Ternary Expansions of Powers of 2

Jeff Lagarias,
University of Michigan

Workshop on Discovery and Experimentation
in Number Theory
Fields Institute, Toronto
(September 25, 2009)

Topics Covered

- Part I. Erdős Problem on ternary expansions of powers of 2

- Part II. Real number generalization and a 3-Adic generalization

- Part III. Intersections of translates of 3-adic Cantor sets

Credits

- Part II reports: J. C. Lagarias, Ternary Expansions of Powers of 2, J. London Math. Soc. **79** (2009), 562–588.
- Part III reports: ongoing work with REU student Will Abram (Univ. of Chicago).
- Work supported by NSF grants DMS-0500555 and DMS-0801029. REU work by W. Abram supported by the National Science Foundation.

Part I. Erdős Ternary Digit Problem

- Problem. Let $(M)_3$ denote the integer M written in ternary (base 3). How many powers 2^n of 2 omit the digit 2 in their ternary expansion?

Examples	Non-examples
$(2^0)_3 = 1$	$(2^3)_3 = 22$
$(2^2)_3 = 11$	$(2^4)_3 = 121$
$(2^8)_3 = 100111$	$(2^6)_3 = 2101$

- Conjecture. (Erdős 1979) There are no solutions for $n \geq 9$.

Paul Erdős

Heuristic for Erdős Ternary Problem

- The ternary expansion $(2^n)_3$ has about

$$\alpha_0 n \quad \text{digits}$$

where

$$\alpha_0 := \log_3 2 = \frac{\log 2}{\log 3} \approx 0.63091$$

- Heuristic. If ternary digits were picked randomly and independently from $\{0,1,2\}$, then the probability of avoiding the digit 2 would be $\approx \left(\frac{2}{3}\right)^{\alpha_0 n}$.

- These probabilities decrease exponentially in n, so their sum converges. Thus expect only finitely many n to have expansion $[2^n]_3$ that avoids the digit 2.

Original Erdős (et al.) Problem

- Problem When is the binomial coefficient $\binom{2n}{n}$ squarefree?

- Known squarefree solutions: $\binom{2}{1} = 2$

$$\binom{4}{2} = 6$$

$$\binom{8}{4} = 70$$

- Conjecture (Erdős, Graham, Rusza and Straus (1975))
  There are no squarefree solutions for $n \geq 5$.

Original Erdős Problem-2

- Lucas's theorem (1878) gives a criterion for a prime p to divide a binomial coefficient $\binom{k}{l}$ in terms of the digits in the base p expansion of k and l.

- Lucas's theorem shows the prime 2 always divides $\binom{2n}{n}$, for $n \geq 1$.

- Question: When does $2^2 = 4$ NOT divide $\binom{2n}{n}$?

- Answer: This happens only when $n = 2^k$ for some $k \geq 0$.

Original Erdős et al Problem-3

- Erdős then asked: What happens for the prime 3?

- Answer: Lucas's theorem shows 3 does not divide $\binom{2^{k+1}}{2^k}$ if and only if the base 3 expansion of 2^k omits the digit 2.

- This observation motivated Erdős's 1979 ternary digit conjecture.

Original Erdős et al Problem-4

- One needs more than the ternary digit conjecture to settle squarefree binomial coefficient problem. One needs a criterion for $3^2 = 9$ to divide $\binom{2^{k+1}}{2^k}$!

- Sufficient condition for 3^2 to divide $\binom{2n}{n}$: at least two $2's$ in the ternary number $(2^n)_3$.

- Thus: should determine all powers $(2^n)_3$ with: at most one 2 in their ternary expansion.

Original Erdős et al Problem-5

- Don't bother! The squarefree binomial coefficient conjecture is completely solved!

- This was shown for all sufficiently large n by Sarkozy (1985). Later shown for all $n \geq 5$, independently, by Velammal (1995) and Granville and Ramaré (1996).

- However: Erdős ternary expansion conjecture is unsolved!

- Assertion: Ternary expansion conjecture appears very hard!

