Polyharmonic Maass Forms for $PSL(2, \mathbb{Z})$

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- **Special Session on Modular Forms and Modular Integrals in Memory of Marvin Knopp**, (Temple University, Oct. 12, 2013)

- This talk describes work in progress.

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Topics Covered

- Part I. Marvin Knopp and Modular Forms
- Part II. Polyharmonic Functions
- Part III. Polyharmonic Maass Forms
- Part IV. Concluding Remarks

Part I. Marvin Knopp and Modular Forms

I worked at the National Bureau of Standards, summer 1967 as an intern under the direction of Morris Newman. I was exposed to number theory and I could have learned some classical modular forms... but didn't.

A neighboring office was assigned to Joseph Lehner, who seemed never there the entire time–I didn't see him but saw his book on *Discontinuous Groups and Automorphic Functions.* It was lying on a table (Now I think it is a lovely book.)

Marvin Knopp wrote five papers with Morris Newman, four of them in 1965, and 9 papers with Joseph Lehner, so he could have been around. Maybe I even met him at NBS (now NIST). Regardless I feel I knew him a long time and was accepted in the (classical) modular forms "family".

Marvin Knopp-2

At a Temple University conference in 1980, he showed hospitality, was a guide and mentor, and gave encouragement. I remember Marvin's warmth and kindness, and love of music.

Attended a number theory conference at Temple University, March 1996, in honor of D. J. Newman. This conference had Louis de Branges speaking, the only time I met him. Now he is known as: Louis de Branges de Bourcia.

Marvin Knopp-3

Grosswald lectures in 2002: Marvin and put me up in his personal apartment. Marvin told me how Emil Grosswald (1912–1989) wrote up lectures of Hans Rademacher (1892–1969) on Dedekind sums, after Rademacher died. How Emil went only to graduate school when he was in his late 30's and how much he loved mathematics. (He had published three papers already, written under the name E. G. Garnea.)

I have recently looked at Marvin's two books on Modular Forms and they supplied inspiration.

Modular Forms

- Marvin would be very pleased at what has happened to the field of modular forms in recent times.

- It has moved a long way from 1940:

  G. H. Hardy: "We may seem to be straying into one of the *backwaters of mathematics*, [...but the genesis of $\tau(n)$ as a coefficient in so fundamental a function compels us to treat it with respect.]"
  [Ref: G. H. Hardy, *Ramanujan. Twelve lectures on subjects suggested by his life and work*, Chapter X]

Proliferation of Modular Forms and Functions

- Marvin Knopp told me that someone (Lehner?) said:

  "Anyone who goes into the jungle of modular forms is bound to come back with something."

- Mock theta functions, partial theta functions, quasi theta functions, false theta functions,

- modular forms with poles at cusps,weakly holomorphic modular forms, Maass forms, quantum modular forms,...

- Eichler integrals, Eichler cohomology, rational period functions

Part II. Polyharmonic Functions

- A *polyharmonic function* $f(z)$ (on a domain) is a function annihilated by a power of the Laplacian operator. It is k*-harmonic* if

$$\Delta^k f(z) = 0.$$

- More generally consider *shifted polyharmonic functions*

$$(\Delta - \lambda)^k f(z) = 0$$

for $z = x + iy \in \mathbb{H}$ for the (weight 0) hyperbolic Laplacian

$$\Delta = y^2 \Big((\frac{\partial}{\partial x})^2 + (\frac{\partial}{\partial y})^2 \Big)$$

and eigenvalue shift parameter $\lambda = s(s-1)$.

Polyharmonic Functions-History

- Miron Nicolescu (1903–1975) studied *d-bar* operator $\frac{\partial}{\partial \bar{z}}$ (1927–1930), then studied *polyharmonic functions* for the Euclidean Laplacian (1930–1970), then *polycaloric functions* for the Euclidean Laplacian, those annihilated by $\frac{\partial}{\partial t} - \Delta^k$ (1932–1966). Book: Nicolescu, *Les Fonctions polyharmoniques,* Hermann: Paris 1936.

- Nachman Aronszajn (1907–1980), known for inventing *reproducing kernel Hilbert spaces*, studied polyharmonic functions of infinite order for the Euclidean Laplacian on $\mathbb{R}^n$ in 1930's and later. Book: Aronszajn, Creese, Lipkin, *Polyharmonic Functions,* Oxford U. Press 1981.

Polyharmonic Functions-Properties

- Polyharmonic functions on $\mathbb{R}^n$: " Behave like harmonic functions in some ways, and like entire functions of finite order in other ways."

- Studies: Nature of singularities; Harnack bounds, general Almansi series expansions

- 2-harmonic functions (biharmonic functions) occur in continuum mechanics. (Almansi strain tensor)

Part III. Polyharmonic Maass forms-1

- *Objective:* To study the family of weight $2k$ polyharmonic Maass forms (with eigenvalue λ on the full modular group $PSL(2, \mathbb{Z})$, which have moderate growth at the cusp.

- *Approach:* Treat the weight 0 case, and get results for the other weights by analyzing the effect of weight raising and lowering differential operators. Treat $\lambda = 0$ and then general λ.

- *Expectation.* Spaces of such forms are finite-dimensional.

Weight 0 Polyharmonic Maass forms-1

The weight 0 non-holomorphic Eisenstein series is

$$E_0^*(z,s) = \frac{1}{2}s(1-s)\pi^{-s}\Gamma(s)\left(\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{y^s}{|mz+n|^{2s}}\right).$$

It satisfies $E_0^*(z,s) = E^*(z,1-s)$.

