product of binomial coefficients in the n-th row of Pascal's triangle.*

$$\overline{G}_n := \prod_{k=0}^{n} \binom{n}{k}$$

Data: $\overline{G}_1 = 1$, $\overline{G}_2 = 2$, $\overline{G}_3 = 9$, $\overline{G}_4 = 96$, $\overline{G}_5 = 2500$, ,$\overline{G}_6 = 162000$, $\overline{G}_7 = 26471025$. (On-Line Encylopedia of Integer Sequences (OEIS): Sequence *A001142*.)

Binomial Products: Questions

- *What is the growth of $\overline{G}_n$ as real number?*
  Measure size by
  $$g_\infty(n) := \log(\overline{G}_n).$$

- *What is the behavior of their prime factorizations?*
  At a prime p, measure size by divisibility exponent
  $$g_p(n) := \mathrm{ord}_p(\overline{G}_n).$$
  Prime factorization is:
  $$\overline{G}_n = \prod_p p^{\,g_p(n)}.$$
  Here $g_p(n) \geq 0$ since $\overline{G}_n$ is an integer.

“Unreduced Farey” Riemann hypothesis

• **Theorem** (“Unreduced Farey” Riemann hypothesis)
The reciprocal unreduced Farey products $\overline{G}_n$ *satisfy*

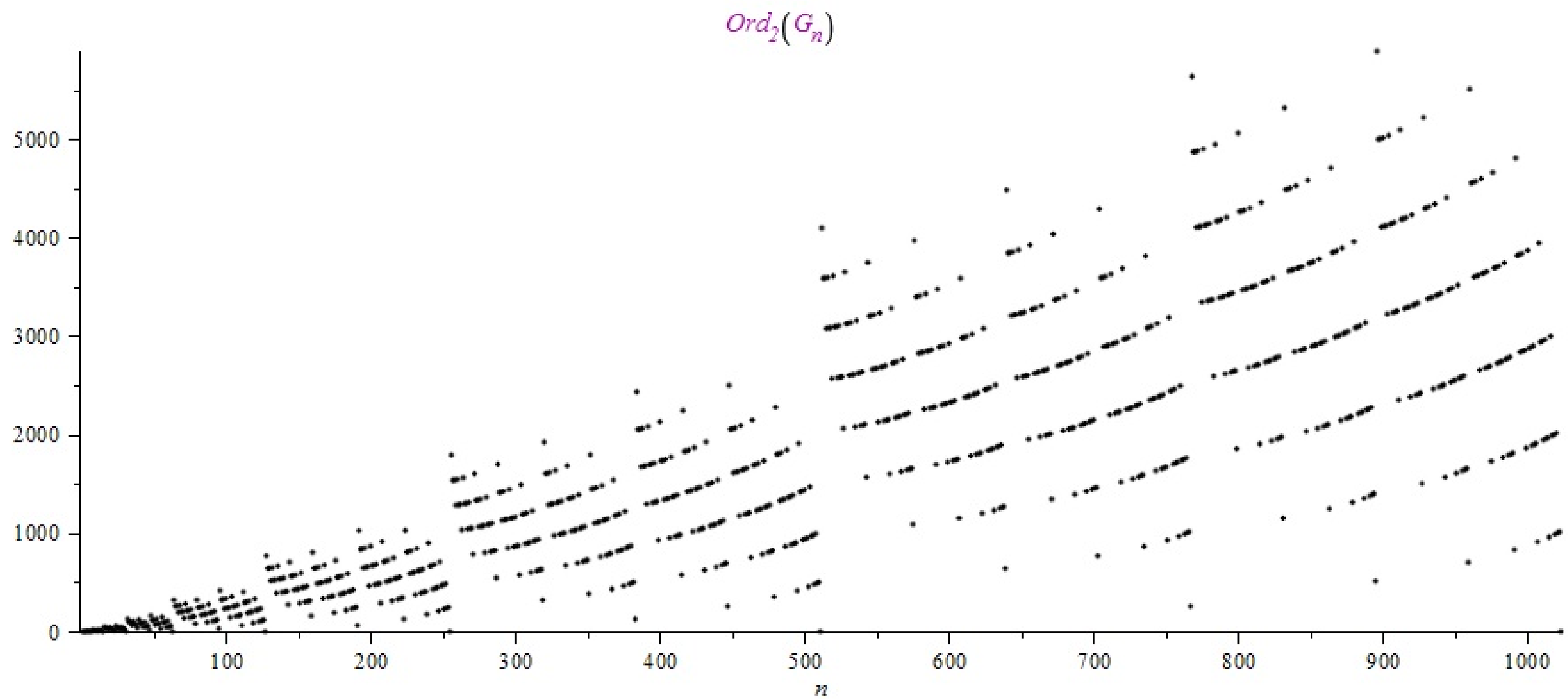
$$\log(\overline{G}_n) \;=\; \Phi^*(n) - \frac{1}{2}n\log n + \left(\frac{1}{2} - \frac{1}{2}\log(2\pi)\right)n + \\ +O(\log n).$$