Narkiewicz's Result

- Definition. The Erdős intersection set is

$$N(1) \ := \ \{n \geq 1 : \text{ternary expansion } (2^n)_3 \\ \text{omits the digit } 2\}$$

- Theorem (Narkiewicz (1980)) (Count Bound) The set of integers in the Erdős intersection set $N(1)$ satisfies

$$\#(\{n \leq x : \ n \in N(1)\}) \leq 1.62 \ x^{\alpha_0}$$

where $\alpha_0 = \log_3 2 \sim 0.63092$

- This result does not exclude the set $N(1)$ being infinite, but shows there are not too many integers in it.

Wladyslaw Narkiewicz

Part II. Dynamical System Generalizations of Erdős Ternary Digit Problem

- Approach: View the set $\{1, 2, 4, ...\}$ as a forward orbit of the discrete dynamical system $T : x \mapsto 2x$.

- The forward orbit $\mathcal{O}(x_0)$ of x_0 is

$$\mathcal{O}(x_0) := \{x_0, T(x_0), T^{(2)}(x_0) = T(T(x_0), \cdots\}$$

Thus: $\mathcal{O}(1) = \{1, 2, 4, 8, \cdots\}$.

- New Problem. Study the forward orbit $\mathcal{O}(\lambda)$ of an arbitrary initial starting value λ. How big can its intersection be, with the "Cantor set"?

General Framework-2

- There are two different places where the dynamical system can live:

- Model 1. Dynamical system lives on positive real numbers $\mathbb{R}^+$.

- Model 2. Dynamical system lives on the 3-adic integers $\mathbb{Z}_3$.

General Framework-3

- Key Fact: (i) The ternary expansion of 2^n is identical to the 3-adic expansion of 2^n.
  (However the dynamical system $x \mapsto 2x$ acts differently in the two models.)

- Key Fact: (ii) The Cantor set makes sense in both models! It also has a dynamical systems interpretation.

  It has the same size: Hausdorff dimension

$$\alpha_0 = \log_3 2 = \frac{\log 2}{\log 3} \approx 0.63092.$$

Real Number Dynamical System-1

- Regard $\{1, 2, 4, 8, ...\}$ as a subset of the positive real numbers.

- The (usual) ternary Cantor set Σ_3 is the set of all real numbers whose ternary expansion has digits 0 and 2 (omits 1)

- The (modified)ternary Cantor set $\Sigma_{3,\bar{2}}$ is the set of all positive real numbers whose ternary expansion omits 2. It satisfies

$$\Sigma_{3,\bar{2}} = \frac{1}{2}\Sigma_3.$$

Real Number Dynamical System-2

- If $\lambda 2^n$ belongs to the Cantor set Σ_3 , then $\lambda 2^{n-1}$ belongs to the modified Cantor set $\Sigma_{3,\bar{2}}$, and vice versa.

- From now on: We consider: intersections of orbits with $\Sigma_{3,\bar{2}}$ (i.e., ternary expansions that omit the digit 2).

Real Number Dynamical System-3

- The real intersection set for $\lambda \in \mathbb{R}$ is:

$$N(\lambda; \mathbb{R}) := \{n \geq 1 : \quad ([\lambda 2^n])_3 \ \text{omits the digit 2}\}$$

  Here: $[x]$ is "greatest integer function."

- $N(1; \mathbb{R}) = N(1)$ is the Erdős intersection set.

- The real truncated exceptional set is

$$\mathcal{E}_t(\mathbb{R}) := \{\lambda > 0 : \quad \text{real intersection set } N(\lambda, \mathbb{R}) \text{ is infinite.}\}$$

Real Number Model: Intersection set Size-1

- Theorem. (Real Model Count Bound) For all $\lambda > 0$ the real intersection set $N(\lambda; \mathbb{R})$ satisfies, for all sufficiently large x,

$$\#(\{n \le x : \ n \in N(\lambda; \mathbb{R})\}) \le 25 \ x^{\alpha_0}$$

where $\alpha_0 = \log_3 2 \sim 0.63092$

- The result is the same strength as that of Narkiewicz, but applies to all initial values.

Real Number Model: Intersection set Size-2

- Remarks on proof: Study the $O(\log x)$ highest order ternary digits of $([\lambda 2^n])_3$. Knock out all those that contain a 2.

