

The Arithmetic of the Spheres

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Topics Covered

- Part 1. The Harmony of the Spheres
- Part 2. Lester Ford and Ford Circles
- Part 3. The Farey Tree and Minkowski ψ -Function
- Part 4. Farey Fractions
- Part 5. Products of Farey Fractions

Part I. The Harmony of the Spheres

Pythagoras (c. 570–c. 495 BCE)

- To [Pythagoras](#) and followers is attributed: pitch of note of vibrating string related to length and tension of string producing the tone. Small integer ratios give pleasing harmonics.
- [Pythagoras](#) or his mentor [Thales](#) had the idea to explain phenomena by mathematical relationships. “All is number.”
- A fly in the ointment: Irrational numbers, for example $\sqrt{2}$.

Harmony of the Spheres-2

- Q. “Why did the Gods create us?”
A. “To study the heavens.” .
- Celestial Sphere: The universe is spherical: Celestial spheres. There are concentric spheres of objects in the sky; some move, some do not.
- *Harmony of the Spheres*. Each planet emits its own unique (musical) tone based on the period of its orbital revolution.

Also: These tones, imperceptible to our hearing, affect the quality of life on earth.

Democritus (c. 460–c. 370 BCE)

Democritus was a pre-Socratic philosopher, some say a disciple of Leucippus. Born in Abdera, Thrace.

- Everything consists of moving *atoms*. These are geometrically indivisible and indestructible.
- Between lies empty space: the *void*.
- Evidence for the void: Irreversible decay of things over a long time, things get mixed up. (But other processes purify things!)
- “By convention hot, by convention cold, but in reality atoms and void, and also in reality we know nothing, since the truth is at bottom.”
- *Summary*: everything is a dynamical system!

Democritus-2

- The earth is round (spherical). The universe started as atoms churning in chaos till collided into larger units, like the earth.
- There are many worlds. Every world has a beginning and an end.
- **Democritus** wrote mathematical books, of which we know titles (all lost): *On Numbers, On Tangencies, On Irrationals*.

Plato (428–348 BCE)

Ideal education. The seven *liberal arts*:

- **Trivium:** (“the three roads”) Grammar, logic (dialectic), and rhetoric.
- **Quadrivium:** (“the four roads”) arithmetic, geometry, music and astronomy

Liberal arts were codified in the classical world:

- **Marcus Terentius Varro** (116 BCE- 27 BCE, Rome)
- **Martianus Capella**, (fl. 410-420 CE, Carthage)

Proclus (417–485 CE)

Neoplatonist philosopher, born in Constantinople, wrote a commentary on the Elements of [Euclid](#) (fl. c. 300 BCE, Alexandria). He said:

- The [Pythagoreans](#) considered all mathematical science to be divided into four parts: one half they marked off as concerned with *quantity*, the other half with *magnitude*; and each of these they posited as twofold.
- A *quantity* can be considered in regard to its character by itself or in its relation to another quantity, *magnitudes* as either stationary or in motion.

Proclus : Quadrivium

- **Arithmetic**, then, studies *quantities* as such,
- **Music**, the relations between *quantities*,
- **Geometry**, *magnitude* at rest,
- **Spherics**, [Astronomy] *magnitude* inherently moving.

Johannes Kepler(1571–1630)

Looking for patterns in the heavens:

- *Mysterium Cosmographicum* (1596) [“The Cosmographic Mystery”] Orbital sizes of the five planets determined by inscribed regular polyhedra [He follows a Platonist cosmology, using polyhedra and spheres]

- *Astronomia Nova* (1609) [“A New Astronomy”]

First two Kepler laws:

1. planets have elliptic orbits with sun at one locus,
2. line segment joining planet and sun sweeps out equal areas in equal times.

- Made nearly 40 attempts for orbit of Mars, elliptic orbit was final try.

Johannes Kepler-3

- *Astronomia Nova*, (1609) Introduction

“Advice for idiots. But whoever is too stupid to understand astronomical science, or too weak to believe Copernicus without [it] affecting his faith, I would advise him that, having dismissed astronomical studies, and having damned whatever philosophical studies he pleases, he mind his own business and betake himself home to scratch in his own dirt patch.”

- Translation: W. H. Donaghue, *Johannes Kepler-New Astronomy*, Cambridge U. Press 1992, page 65.

Johannes Kepler-4

- *Epitome astronomiae Copernicanae* (1615–1621) [“Epitome of Copernican Astronomy”]

Made improvements on Copernican theory.

- *Harmonicis Mundi* (1619) [“Harmony of the World”]

Discusses “music of the spheres”, regular solids, their relation to music.

Book V applies to planetary motion, Kepler’s third law:

3. square of periodic times proportional to cube of planetary mean distances.

In this book, Kepler computed many statistics, comparing orbital periods of various kinds. For some statistics he found no harmony, and said so.

Kepler's Third Law-Modern Data

Planet	a (A. U.)	T (years)	T^2/a^3
Mercury	0.38710	0.24085	1.0001
Venus	0.72333	0.61521	0.9999
Earth	1.00000	1.00000	1.00000
Mars	1.52369	1.88809	1.0079
Jupiter	5.2028	11.8622	1.001
Saturn	9.540	29.4577	1.001

TABLE Modern Values for Orbital Data: a = average of perihelion + aphelion

- Source: Stephen Weinberg, *To Explain the World*, Harper-Collins: New York 2015, page 171.

Johannes Kepler's Dream- "Somnium" (1634)

- Kepler's original conversion to Copernican theory: "What would the motion of the planets in the sky look like if one were looking from the moon?"
- This thought experiment turned out fruitful.
- **Moral.** Examining the consequences of looking at old data from a new viewpoint can lead to new research discoveries.

Part 2. Lester Ford and Ford Circles

Lester R. Ford (1888-1967) grew up in Missouri. Graduated M. A. from Univ. of Missouri-Columbia 1912 [Discontinuous functions]. Another M. A. from Harvard (1913) [Maxime Bôcher, advisor]

Then Univ. of Edinburgh, Scotland 1915–1917.

- L. R. Ford, *Introduction to the theory of automorphic functions*, Edinburgh Math. Tract. No. 6, 1915.
- L. R. Ford, *Rational approximations to irrational complex numbers*, Transactions of the AMS **19** (1918), 1–42.
- L. R. Ford, *Elementary Mathematics for Field Artillery*, Field Artillery Officer's Training School, Camp Zachary Taylor, Kentucky, 1919.

Lester Ford-2

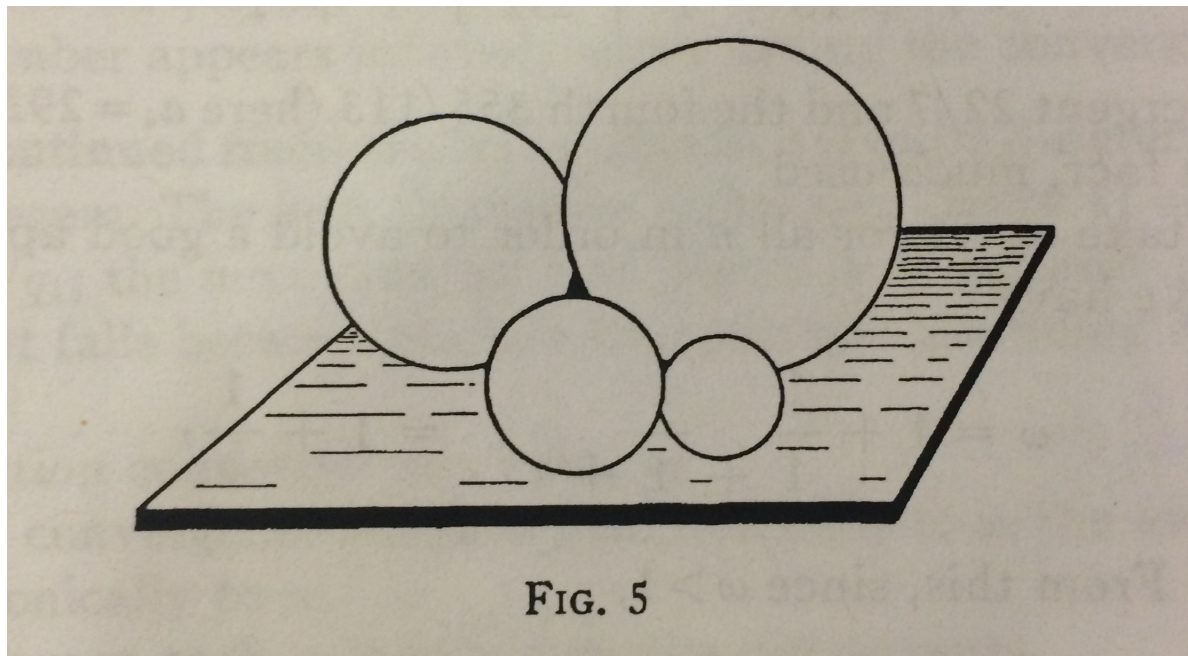
- L. R. Ford, *Automorphic Functions*, McGraw-Hill 1929.
- L. R. Ford, *Fractions*, American Math. Monthly **45** (1938), 586–601.
- Editor, American Mathematical Monthly 1942–1946.
- President of MAA, 1947–1948.
- His son [Lester R. Ford, Jr.](#) is known for network flow algorithms (Ford-Fulkerson algorithm).