Here $\frac{1}{2} - \frac{1}{2}\log(2\pi) \approx -0.41894$ *and* $\Phi^*(n) = \frac{1}{2}n(n+1)$.

• This is “unreduced Farey product” analogy with Mikoläs’s formula, where RH says error term $O(n^{1/2+\epsilon})$.
But here we get instead a tiny error term: $O(\log n)$.

• **Question.** Does this error term $O(\log n)$ mean: there are no “zeros” in the critical strip all the way to $Re(s) = 0$ (of some function)?

Prime $p = 2$ divisibility



Binomial Products-Prime Factorization Patterns

- Graph of $g_2(n)$ shows the function is increasing on average. It exhibits a regular series of stripes.

- Stripe patterns are grouped by powers of 2: Self-similar behavior?

- Function $g_2(n)$ must be highly oscillatory, needed to produce the stripes. Fractal behavior?

- Harder to see: The number of stripes increases by 1 at each power of 2.

Binomial Products-3

- All patterns above can be proved (unconditionally).

- **Method:** We obtained an explicit formula for $\mathrm{ord}_p(\overline{G}_n)$ in terms of the base p radix expansion of n. This formula started from Kummer's formula giving the power of p that divides the binomial coefficient.

- **Theorem** (Kummer (1852)) *Given a prime p, the exact power of divisibility p^e of binomial coefficient $\binom{n}{t}$ by a power of p is found by writing t, $n-t$ and n in base p arithmetic: the power e is the number of carries that occur when adding $n-t$ to t in base p arithmetic, using digits $\{0, 1, 2, \cdots, p-1\}$, working from the least significant digit upward.*

Binomial Products-4

- **Theorem** (L-Mehta 2015)

$$\mathrm{ord}_p(\overline{G}_n) = \frac{1}{p-1}\Big(2S_p(n) - (n-1)d_p(n)\Big).$$

where $d_p(n)$ is the sum of the base p digits of n, and $S_p(n)$ is the running sum of all base p digits of the first $n-1$ integers.

- One can now apply a ("well-known") result of Delange (1975):

$$S_p(n) = \Big(\frac{p-1}{2}\Big) n \log_p n + F_p(\log_p n) n, \tag{1}$$

in which $F_p(x)$ is a continuous real-valued function which is periodic of period 1. The function $F_p(x)$ is everywhere non-differentiable. Its Fourier expansion is given in terms of the Riemann zeta function on the line $Re(s) = 0$ at $s_k = \frac{2\pi i k}{\log p}$.