- Set $f(n) := \frac{\log(\lambda 2^n)}{\log 3} = n\alpha_0 + \log_3 \lambda$.

- Study $f(n)$ (modulo 1), show it is close to uniformly distributed. If so: it spends most of its time in subintervals whose ternary expansion has a 2 in first $\log x$ digits.

Real Number Model: Intersection set Size-3

- To establish uniform distribution:

- Use Diophantine approximation estimates to the number $\alpha_0 = \log_3 2$. Linear forms in logarithms estimates, (due to G. Rhin) show that

$$|\alpha_0 - \frac{p}{q}| \geq \frac{c}{q^{13.3}}$$

with $c = 0.0001$, for all $q \geq 1$.

Georges Rhin

Real Number Model: Hausdorff Dimension

- Theorem. (Truncated Exceptional Set Dimension)
  The Hausdorff dimension of the (truncated) exceptional set $\mathcal{E}_t(\mathbb{R})$ is exactly $\alpha_0 = \log_3 2 \approx 0.63092$.

- Corollary: There exist $\lambda \in \mathbb{R}$ where infinitely many of $([\lambda 2^n])_3$ omit the digit 2.

- Remark: The infinite sets $N(\lambda; \mathbb{R})$ so constructed are extremely sparse, with counting function growing like $\log^* x$!

  ($\log^* x$ counts the number of iterations of taking logarithm to get x smaller than 1.)

Hausdorff Dimension-1

- Defn. Let $X \subset \mathbb{R}^n$. The s-dimensonal Hausdorff content of X is:

$$Vol_s(S) := \liminf_{\delta \to 0} \{ \sum_i (r_i)^s \}$$

where the infimum runs over all coverings of X with a collection of balls having radii $r_i > 0$, and with all$r_i \leq \delta$.

- Defn. The Hausdorff dimension of X is

$$dim_H(X) := \inf\{s \geq 0 : Vol_s(X) = 0\},$$

equivalently,

$$dim_H(X) := \sup\{s \geq 0 : Vol_s(X) = +\infty\}.$$

Hausdorff Dimension-2

- The definition makes sense on any metric space.

- In the critical dimension, the Hausdorff measure $Vol_s(X)$ can be 0, finite, or $+\infty$.

- Example. The Cantor set Σ_3 (inside $[0,1]$) has Hausdorff dimension $\log_3 2 = \frac{\log 2}{\log 3} \approx 0.63092$. It has positive finite Hausdorff measure.

Hausdorff Dimension-3

- Getting an Upper Bound. Find a good family of coverings. For example, one can cover Σ_3 (in $[0,1]$) with 2^k intervals of length $\frac{1}{3^k}$ each. using all ternary expansions of length k with digits 0 and 2.

  Taking $s = (\log_3 2 + \epsilon)$, this covering has content, as $k \to \infty$,

  $$\sum_i (r_i)^{\log_3 2+\epsilon} = 2^k (3^{-k})^{\log_3 2+\epsilon} = 3^{-\epsilon k} \longrightarrow 0.$$

  thus $dim_H(\Sigma_3) \leq \log_3 2$.

- Getting a Lower Bound. Usually harder to show; must consider all coverings!

Hausdorff Dimension Theorem: Proof Idea

- (Upper Bound) By construction. One actually finds a large Hausdorff dimension set with a fixed infinite set $r_1 < r_2 < r_3 < ...$ with all $(\lfloor \lambda 2^{r_k} \rfloor)_3$ omitting digit 2.

- (Lower Bound) Uses a fill-in-levels argument, modifying the covering to a standard form.

3-adic Integer Dynamical System-1

- View the integers $\mathbb{Z}$ as contained in the set of 3-adic integers $\mathbb{Z}_3$. The quotient field of the 3-adic integers is the *3-adic numbers* $\mathbb{Q}_3$

- The 3-adic integers $\mathbb{Z}_3$ are the set of all formal expansions

$$\beta = d_0 + d_1 \cdot 3 + d_2 \cdot 3^2 + \ldots$$

  where $d_i \in \{0, 1, 2\}$. Call this the 3-adic expansion of β.