Fact. *The mth Taylor coefficient $F_m(z)$ of the nonholomorphic Eisenstein series $E_0(z,s)$ at $s=0$ is $(m+1)$-harmonic.*

$$E_0^*(z,s) =: \sum_{n=0}^{\infty} F_n(z)s^n.$$

Weight 0 Polyharmonic Maass forms-2

Theorem 1.

(1) The vector space V_0^m of m-harmonic modular forms of weight 0 for $\mathrm{SL}_2(\mathbb{Z})$ with at most polynomial growth toward the cusp, is exactly m-dimensional.

(2) The set of Taylor coefficients $\{F_0(z), \cdots, F_{m-1}(z)\}$ of $E_0^*(z,s)$ at $s = 0$ is a basis for V_0^m.

Weight 0 Polyharmonic Maass forms-3

$$\begin{array}{c} \vdots \\ \big\downarrow \Delta_0 \end{array}$$

$$\widetilde{F_1}(z) = -\tfrac{1}{2}\gamma + \log(4\pi) + \log\left(\sqrt{y}|\Delta(z)|^{\frac{1}{12}}\right) \qquad \text{[Kronecker limit formula]}$$

$$\big\downarrow \Delta_0$$

$$\widetilde{F_0}(z) = \tfrac{1}{2}$$

$$\big\downarrow \Delta_0$$

$$0$$

where

$$\widetilde{F}_n(z) := \sum_{j=0}^{n} \binom{2n-j}{n-j} F_j(z)$$

Factoring the Laplacian

It is "well known" that the non-Euclidean Laplacian factors into first order operators

$$\Delta = \xi_2 \xi_0,$$

where

$$\xi_k = 2iy^k \overline{\frac{\partial}{\partial \overline{z}}}.$$

This factorization was effectively used by Bruinier-Funke and others.

Moreover, define the weight 2 hyperbolic Laplacian by

$$\Delta_2 := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 4iy \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) = \xi_0 \xi_2$$

Weight 2 Polyharmonic Maass forms-1

The weight 2 (nonholomorphic) Eisenstein series is

$$E_2^*(z,s) := -s(s+1)\pi^{-s}\Gamma(s)\frac{1}{2}\zeta(2s+2)\left(\sum_{\substack{(c,d)\\(c,d)=1}} \frac{y^s}{(cz+d)^2|cz+d|^{2s}}\right)$$

Fact *The m-th Taylor coefficient $G_m(z)$ at $s=0$ of $E_2^*(z,s)$ is m-harmonic for Δ_2 . Here*

$$E_2^*(z,s) =: \sum_{n=0}^{\infty} G_n(z)s^n.$$

and $G_0(z) \equiv 0$.

Weight 2 Polyharmonic Maass forms-2

Theorem.

(1) The vector space, V_2^m, of weight 2 m-harmonic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ with at most polynomial growth toward the cusp, is m-dimensional.

(2) Moreover, $G_0(z) \equiv 0$ and $\{G_1(z), G_2(z), \cdots, G_m(z)\}$ is a basis for V_2^m.

A Polyharmonic Ladder

$$
\begin{tikzcd}
\vdots & \vdots \arrow[dl, "\xi_2"'] \arrow[dd, "\Delta_2"] \\
\widetilde{F_2} \arrow[dr, "\xi_0"] \arrow[dd, "\Delta_0"] & \\
 & \widetilde{G_1} \arrow[dl, "\xi_2"'] \arrow[dd, "\Delta_2"] \\
\widetilde{F_1} \arrow[dr, "\xi_0"] \arrow[dd, "\Delta_0"] & \\
 & \widetilde{G_0} \arrow[dl, "\xi_2"'] \arrow[dd, "\Delta_2"] \\
\widetilde{F_0} \arrow[dr, "\xi_0"] & \\
 & 0
\end{tikzcd}
$$

A Polyharmonic Ladder-2

In this ladder

$$\widetilde{G}_1(z) = -\frac{\pi}{6}\left(1 - 24\sum_{n=1}^{\infty}\left(\sum_{d|n} d\right) e^{2\pi i n z} + \frac{3}{\pi y}\right)$$

and the "holomorphic part"

$$1 - 24\sum_{n=1}^{\infty}\left(\sum_{d|n} d\right) e^{2\pi i n z}$$

is a quasimodular form.

Ladder versus Tower

- The ladder structure is special to eigenvalue $\lambda = s(s-1) = 0$, where the Laplacian factorizes. In the general λ case there is only a "tower" in d/ds.

- In the case of general λ, the Taylor series coefficients give shifted polyharmonic functions. But now there are some extra functions for special λ, e.g. Maass cusp forms.

Part IV. Conclusion

Observation:

We are given a family of eigenfunctions $E(z,s)$ of Laplacian depending on eigenvalue parameter $\lambda = s(s-1)$. Can "switch" from Taylor coefficients in d/ds to polyharmonic functions in other variables z because the family $E(z,s)$ has "curvature" in the s-variable with respect to the Laplacian (even though d/ds and Δ commute as operators.

Which polyharmonic forms do these constructions account for?

Future work includes:

1. other weights
2. other eigenvalues
3. Fourier expansions
4. arithmetic interpretations/consequences

Thank you for your attention!