4. Products of Farey Fractions-2

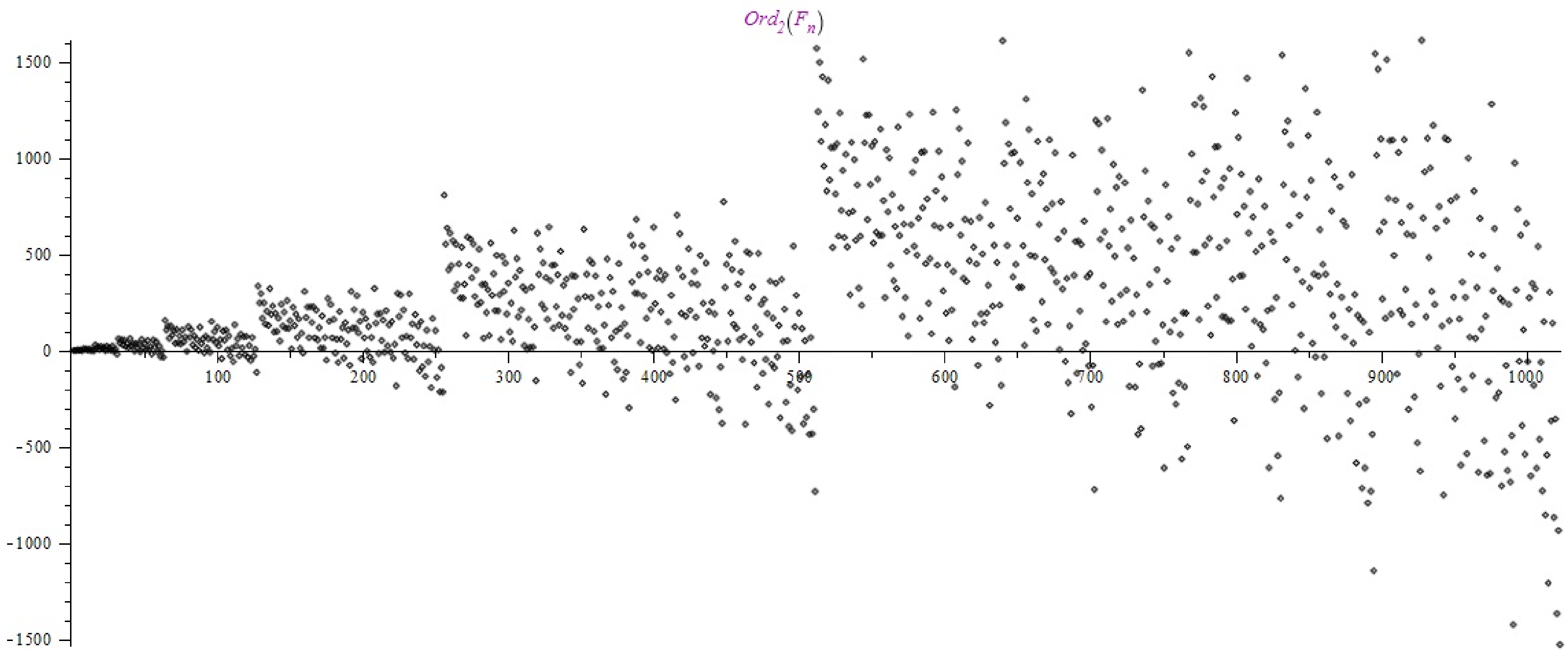
- We return to products of Farey fractions $\overline{F}_n$.

- The asymptotic behavior of (the logarithm of) Farey products encodes the Riemann hypothesis.

- What about divisibility patterns by a fixed prime?

- The next slide presents data on distribution of divisibility for $p = 2$. (Other small primes behave similarly).

Farey products- ord$_2(\overline{F}_n)$ data to n=1023



Observations on Farey Product $\mathrm{ord}_2(\overline{F}_n)$ data

• Negative values of $f_2(n)$ seem to occur often, perhaps a positive fraction of the time. (**UNPROVED!**)

• Just before each (small) power of 2, at $n = 2^k - 1$, we observe $f_2(n) \le 0$, while at $n = 2^k$ a big jump occurs (of size $\gg n \log_2 n$, leading to $f_2(n+1) > 0$. –see next slide–
(**UNPROVED!**)

• For small primes the quantity $f_p(n)$ appears to be both positive and negative on each interval p^k to p^{k+1}.
(**UNPROVED!**)