- Set $ord_3(0) := +\infty$ and $ord_3(\beta) := \min\{j : d_j \neq 0\}$.

  The 3-adic size of $\beta \in \mathbb{Q}_3$ is:

$$||\beta||_3 = 3^{-ord_3(\beta)}$$

3-adic Integer Dynamical System-2

- Now view $\{1, 2, 4, 8, ...\}$ as a subset of the 3-adic integers.

- The (usual) 3-adic Cantor set $\tilde{\Sigma}$ is the set of all 3-adic integers whose 3-adic expansion omits the digit 1.

- The modified 3-adic Cantor set $\tilde{\Sigma}_{3,\bar{2}}$ is the set of all 3-adic integers whose 3-adic expansion omits the digit 2.

- The Hausdorff dimension of $\tilde{\Sigma}_{3,\bar{2}}$ is $\log_3 2$.

3-adic Integers versus Real Numbers-1

- The map $j : \mathbb{Z}_3 \to [0,1] \subset \mathbb{R}$ that maps a 3-adic integer to the real number whose ternary digit expansion matches the 3-adic expansion, has the properties:

- (1) This map is continuous, and almost invertible: every number has one preimage except dyadic rationals, which have two preimages.

- (2) It is a Lipschitz map

$$|j(x) - j(y)| \leq 3||x - y||_3.$$

3-adic Integers versus Real Numbers-2

- The map $j: \mathbb{Z}_3 \to [0,1]$ preserves Hausdorff dimension.

- The 3-adic Cantor set maps under j to the real Cantor sets in $[0,1]$.

General Framework: 3-adic Model-1

- A general 3-adic number $\alpha \in \mathbb{Q}_p$ has "Laurent expansion":

$$\alpha = b_{-j}\frac{1}{3^j} + \cdots + b_{-1} \cdot \frac{1}{3} + b_0 + b_1 \cdot 3 + \cdots .$$

- The polar part of the number α is:

$$PP(\alpha) := b_{-j}3^{-j} + \cdots + b_{-1} \cdot 3^{-1}.$$

General Framework: 3-adic Model-2

- The 3-adic (truncated) intersection set for $\lambda \in \mathbb{Z}_3$ is:

$$\begin{aligned} N(\lambda; \mathbb{Z}_3) \;:=\; & \{n \geq 1: \text{ The polar part} \\ & PP(\lambda 2^n / 3^{\lfloor \alpha_0 n \rfloor}) \text{ omits the digit 2}\} \end{aligned}$$

  Again $N(1; \mathbb{Z}_3)$ recovers the Erdős intersection set.

- The 3-adic truncated exceptional set is

$$\mathcal{E}_t(\mathbb{Z}_3) := \{\lambda > 0 : \;\; \text{intersection set } N(\lambda; \mathbb{Z}_3) \text{ is infinite.}\}$$

3-adic model: Intersection set size

- Theorem. (3-adic Model Count Bound) For all nonzero 3-adic integers λ the general intersection set $N(\lambda; \mathbb{Z}_3)$ satisfies, for all sufficiently large x,

$$\#(\{n \le x : \; n \in N(\lambda; \mathbb{Z}_3)\}) \le 2.5 \; x^{\alpha_0}$$

  where $\alpha_0 = \log_3 2 \sim 0.63092$

- Narkiewicz's theorem had a 3-adic proof. His proof extends to all initial values.

Punchline-1

- Both the real number model and the 3-adic model give restrictions on the set of integers in the Erdős intersection set $N(1)$.

- The models give restrictions of roughly equal strength on $N(1)$, cutting the number of possible integers down to $O(x^{\alpha_0})$.

- The real number information on $N(1;\mathbb{R})$ excludes $2's$ in the top $O(\log n)$ ternary digits of $(2^n)_3$. The 3-adic information on $N(1;\mathbb{Z}_3)$ excludes $2's$ in the bottom $O(\log n)$ 3-adic digits of $(2^n)_3$.

Punchline-2

- Heuristic: The top $O(\log n)$ ternary digits ought to be "independent" of the bottom $O(\log n)$ ternary digits!