Ford Circles

Lester Ford, *Fractions*, American Math. Monthly **45** (1938), 586–601.

“The idea of representing a fraction by a circle is one which the author arrived at by an exceedingly circuitous journey. It began with the Group of [Picard](#). In the treatment of this group as carried on by [Bianchi](#), in accordance with the general ideas of [Poincaré](#), certain invariant families of spheres appear. These spheres, which are found at the complex rational fractions, [...] suggest analogous known invariant families of circles at real rational points in the the theory of the Elliptic Modular Group in the complex plane. Finally it became plain that this intricate scaffolding of group theory could be dispensed with and the whole subject be built up in a completely elementary fashion.”

Ford Spheres- Picard Group $SL_2(\mathbb{Z}[i])$



Ford Circles-2

- The *Ford circle* $C(\frac{p}{q})$ attached to rational $\frac{p}{q}$ (in lowest terms $\gcd(p, q) = 1$) is the circle tangent to the x -axis having radius $\frac{1}{2q^2}$.
- All Ford circles are disjoint.
- The *neighboring Ford circles* are those Ford circles $C(\frac{p'}{q'})$ that are tangent to it. They form a singly infinite chain...

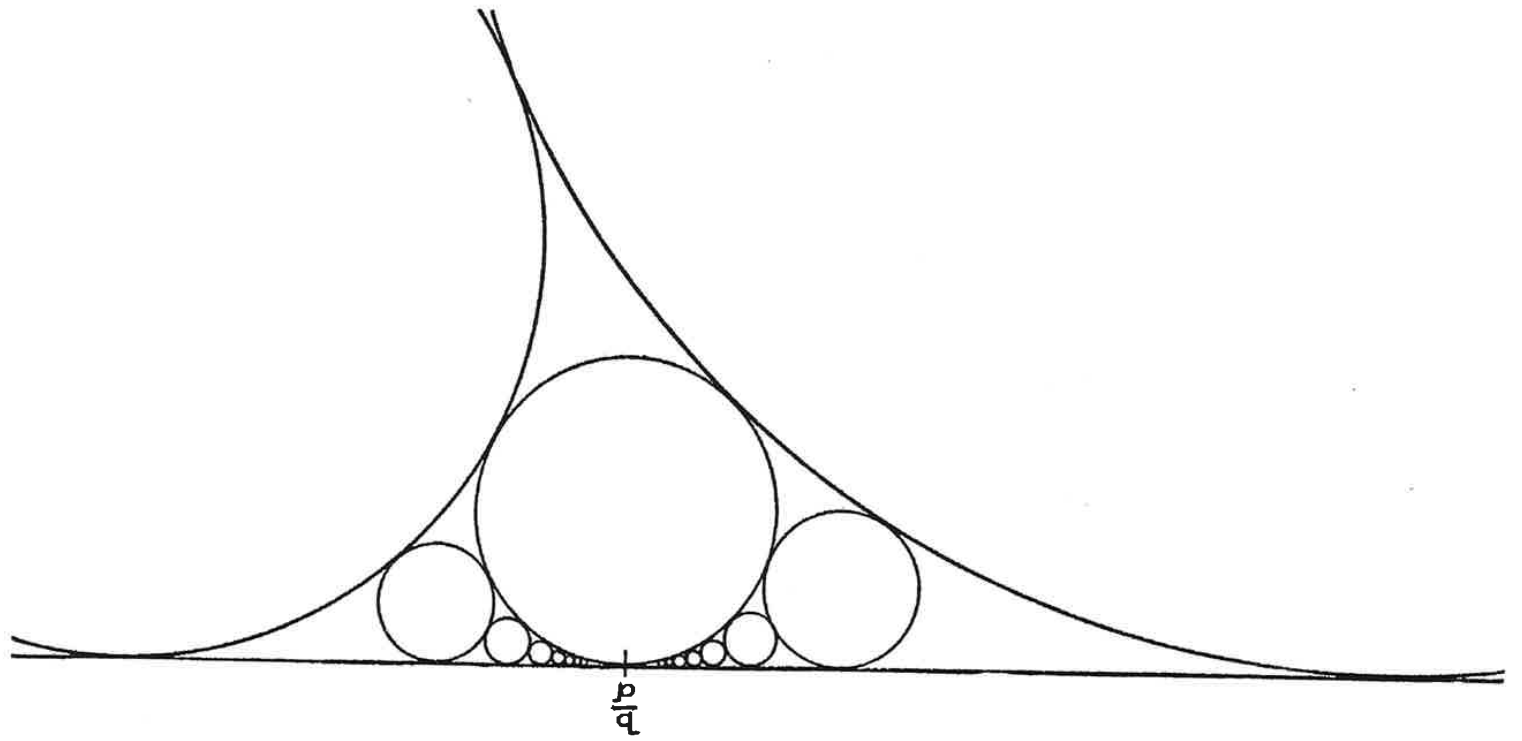


FIG. 2

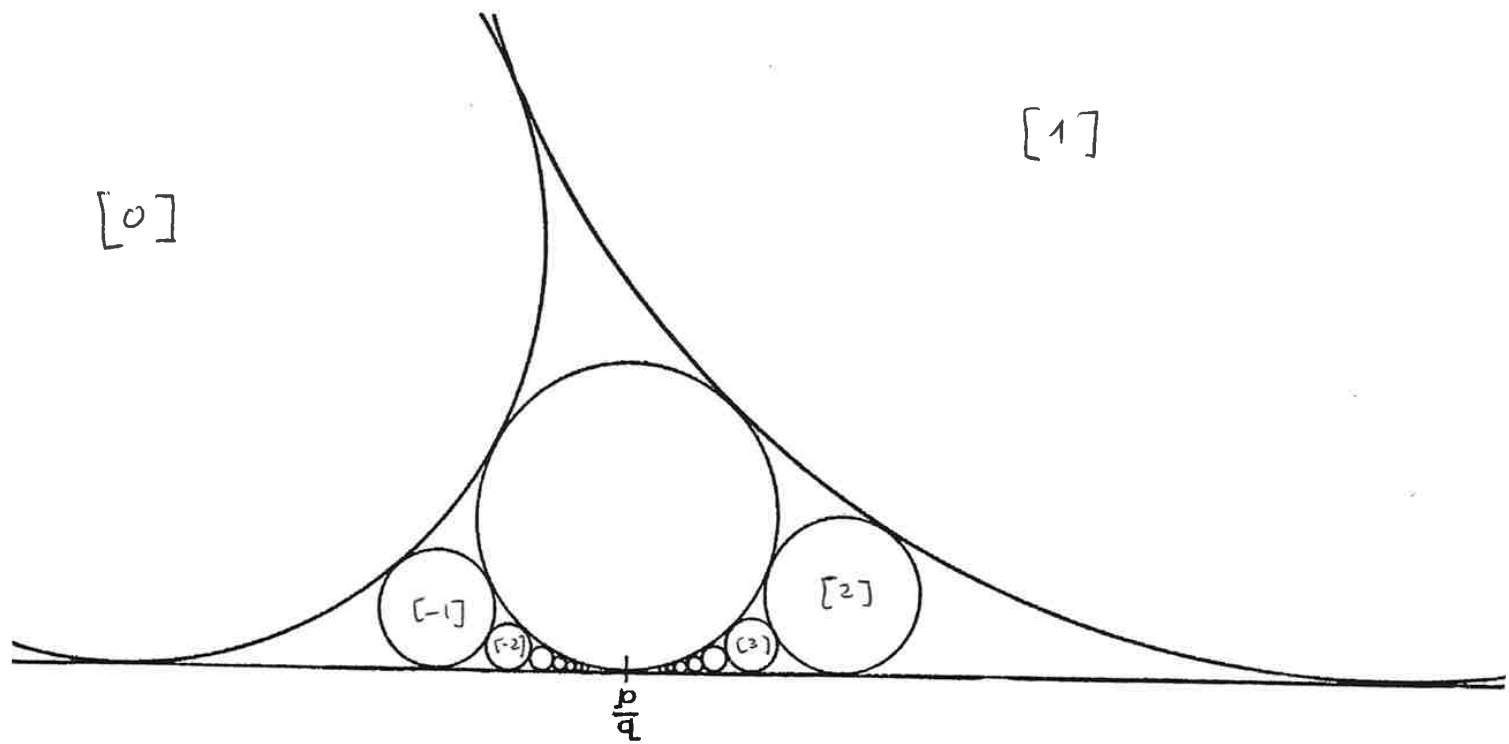


FIG. 2

Farey Sum-1

- Two touching Ford cycles at $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ define a third Ford circle touching each of them and the x -axis. It has value