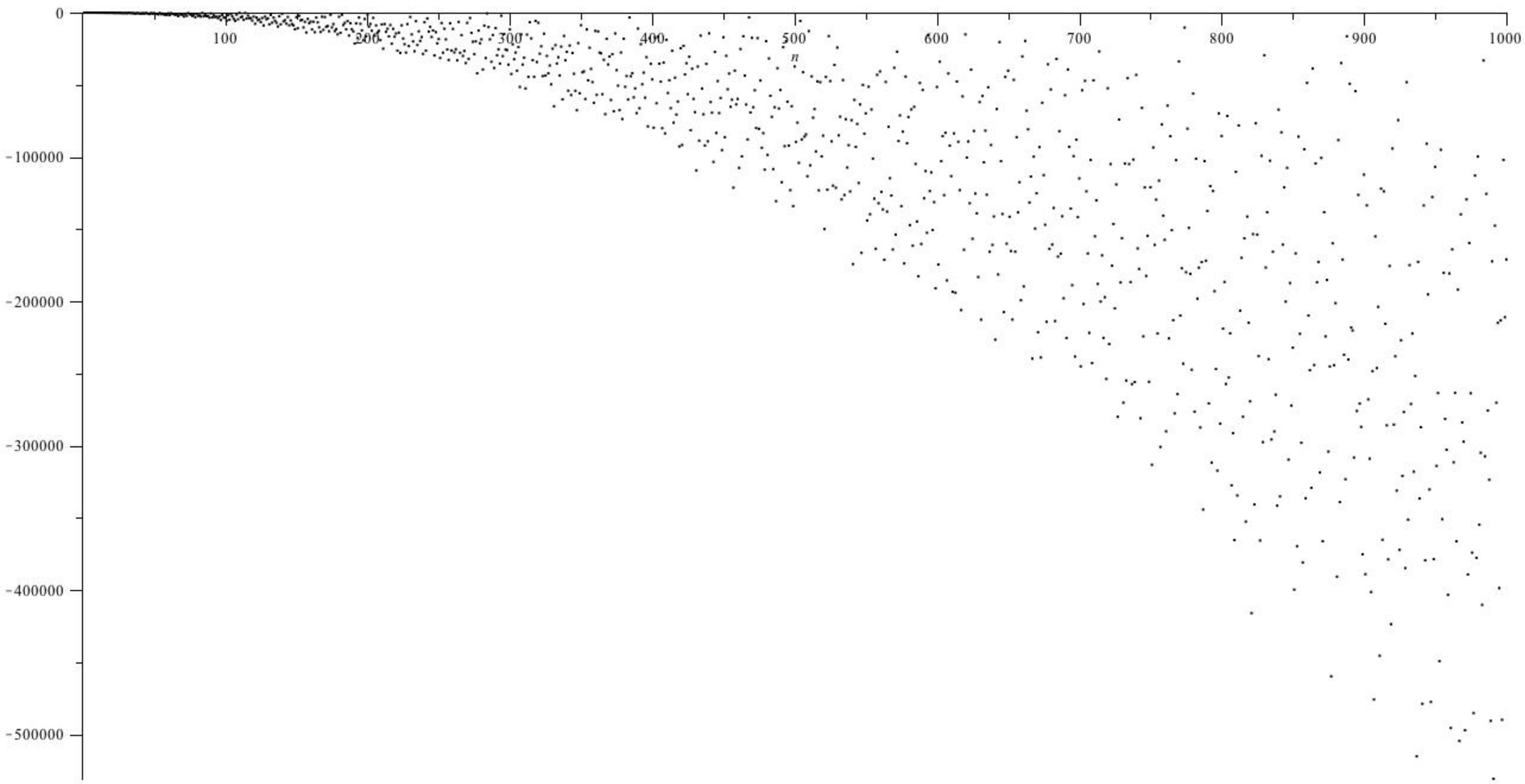
Power r	$N = 2^r - 1$	$\mathrm{ord}_2(\overline{F}_{2^r-1})$	$-\frac{\mathrm{ord}_2(F_{2^r-1})}{N}$	$-\frac{\mathrm{ord}_2(F_{2^r-1})}{N \log_2 N}$
1	1	0	0.0000	0.0000
2	3	0	0.0000	0.0000
3	7	-1	0.1429	0.0509
4	15	-2	0.1333	0.0341
5	31	-19	0.6129	0.0586
6	63	-35	0.5555	0.0929
7	127	-113	0.8898	0.1273
8	255	-216	0.8471	0.1095
9	511	-733	1.4344	0.1594
10	1023	-1529	1.4946	0.1495
11	2047	-3830	1.8710	0.1701
12	4095	-7352	1.7953	0.1496
13	8191	-20348	2.4842	0.1910
14	16383	-41750	2.5484	0.1820
15	32767	-89956	2.7453	0.1830

Simplest Case: $n = p^2 - 1$

- A very special case of sign changes:

Experimentally $\mathrm{ord}_p(\overline{F}_{p^2-1}) \leq 0$ for all primes $p \leq 2000$.
(**UNPROVED!**)

Simplest Case: $n = p^2 - 1$ data



Relating Unreduced and Reduced Farey Products:

• One can study Farey products $\mathrm{ord}_p(\overline{F}_n)$ using $\mathrm{ord}_p(\overline{G}_n)$ using Möbius inversion: We have

$$\overline{G}_n = \prod_{k=1}^{n} \overline{F}_{\lfloor n/k \rfloor},$$

which implies

$$\overline{F}_n = \prod_{k=1}^{n} (\overline{G}_{\lfloor n/k \rfloor})^{\mu(k)}.$$

• **Idea.** Combine this identity with ideas from the Dirichlet hyperbola method, to get new formulation of Riemann hypothesis having (possible) p-adic analogues.