- Thus: the information in the two models ought to non-trivially combine to give a better result. But we observe...

Punchline-3

- Observation: No one knows how to combine the information in the two methods to do better than either one separately!

- Observation: No one knows how to estimate the number of $2's$ in the $\alpha n - O(\log n)$ middle ternary digits in $(2^n)_3$!

- I bring these puzzling observations to your attention!

Part III. Complete 3-adic Exceptional Set

- We revisit the problem, imposing a stronger condition: avoid the digit 2 on an infinite set of digits.

- Define the complete (i.e. non-truncated) intersection set

$$N^*(\lambda; \mathbb{Z}_3) := \{n \geq 1 : \text{ the complete 3-adic expansion } (\lambda 2^n)_3 \text{ omits the digit 2}\}$$

Complete 3-adic Exceptional Set-2

- The 3-adic complete exceptional set is

$$\mathcal{E}^*(\mathbb{Z}_3) \;:=\; \{\lambda > 0 : \text{ the complete intersection set } N^*(\lambda; \mathbb{Z}_3) \text{ is infinite.}\}$$

- The set $\mathcal{E}^*(\mathbb{Z}_3)$ ought to be "much smaller" than the truncated exceptional set $\mathcal{E}_t(\mathbb{Z}_3)$. Concievably it is just one point $\{0\}$. If it is larger, then it must be infinite!

Complete Exceptional Set Conjecture

- Complete Exceptional Set Conjecture.
  The 3-adic complete exceptional set $\mathcal{E}^*(\mathbb{Z}_3)$ has Hausdorff dimension 0.

- A similar conjecture can be made for the real complete exceptional set, $\mathcal{E}^*(\mathbb{R})$, defined analogously.

- The 3-adic version of the conjecture is approachable, due to nice symbolic dynamics!

Some subproblems

- The Level k exceptional set $\mathcal{E}_k^*(\mathbb{Z}_3)$ has those λ that have at least k distinct powers of 2 with $\lambda 2^k$ in the Cantor set, i.e.

$$\mathcal{E}_k^*(\mathbb{Z}_3) := \{\lambda > 0 : \text{ the set } N^*(\lambda; \mathbb{Z}_3) \geq k.\}$$

- Level k exceptional sets are nested by increasing k:

$$\mathcal{E}^*(\mathbb{Z}_3) \subset \cdots \subset \mathcal{E}_3^*(\mathbb{Z}_3) \subset \mathcal{E}_2^*(\mathbb{Z}_3) \subset \mathcal{E}_1^*(\mathbb{Z}_3)$$

- Goal: Study the Hausdorff dimension of $\mathcal{E}_k^*(\mathbb{Z}_3)$; it gives an upper bound on $dim_H(\mathcal{E}^*(\mathbb{Z}_3))$.

Upper Bounds on Hausdorff Dimension

- Theorem. (Upper Bound Theorem)

$$(1).\qquad dim_H(\mathcal{E}_1^*(\mathbb{Z}_3)) = \alpha_0 \approx 0.63092.$$

$$(2).\qquad dim_H(\mathcal{E}_2^*(\mathbb{Z}_3)) \leq 0.5.$$

- Remark. There is a lower bound:

$$dim_H(\mathcal{E}_2^*(\mathbb{Z}_3)) \geq \log_3(\frac{1+\sqrt{5}}{2}) \approx 0.438$$

Upper Bounds on Hausdorff Dimension

- Question. Could it be true that

$$\lim_{k\to\infty} dim_H(\mathcal{E}_k^*(\mathbb{Z}_3)) = 0?$$

- If so, this would imply that the complete exceptional set $\mathcal{E}^*(\mathbb{Z}_3)$ has Hausdorff dimension 0.