$$\frac{p_3}{q_3} := \frac{p_1 + q_1}{p_2 + q_2}.$$

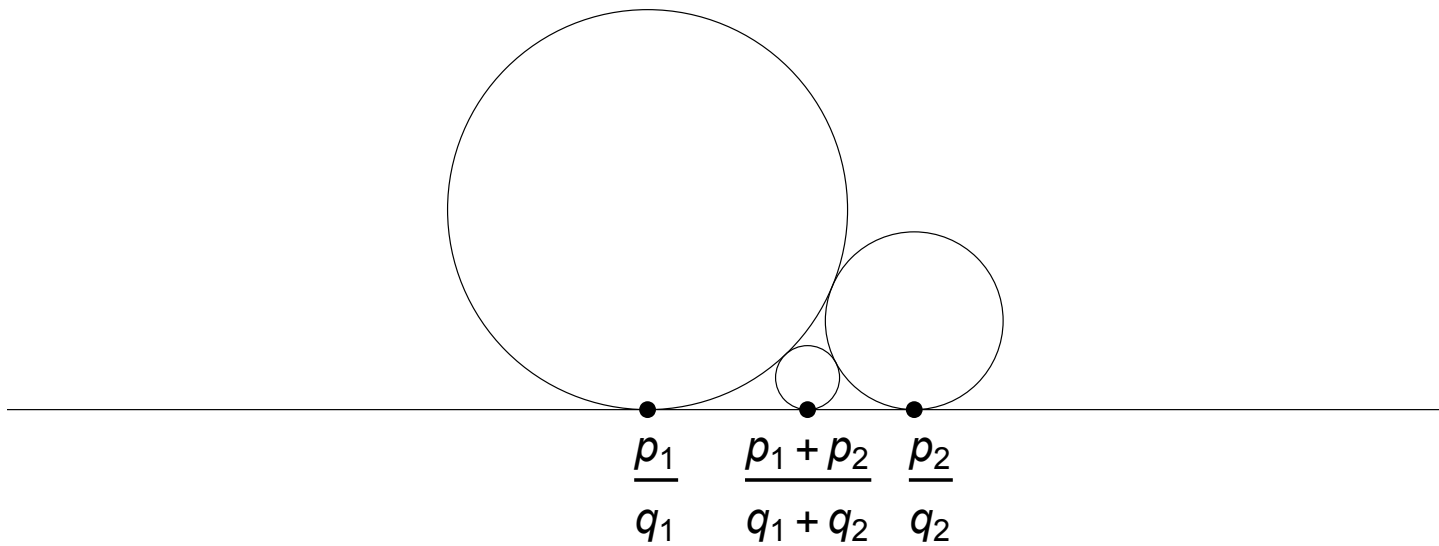
- We call this combination

$$\frac{p_1}{q_1} \oplus \frac{p_2}{q_2} := \frac{p_1 + p_2}{q_1 + q_2}$$

the **Farey sum** operation.

Farey Sum

$$\frac{p_1}{q_1} \oplus \frac{p_2}{q_2} = \frac{p_1 + p_2}{q_1 + q_2}$$

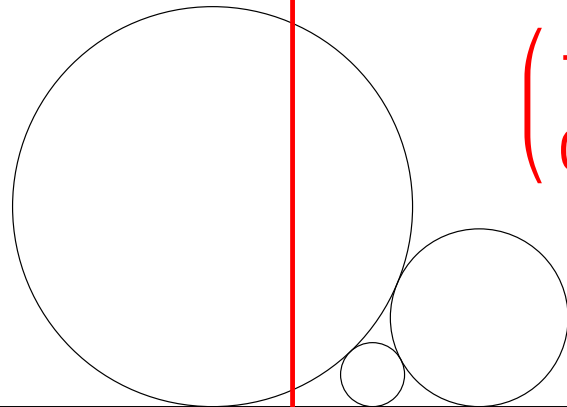


Ford Circles-Geodesic Flow and Continued Fractions

- A *vertical line* L going to a value $x = \theta$ on the x -axis, it is a **geodesic** in the hyperbolic metric on the upper half plane.
- The “geodesic flow” of a point along the line L defines an orbit of a dynamical system, described by the sequence of Ford cycles it cuts through. It is closely related to the continued fraction algorithm.
- Each new circle cut along the line produces a good rational approximation to θ , satisfying

$$\left| \theta - \frac{p}{q} \right| \leq \frac{1}{q^2}.$$

Geodesic



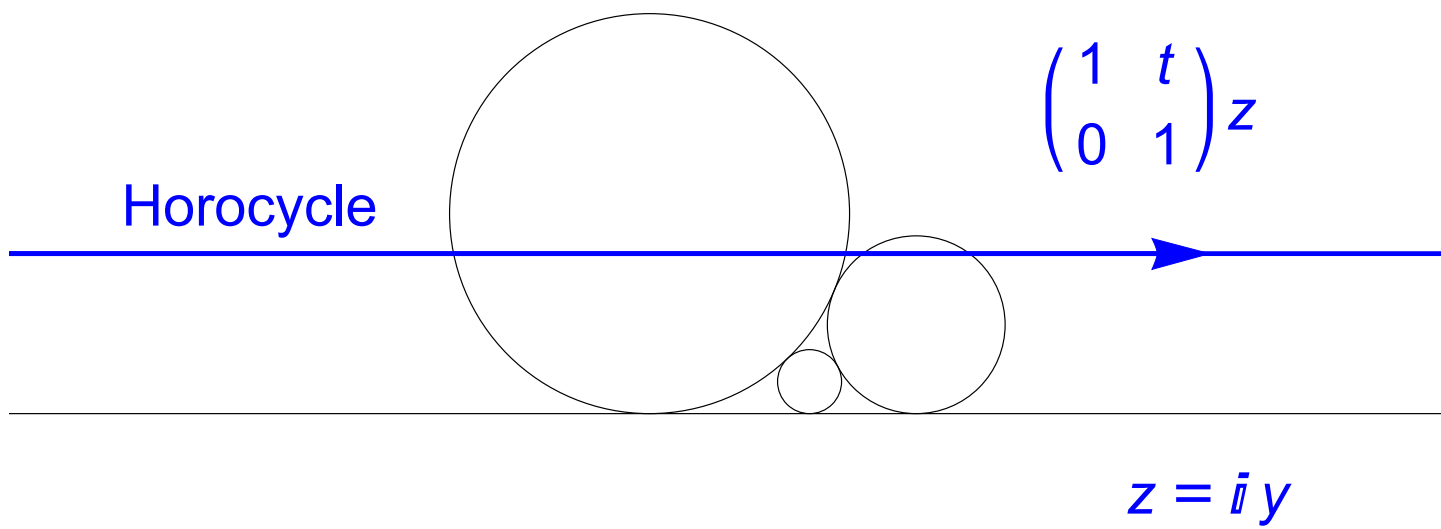
$$\begin{pmatrix} \frac{1}{t} & \theta(t - \frac{1}{t}) \\ 0 & t \end{pmatrix} z$$