Relating Unreduced and Reduced Farey Products-2

- **Möbius inversion** gives:

$$\log(\overline{F}_n) = \sum_{k=1}^{n} \mu(k) \log(\overline{G}_{\lfloor n/k \rfloor})$$

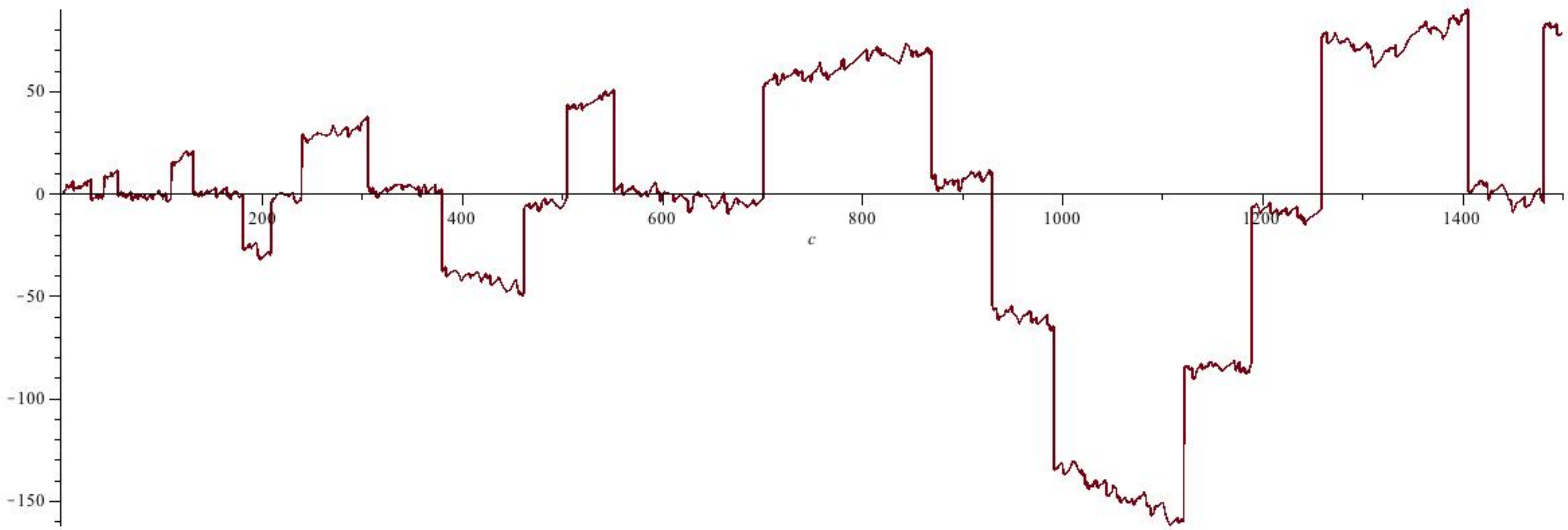
- **Main Term.** (concocted starting from above formula)

$$\Phi_\infty(\overline{F}_n) := \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \mu(k) \left(\log(\overline{G}_{\lfloor n/k \rfloor}) - \frac{1}{2} \lfloor \frac{n}{k} \rfloor^2) \right) + \sum_{k=1}^{n} \mu(k) \left(\frac{1}{2} \lfloor \frac{n}{k} \rfloor^2 \right),$$

- **Remainder Term.** (definition)

$$\overline{R}_\infty(n) := \log \overline{F}_n - \Phi_\infty(\overline{F}_n)$$

Plot of $\overline{R}_\infty(n)$



Relating Unreduced and Reduced Farey Products-2

• The term $\Phi_\infty(n)$ was constructed to reproduce the main term $\Phi(n) - \frac{1}{2}n$ in the formula of Mikolás.

• **Theorem** (L-Mehta (2016)) *If the Riemann hypothesis is true, then the remainder term has*

$$\overline{R}_\infty(n) = O(n^{3/4+\epsilon})$$

• **Followup:** A converse assertion holds: The bound

$$\overline{R}_\infty(n) = O(n^{3/4+\epsilon})$$

implies the Riemann hypothesis.

Relating Unreduced and Reduced Farey Products-3

• p**-adic analogue:** Replace $\log \overline{G}_n$ with $\mathrm{ord}_p(\overline{G}_n)$. ($d_p(n) =$ sum of base p arithmetic digits of n, cf. Kummer's theorem.)