Upper Bound Theorem: Proof Idea

- The set $\mathcal{E}_k^*(\mathbb{Z}_3)$ is a countable union of closed sets

  $$\mathcal{E}_k^*(\mathbb{Z}_3) = \bigcup_{r_1 < r_2 < \ldots < r_k} \mathcal{C}(2^{r_1}, 2^{r_2}, \ldots, 2^{r_k}),$$

  given by

  $$\mathcal{C}(2^{r_1}, 2^{r_2}, \ldots, 2^{r_k}) := \{\lambda : \ (2^{r_i}\lambda)_3 \ \text{ omits digit } 2\}.$$

- We have

  $$dim_H(\mathcal{E}_k^*(\mathbb{Z}_3)) = \sup\{dim_H(\mathcal{C}(2^{r_1}, 2^{r_2}, \ldots, 2^{r_k}))\}$$

- Proof for $k = 1, 2$: obtain upper bounds on Hausdorff dimension of all the sets $\mathcal{C}(2^{r_1}, 2^{r_2}, \ldots, 2^{r_k})$.

Discovery and Experimentation-1

- New Problem. For positive integers $r_1 < r_2 < \cdots < r_k$ set

$$\mathcal{C}(2^{r_1}, 2^{r_2}, ..., 2^{r_k}) := \{\lambda : (2^{r_i}\lambda)_3 \text{ omits the digit } 2\}$$

Determine the Hausdorff dimension of $\mathcal{C}(2^{r_1}, 2^{r_2}, ..., 2^{r_k})$.

- More generally, allow arbitrary positive integers $N_1, N_2, ..., N_k$. Determine the Hausdorff dimension of:

$$\mathcal{C}(N_1, N_2, \cdots, N_k) := \{\lambda : \text{all } (N_i\lambda)_3 \text{ omit the digit } 2\}$$

Discovery and Experimentation-2

- The Hausdorff dimension of sets $\mathcal{C}(N_1, N_2, ..., N_k)$ can in principle be determined exactly!

- Mainly discuss special case $\mathcal{C}(1, N)$, for simplicity.

- This special case already has a complicated and intricate structure!

Basic Structure of the answer-1

- The 3-adic expansions of members of sets $\mathcal{C}(N_1, N_2, ..., N_k)$ are describable dynamically as having the symbolic dynamics of a sofic shift, given as the set of allowable infinite paths in a suitable labelled graph (finite automaton).

- The sequence of allowable paths is characterized by the topological entropy of the dynamical system. This is the growth rate ρ of the number of allowed label sequences of length n. It is the maximal (Perron-Frobenius) eigenvalue ρ of the weight matrix of the labelled graph, a non-negative integer matrix. (Adler-Konheim-McAndrew (1965))

Basic Structure of the answer-2

- The Hausdorff dimension of the associated "fractal set" $\mathcal{C}(N_1, ..., N_k)$ is given as the base 3 logarithm of the topological entropy of the dynamical system.

- This is $\log_3 \rho$ where ρ is the Perron-Frobenius eigenvalue of the symbol weight matrix of the labelled graph.

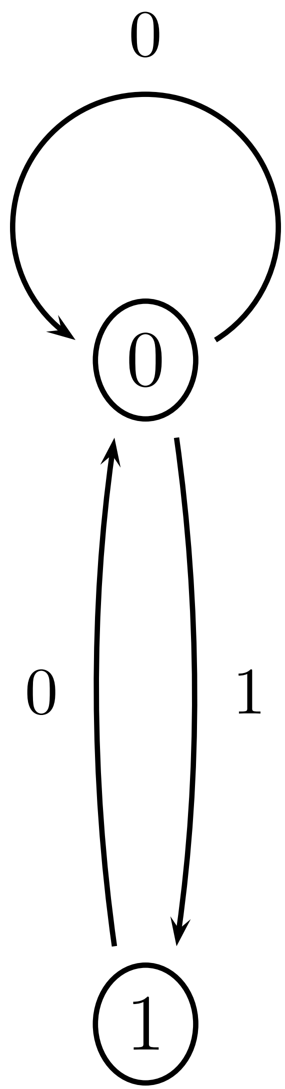
- Remark. These sets are "self-similar fractals" in sense of Hutchinson (1981), as extended in Mauldin-Williams (1985). It is given as a fixed point of a system of set-valued functional equations.

Basic Structure of the answer-3

- If some $N_j \equiv 2 \pmod 3$ occurs, then Hausdorff dimension $\mathcal{C}(N_1, N_2, ..., N_k)$ will be 0.