θ

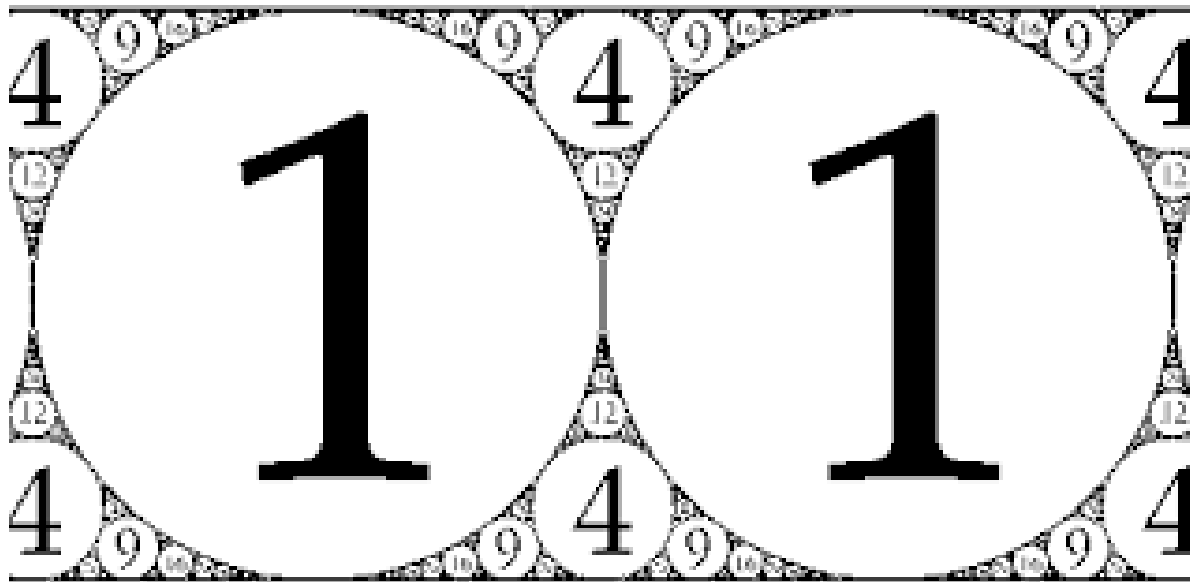
$$z = \theta + i$$

Ford Circles- Horocycle Flow and Farey Fractions

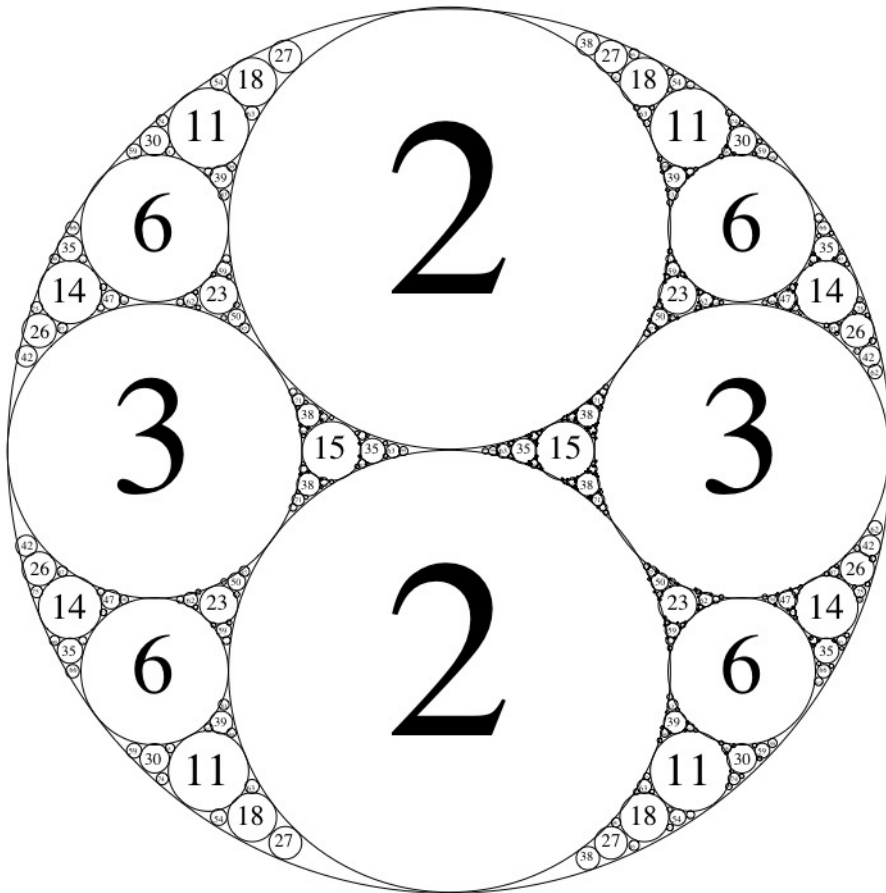
- When L is a horizontal line at a vertical height y , it is called a **horocycle**. The flow of a point along a horizontal line also defines an orbit of a dynamical system, called the “horocycle flow” .
- The Ford circles cut through by such a horocycle L are related to Farey fractions at value $N \approx \sqrt{y}$.



Apollonian Circle Packing: Strip Packing (0,0, 1,1)



Apollonian Circle Packing: $(-1, 2, 2, 3)$



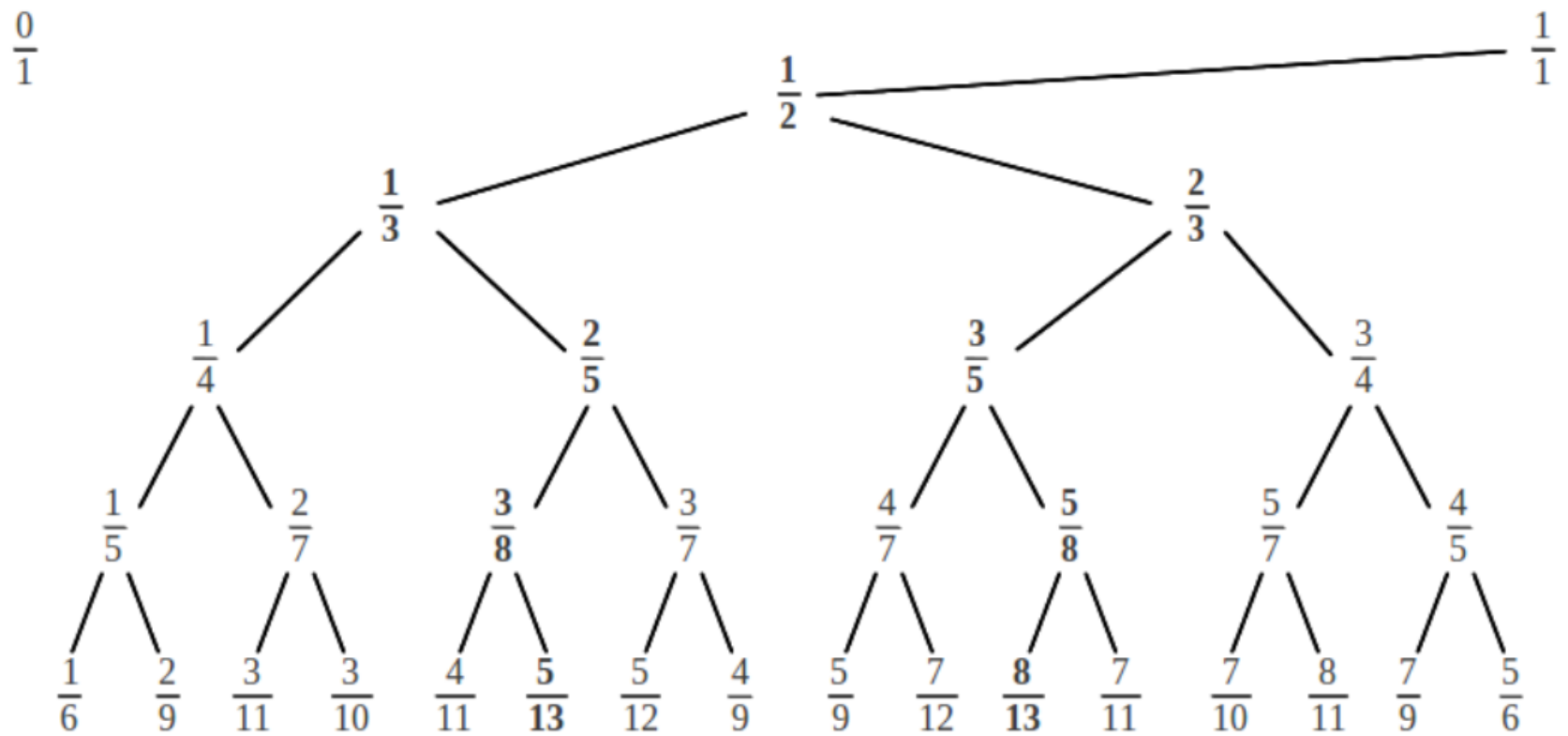
Part 3. Farey Tree

The *Farey tree* is an infinite tree of rational numbers connected by the *Farey sum operation*.

It is related to:

- (1) Stern's diatomic sequence,
- (2) Geodesic Flow;
- (3) The continued fraction algorithm.

Farey Tree-1



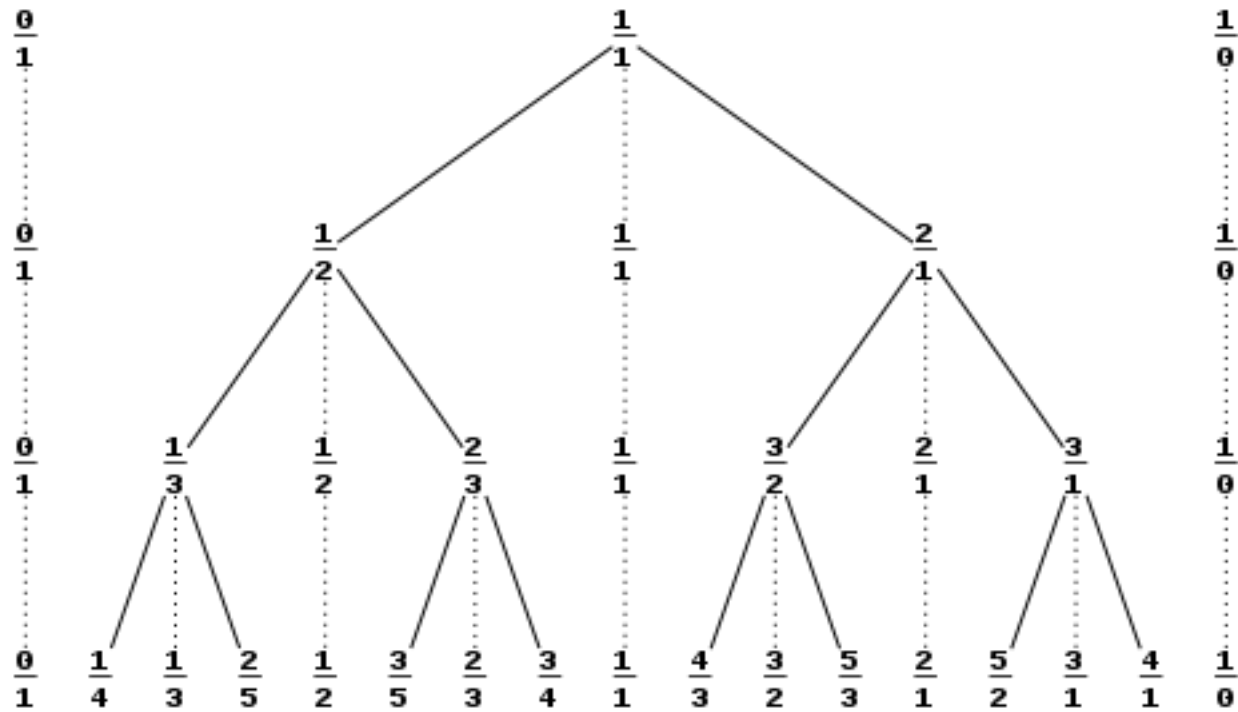
Farey Tree-2

- The Farey tree computes values at each level using the Farey sum (mediant) of two adjacent at earlier levels:

$$\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d}.$$

- The leaves of the Farey tree at levels below a given k respect the ordering $<$ on the real line.
- The Farey tree forms “half” of a larger tree that enumerates all positive rationals. The other “half” gives the rationals larger than one. The new root node is $\frac{1}{1}$.