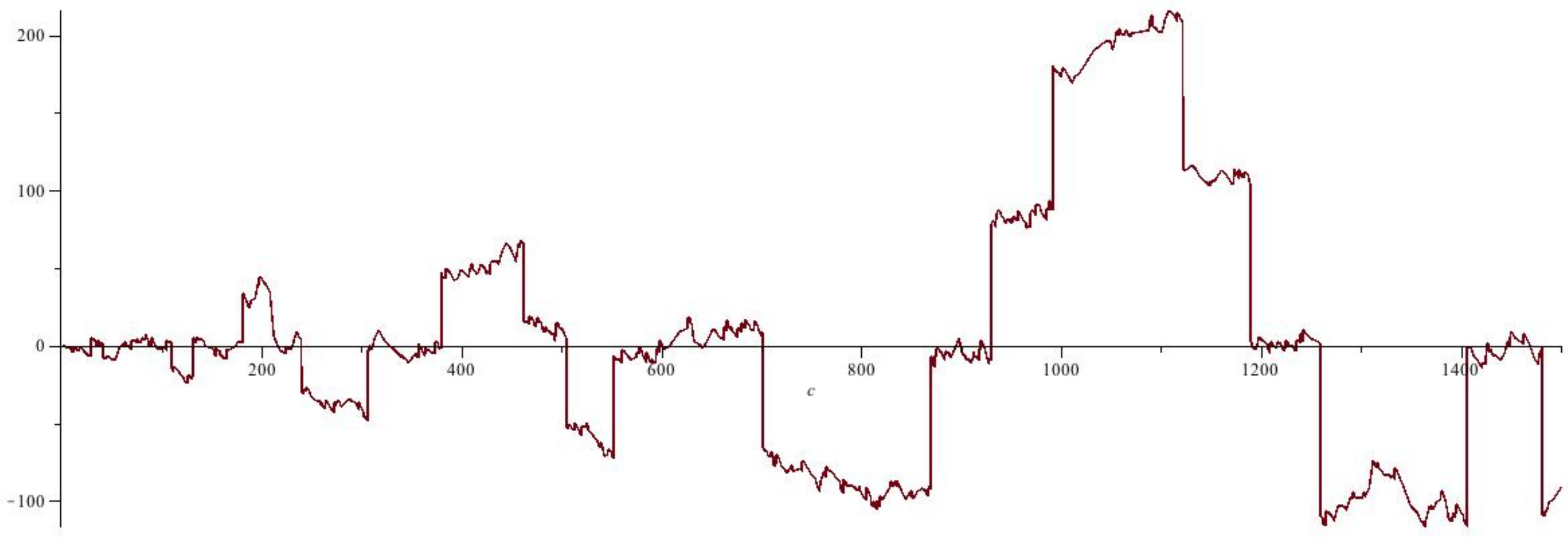
• **Main Term.** Set:

$$\begin{aligned} \Phi_{p,1}(\overline{F}_n) \; := \; & -\frac{n+1}{p-1}\Big(\sum_{k=1}^{n} \mu(k) d_p(\lfloor \frac{n}{k} \rfloor)\Big) \\ & + \sum_{k=1}^{\sqrt{n}} \mu(k) \left(\mathrm{ord}_p(\overline{G}_{\lfloor n/k \rfloor}) + \frac{n+1}{p-1} d_p(\lfloor \frac{n}{k} \rfloor) \right) \end{aligned}$$

• **Remainder Term**. (definition)

$$R_{p,1}(n) := \mathrm{ord}_p(\overline{F}_n) - \Phi_{p,1}(\overline{F}_n)$$

Plot of 3-adic remainder term $R_{3,1}(n)$



Relating Unreduced and Reduced Farey Products-4

• The 3-adic plot, if turned upside down, has an *amazingly similar* appearance to the plot for $\overline{R}_{\infty}(n)$. (But it is slightly different.)

• Very similar appearance of the plots turns out to be related to the hyperbola method, not related to the Riemann hypothesis.

• Is the growth rate of this error term $R_{p,1}(n)$ related to the Riemann hypothesis? **We don't know. (But it might be!)**

Conclusion

- Since many of these problems relate to the Riemann hypothesis, proving even simple looking things may turn out to be very difficult!

- So — *start scribbling...*

The Last Slide...

Thank you for your attention!

Credits and References

• H. Mehta and J. C. Lagarias,
Products of binomial coefficients and unreduced Farey fractions, Intl. J. Number Theory, **12** (2016), No. 1, 57–91.

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Experimental Math., (2016), to appear.
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Don’t Listen to This Talk

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