Full Farey Tree: Positive Rationals



Stern's Diatomic Sequence

Moritz A. Stern, *Ueber eine zahlentheoretische Funktion*,
J. reine Angew Math. **55** (1858), 193–220.

- The sequence a_n begins

0, 1; 1, 2; 1, 3, 2, 3; 1, 4, 3, 5, 2, 5, 3, 4; 1, 5, 4, 7, 3, 8, ...

- It is determined by the initial conditions

$$a_0 = 0 \quad a_1 = 1$$

and the recursion rules

$$a_{2n} = a_n$$

$$a_{2n+1} = a_n + a_{n+1}$$

Stern's Diatomic Sequence-2

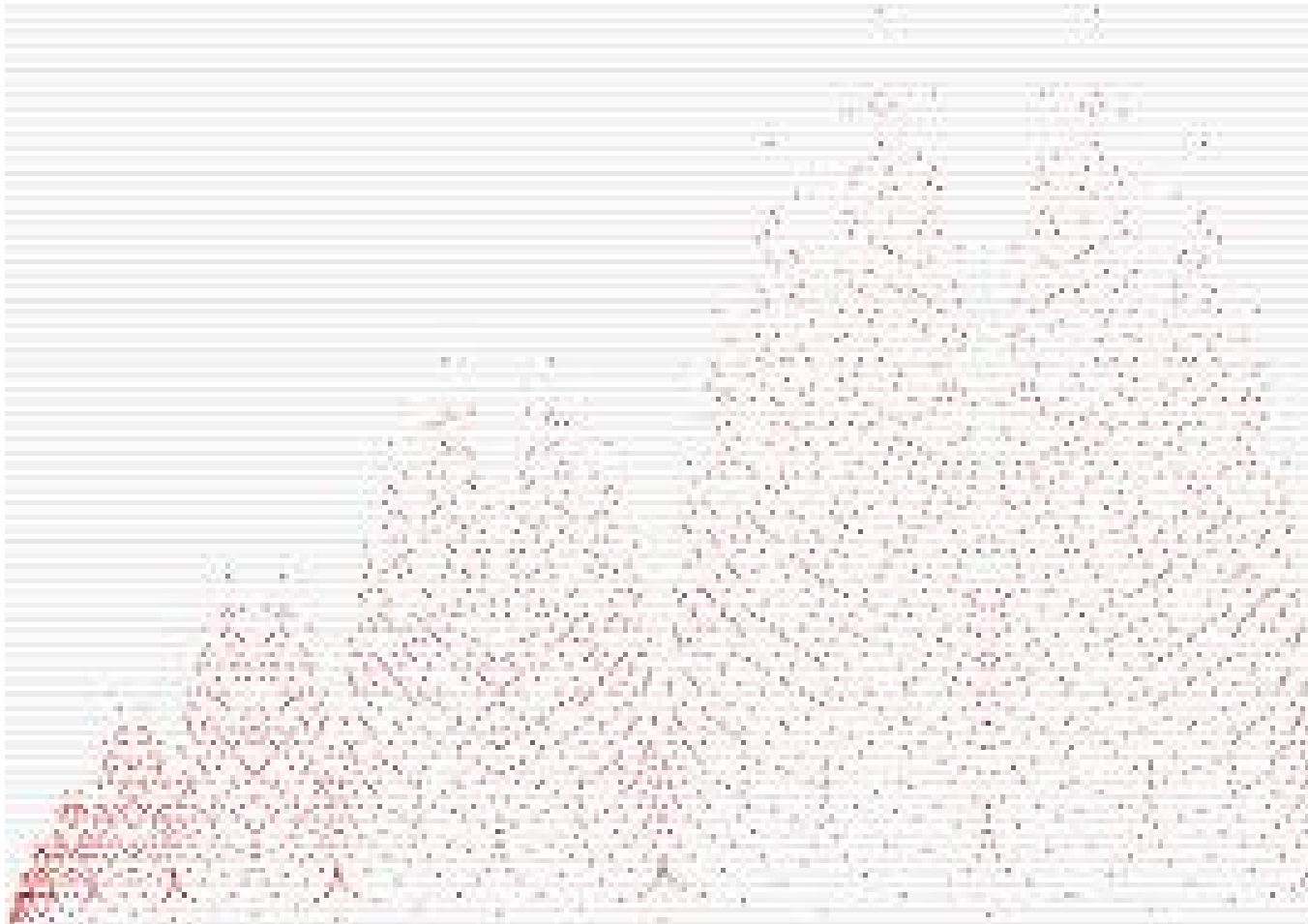
A very useful reference:

[Sam Northshield](#), *Stern's Diatomic Sequence*

0, 1, 1, 2, 1, 3, 2, 3, 1, 4, ..., Amer. Math. Monthly **117** (2010), 581–598.

- The sequence a_n breaks into blocks of length 2^k (indicated by semicolons) which give the sequence of *denominators* of the Farey tree at the k -th level.
- Sequence of *numerators* of the Farey tree can also be understood.

Stern's Diatomic Sequence: Plot

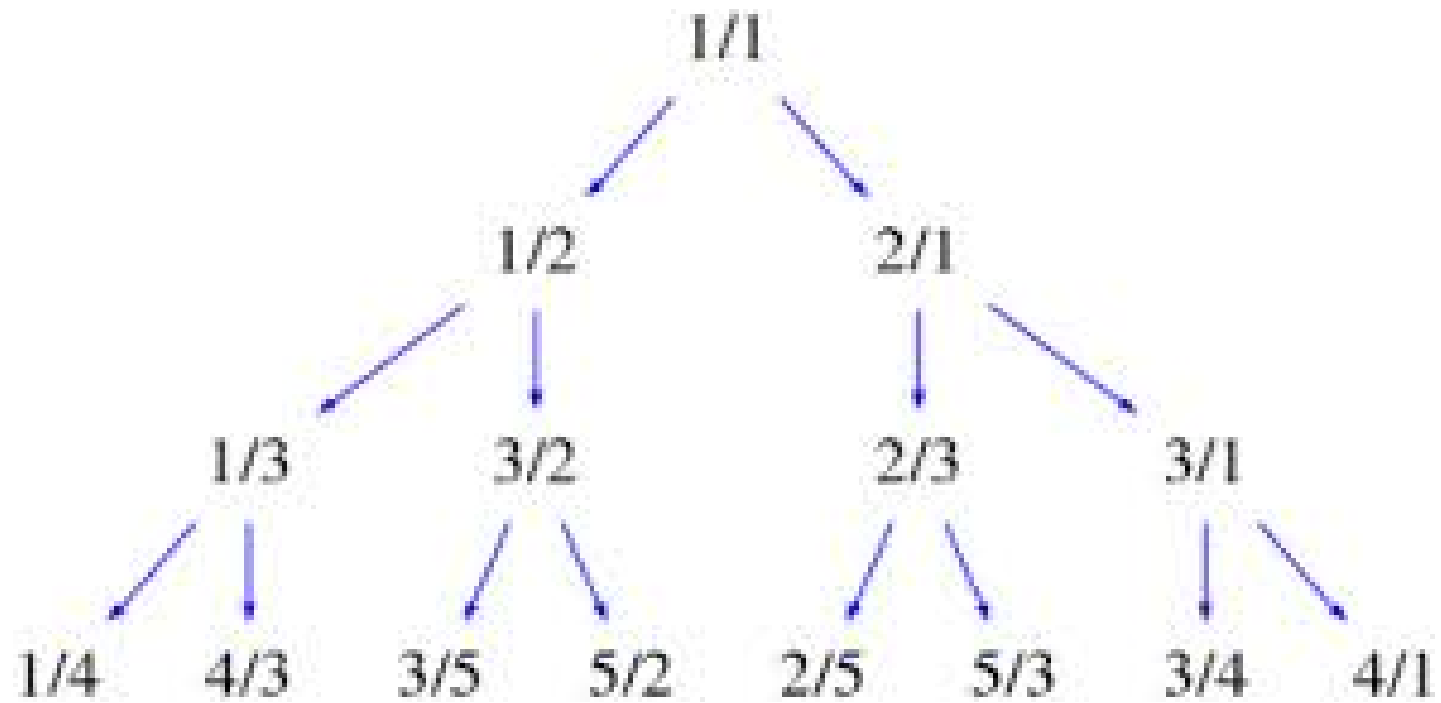


Calkin-Wilf Tree

A different, related ordering of positive rationals is the [Calkin-Wilf tree](#). See: [Neil Calkin and Herb Wilf](#), *Recounting the rationals*, Amer. Math. Monthly **107** (2000), 360–363

- This tree lists all positive rationals in a different order than in the full Farey tree. The totality of elements on each level also are same (as a set), and the denominators are in same order. However the numerators appear in a different order;
- The order of tree elements from left to right (below a fixed level) are fractions a_n/a_{n+1} with a_n being [Stern's diatomic sequence](#).

Calkin-Wilf Tree



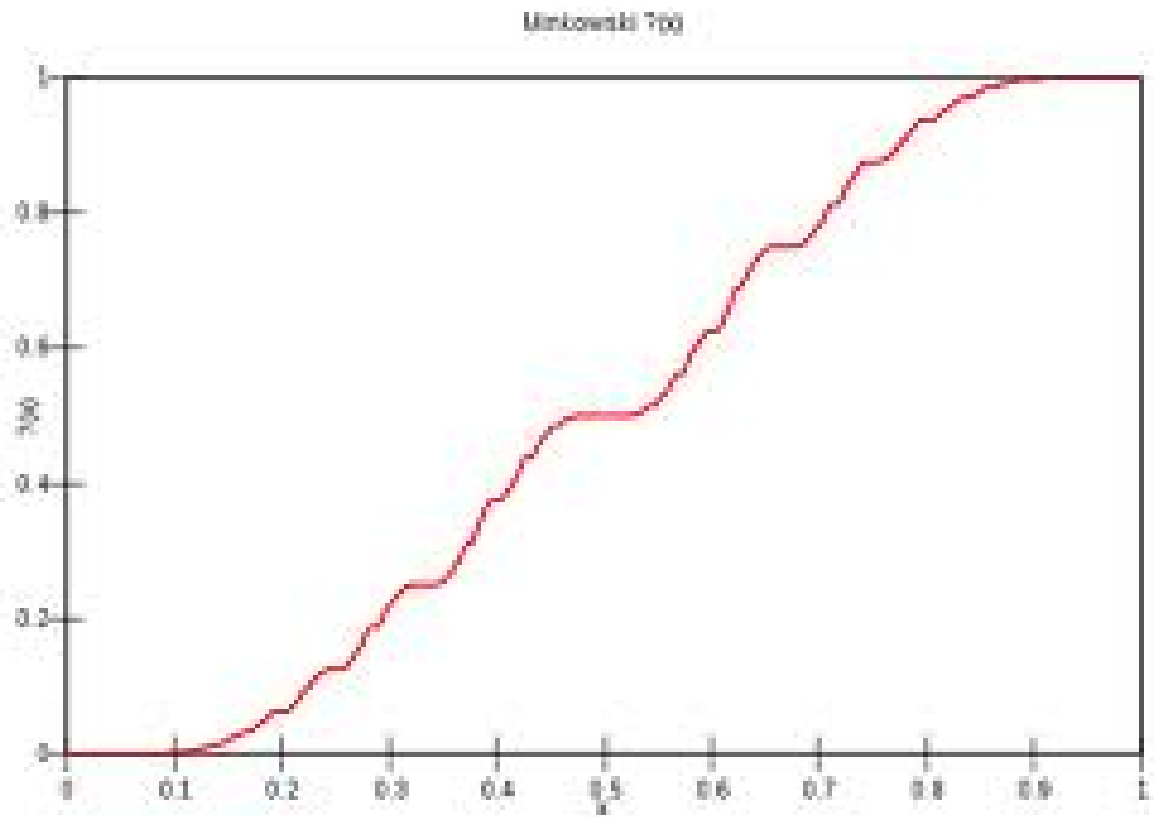
Farey Tree and Question-Mark Function

Theorem. (1) A uniform delta function measure (equally weighted point masses) on the values of the Farey tree at levels below k converge (weakly) as $k \rightarrow \infty$ to a limit probability measure μ on $[0, 1]$.

(2) The limit measure μ is purely singular measure. It is supported on a set S of Hausdorff dimension less than 1. (It is between 0.8746 and 0.8749)

(3) The cumulative distribution function $F(x) = \int_0^x d\mu(t)$ is the Minkowski question-mark function.

Minkowski Question-Mark Function-1



Minkowski Question-Mark Function-2

- **Hermann Minkowski** (1904) introduced this function for a different reason. He showed it maps rational numbers to rationals or to real-quadratic irrational numbers.

- Suppose $0 < \theta < 1$ has continued fraction expansion

$$\theta = [0, a_1, a_2, a_3, \dots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}.$$

Then

$$?(\theta) := 2^{1-a_1} - 2^{1-a_1-a_2} + 2^{1-a_1-a_2-a_3} - 2^{1-a_1-a_2-a_3-a_4} + \dots$$

Continued Fractions and Astronomical Dynamics

- Some dissipative dynamical processes seem to exhibit behavior with a series of bifurcations at small rational numbers following the [Farey tree](#) structure.
- These occur in “mode-locking” processes in astronomy, leading to “harmony of spheres” where certain orbital parameters of different objects satisfy linear dependencies with small rational numbers.
- Resonances given by small rational numbers can also lead to certain orbit parameters being unstable and to regions being cleared of objects. Saturn’s rings exhibit various gaps perhaps explainable by such mechanisms.

Saturn's Rings



Part 4. Farey Fractions

- The **Farey fractions** \mathcal{F}_n of order n are fractions $0 \leq \frac{h}{k} \leq 1$ with $\gcd(h, k) = 1$. Thus

$$\mathcal{F}_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}.$$

The non-zero **Farey fractions** are

$$\mathcal{F}_4^* := \left\{ \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}.$$

- The number $|\mathcal{F}_n^*|$ of nonzero **Farey fractions** of order n is

$$\Phi(n) := \phi(1) + \phi(2) + \cdots + \phi(n).$$

Here $\phi(n)$ is *Euler totient function*. One has

$$\Phi(n) = \frac{3}{\pi^2} N^2 + O(N \log N).$$

Farey Fractions-2

- The Farey fractions have a limit distribution as $N \rightarrow \infty$. They approach the uniform distribution on $[0, 1]$.

- **Theorem.** *The distribution of Farey fractions described by sum of (scaled) delta measures at members of \mathcal{F}_n , weighted by $\frac{1}{\Phi(n)}$. Let*

$$\mu_n := \frac{1}{\Phi(n)} \sum_{r=1}^{\Phi(n)} \delta(\rho_r)$$

Then these measures μ_n converge weakly as $n \rightarrow \infty$ to the uniform (Lebesgue) measure on $[0, 1]$.

Farey Fractions-3

- The rate at which Farey fractions approach the uniform distribution is related to the [Riemann hypothesis](#)!
- **Theorem.** ([Franel's Theorem](#) (1924)) *Consider the statistic*

$$S_n = \sum_{j=1}^{\Phi(n)} \left(\rho_j - \frac{j}{\Phi(n)} \right)^2$$

Then as $n \rightarrow \infty$

$$S_n = O(n^{-1+\epsilon})$$

for each $\epsilon > 0$ if and only if the Riemann hypothesis is true.

- One knows unconditionally that $S_n \rightarrow 0$ as $n \rightarrow \infty$. This fact is equivalent to the Prime Number Theorem.

Part 5. Products of Farey Fractions

- There is a mismatch in scales between addition and multiplication in the rationals \mathbb{Q} , which in some way influences the distribution of prime numbers. To understand this better one might study (new) *arithmetic statistics* that mix addition and multiplication in an interesting way.
- The Farey fractions \mathcal{F}_n encode data that seems “additive”. So why not study the **product** of the Farey fractions? (We exclude the Farey fraction $\frac{0}{1}$ in the product!)
- Define the **Farey product** $F_n := \prod_{r=1}^{\Phi(n)} \rho_r$, where ρ_r runs over the nonzero Farey fractions in increasing order.

Products of Farey Fractions-2

- It turns out convenient to study instead the **reciprocal Farey product** $\overline{F}_n := 1/F_n$.
- Studying **Farey products** seems interesting because will be a lot of cancellation in the resulting fractions. There are about $\frac{3}{\pi^2}n^2$ terms in the product, but all numerators and denominators of ρ_r contain only primes $\leq n$, and there are certainly at most n of these. So there *must* be enormous cancellation in product numerator and denominator!
- This research project was done with REU student **Harsh Mehta** (now grad student at U. South Carolina). Questions about Farey products arose in discussion with **Harm Derksen** (Michigan) some years ago.

Products of Farey Fractions-3

- *Idea.* The products of all (nonzero) Farey fractions

$$F_n := \prod_{\rho_r \in \mathcal{F}_n^*} \rho_r.$$

give a single statistic for each n . Is the Riemann hypothesis encoded in its behavior?

- Amazing answer: [Mikolás \(1952\)](#) Yes!

- **Theorem.** ([Mikolás \(1952\)](#)- rephrased) *Let $\bar{F}_n = 1/F_n$. The Riemann hypothesis is equivalent to the assertion that*

$$\log(\bar{F}_n) = \Phi(n) - \frac{1}{2}n + O(n^{1/2+\epsilon}).$$

(Here $\Phi(n) \sim \frac{3}{\pi^2}n^2$ counts the number of Farey fractions.) The RH concerns the size of the remainder term.

Products of Farey Fractions-4

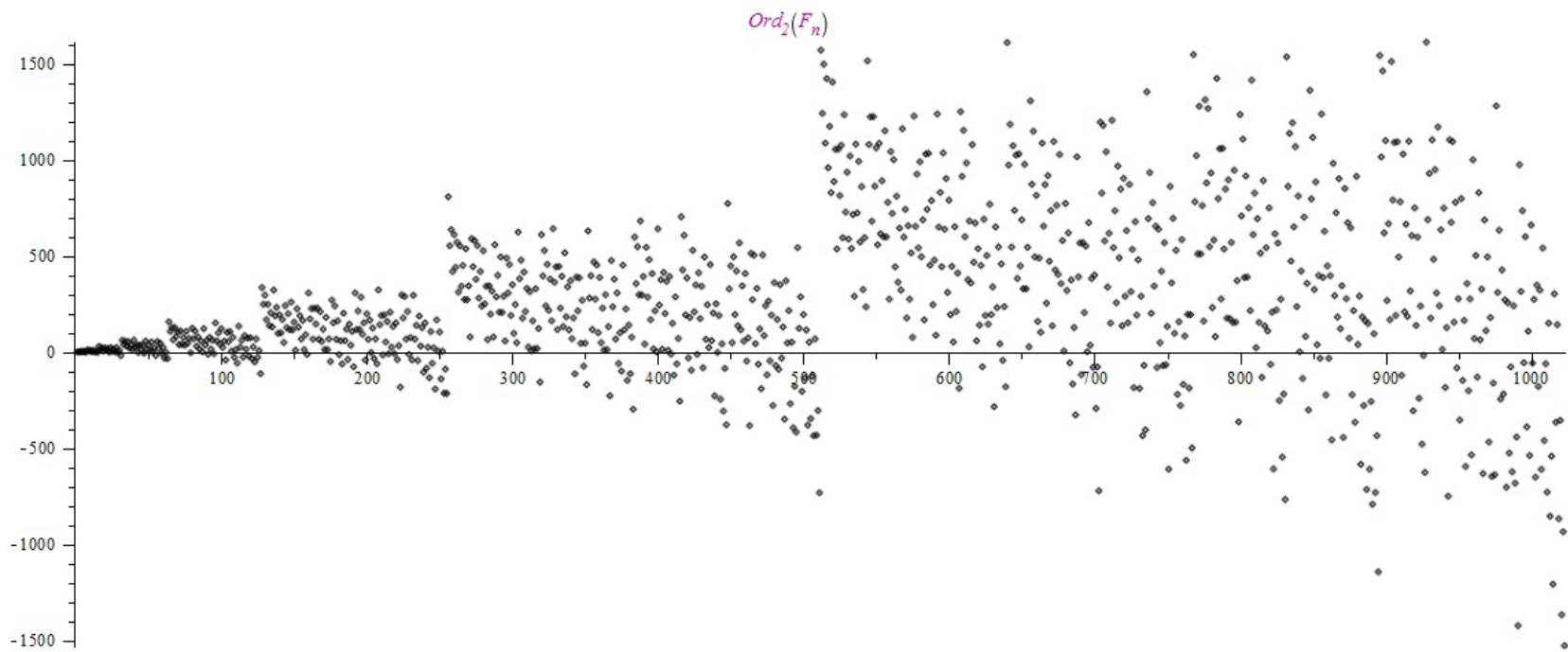
- For Farey products we can ask some *new questions*: what is the behavior of the divisibility of \overline{F}_n by a fixed prime p : What power of p divides \overline{F}_n ? Call it

$$f_p(n) := \text{ord}_p(\overline{F}_n)$$

This value can be positive or negative, because \overline{F}_n is a rational number in general.

- Could some information about RH be encoded in the individual functions $f_p(n)$ for a single prime p ?
- Study this question experimentally by computation for small n and small primes.

Farey products- $\text{ord}_2(\overline{F}_n)$ data to $n=1023$



Observations on $\text{ord}_2(\overline{F}_n)$ data

- Negative values of $f_2(n)$ occur often, perhaps a positive fraction of the time.
- Just before each (small) power of 2, at $n = 2^k - 1$, we observe $f_2(n) \leq 0$, while at $n = 2^k$ a big jump occurs (of size $\gg n \log_2 n$, leading to $f_2(n + 1) > 0$).
- For small primes discover an interesting fractal-like pattern of oscillations. The quantity $f_p(n)$ appears to be both positive and negative on each interval p^k to p^{k+1} .

Power r	$N = 2^r - 1$	$\text{ord}_2(\overline{F}_{2^r-1})$	$-\frac{\text{ord}_2(F_{2^r-1})}{N}$	$-\frac{\text{ord}_2(F_{2^r-1})}{N \log_2 N}$
1	1	0	0.0000	0.0000
2	3	0	0.0000	0.0000
3	7	-1	0.1429	0.0509
4	15	-2	0.1333	0.0341
5	31	-19	0.6129	0.0586
6	63	-35	0.5555	0.0929
7	127	-113	0.8898	0.1273
8	255	-216	0.8471	0.1095
9	511	-733	1.4344	0.1594
10	1023	-1529	1.4946	0.1495
11	2047	-3830	1.8710	0.1701
12	4095	-7352	1.7953	0.1496
13	8191	-20348	2.4842	0.1910
14	16383	-41750	2.5484	0.1820
15	32767	-89956	2.7453	0.1830

More Observations

- A very special case:

Experimentally $\text{ord}_p(\overline{F}_{p^2-1}) \leq 0$ for all primes $p \leq 2000$.

- We cannot prove this holds in general!
- Since some of these problems relate to the [Riemann hypothesis](#), even simple looking things may turn out very difficult!

Toy Model: Products of Unreduced Farey Fractions

- *Idea.* Why not study a simpler “toy model”, products of unreduced Farey fractions?
- The (nonzero) unreduced Farey fractions \mathcal{G}_n^* of order n are all fractions $0 \leq \frac{h}{k} \leq 1$ with $1 \leq h \leq k \leq n$ (no gcd condition imposed).

$$\mathcal{G}_4^* := \left\{ \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{4}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4} \right\}.$$

- The number of unreduced Farey fractions is

$$|\mathcal{G}_n^*| = \Phi^*(n) := 1 + 2 + 3 + \cdots + n = \binom{n+1}{2} = \frac{1}{2}n(n+1).$$

Unreduced Farey Products are Binomial Products

- The reciprocal unreduced Farey product $\overline{G}_n := 1/G_n$ is always an integer.
- **Proposition.** *The reciprocal product \overline{G}_n of unreduced Farey fractions is the product of binomial coefficients in the n -th row of Pascal's triangle.*

$$\overline{G}_n := \prod_{k=0}^n \binom{n}{k}$$

Data: $\overline{G}_1 = 1$, $\overline{G}_2 = 2$, $\overline{G}_3 = 9$, $\overline{G}_4 = 96$,
 $\overline{G}_5 = 2500$, $\overline{G}_6 = 162000$, $\overline{G}_7 = 26471025$. (On-Line
Encyclopedia of Integer Sequences (OEIS): Sequence A001142.)

Binomial Products

- Can now ask same questions as for Farey products.
- We consider the size of \overline{G}_n as real numbers.
Measure size by

$$g_\infty(n) := \log(\overline{G}_n).$$

- We consider behavior of their prime factorizations.
At a prime p , measure size by divisibility exponent

$$g_p(n) := \text{ord}_p(\overline{G}_n).$$

Factorization is: $\overline{G}_n = \prod_p p^{g_p(n)}$.

“Unreduced Farey” Riemann hypothesis

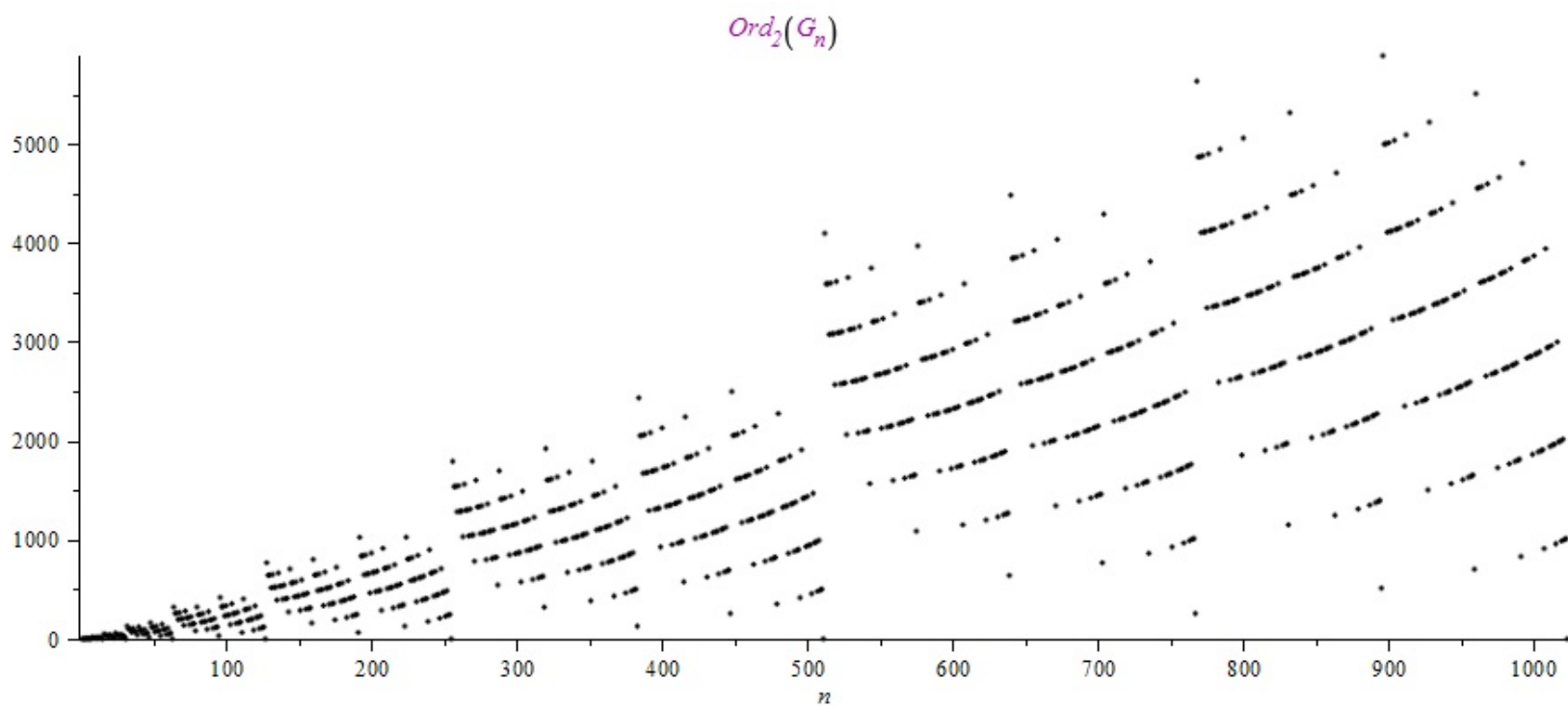
- **Theorem** (“Unreduced Farey” Riemann hypothesis)
The reciprocal unreduced Farey products \overline{G}_n satisfy

$$\log(\overline{G}_n) = \Phi^*(n) - \frac{1}{2}n \log n + \left(\frac{1}{2} - \frac{1}{2} \log(2\pi)\right) n + O(\log n).$$

Here $\frac{1}{2} - \frac{1}{2} \log(2\pi) \approx -0.41894$.

- This is “Unreduced Farey” analogy with Mikoläs formula, where RH says error term $O(n^{1/2+\epsilon})$. In fact: $O(\log n)$.
- This error term $O(\log n)$ says: there are no “zeros” in the critical strip all the way to $Re(s) = 0$! (of some function)

Prime $p = 2$



Binomial Products-Prime Factorization Patterns

- Graph of $g_2(n)$ shows the function is increasing on average. It exhibits a regular series of stripes.
- Stripe patterns are grouped by powers of 2: Self-similar behavior?
- Function $g_2(n)$ must be highly oscillatory, needed to produce the stripes. Fractal behavior?
- Harder to see: The number of stripes increases by 1 at each power of 2.

Binomial Products-3

- We obtained an explicit formula for $\text{ord}_p(\overline{G}_n)$ in terms of the base p radix expansion of n . This formula started from Kummer's formula giving the power of p that divides the binomial coefficient.
- **Theorem (Kummer (1852))** *Given a prime p , the exact divisibility p^e of $\binom{n}{t}$ by a power of p is found by writing t , $n - t$ and n in base p arithmetic. Then e is the number of carries that occur when adding $n - t$ to t in base p arithmetic, using digits $\{0, 1, 2, \dots, p - 1\}$, working from the least significant digit upward.*

Binomial Products-4

- **Theorem** (L.- Mehta 2015)

$$\text{ord}_p(\overline{G}_n) = \frac{1}{p-1} \left(2S_p(n) - (n-1)d_p(n) \right).$$

where $d_p(n)$ is the sum of the base p digits of n , and $S_p(n)$ is the running sum of the base p digits of the first $n-1$ integers.

- One can now apply a result of [Delange](#) (1975):

$$S_p(n) = \left(\frac{p-1}{2} \right) n \log_p n + F_p(\log_p n) n, \quad (1)$$

in which $F_p(x)$ is a continuous real-valued function which is periodic of period 1. The function $F_p(x)$ is continuous but everywhere non-differentiable. Its Fourier expansion is given in terms of the Riemann zeta function on the line $\text{Re}(s) = 0$.

Binomial Products-5

- Further work: One can study Farey products $\text{ord}_p(\overline{F}_n)$ using $\text{ord}_p(\overline{G}_n)$ using [Möbius inversion](#): We have

$$\overline{G}_n = \prod_{k=1}^n \overline{F}_{\lfloor n/k \rfloor},$$

which implies

$$\overline{F}_n = \prod_{k=1}^n (\overline{G}_{\lfloor n/k \rfloor})^{\mu(k)}.$$

- By combining this identity with ideas from the [Dirichlet hyperbola method](#), we obtained some striking experimental empirical results, possibly relating $\text{ord}_p(\overline{F}_n)$ for a single prime p (e.g. $p = 2$) to the Riemann hypothesis.

The Last Slide...

Thank you for your attention!

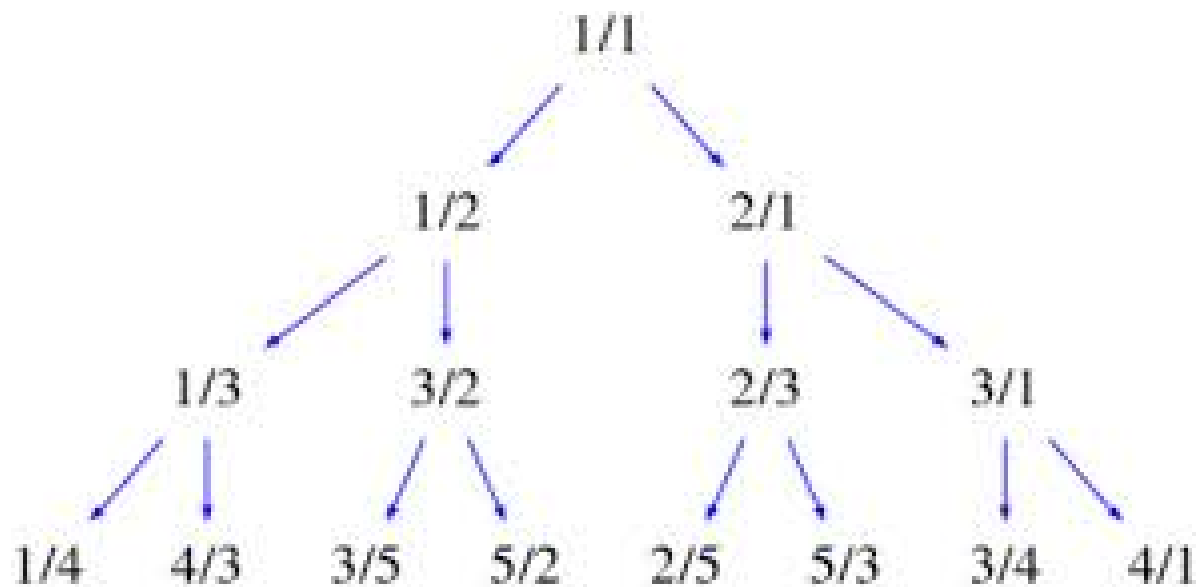
Afterword: Credits and References

- H. Mehta and J. C. Lagarias ,
Products of binomial coefficients and unreduced Farey fractions, Intl. J. Number Theory, to appear 2016
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Calkin-Wilf Tree: All Positive Rationals



Xenophanes (c. 570- c. 478 BCE)

Xenophanes was a pre-Socratic philosopher.

- “The substance of God is spherical, in no way resembling man. He is all eye and all ear, but does not breathe; he is the totality of mind and thought, and is eternal.”

- Empedocles said: *“It is impossible to find a wise man.”*

Xenophanes replied: *“Naturally, for it takes a wise man to recognize a wise man!”*

- Source: Diogenes Laertius, *Lives of the Philosophers*, Book IX, 18–20. [Second century ACE]