- If one replaces N_j with $3^k N_j$ then the Hausdorff dimension does not change.

- Can therefore reduce to case: All $N_j \equiv 1 \pmod 3$.



Graph: $N = 2^2 = 4$

Associated Matrix $N = 4$

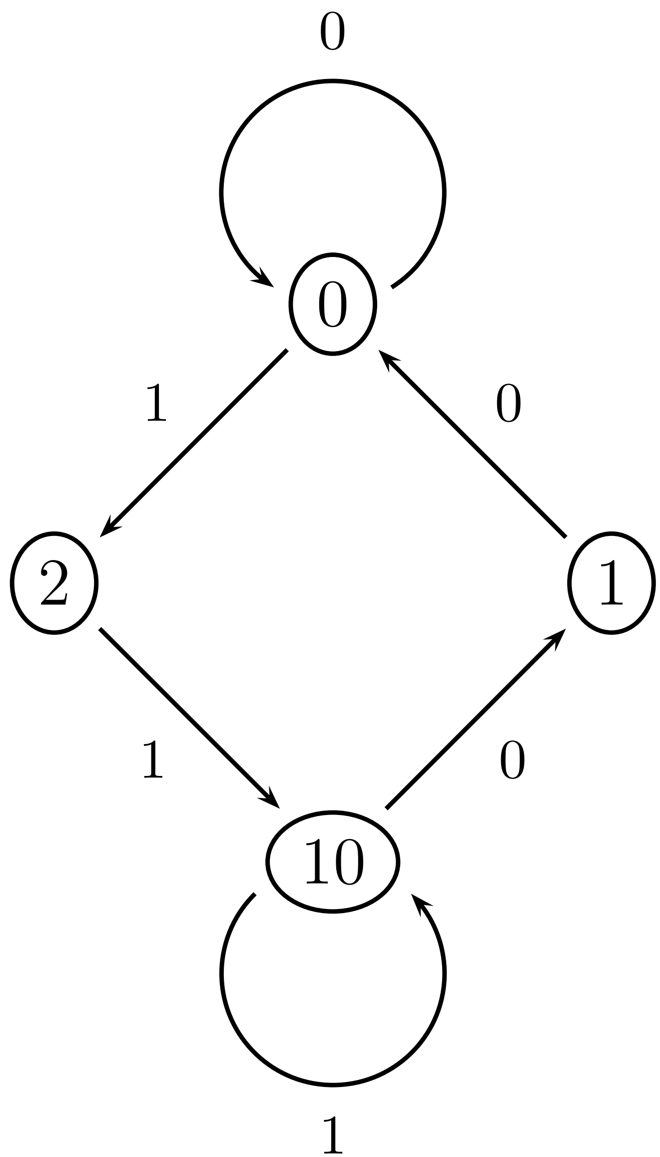
- Weight matrix is:

	state 0	state 1
state 0	[1	1]
state 1	[0	1]

- This is Fibonacci shift. Perron-Frobenius eigenvalue is:

$$\rho = \frac{1+\sqrt{5}}{2} = 1.6180...$$

- Hausdorff Dimension $= \log_3 \rho \approx 0.438$.



Graph: $N = 7 = (21)_3$

Associated Matrix $N = 7$

- Weight matrix is:

		state 0	state 2	state 10	state 1	
state 0	[	1	1	0	0	]
state 2	[	0	0	1	0	]
state 10	[	0	0	1	1	]
state 1	[	1	0	0	0	]

- Perron-Frobenius eigenvalue is : $\rho = \frac{1+\sqrt{5}}{2} = 1.6180...$

- Hausdorff Dimension $= \log_3 \rho \approx 0.438$.

Graphs for $N = (10^k 1)_3$

- Theorem. ("Fibonacci Graphs")
  For $N = (10^k 1)_3$, (i.e. $N = 3^{k+1} + 1$)

$$dim_H(\mathcal{C}(1, N)) := dim_H(\Sigma_{3,\bar{2}} \cap \frac{1}{N}\Sigma_{3,\bar{2}}) = \log_3(\frac{1+\sqrt{5}}{2}) \approx 0.438$$

- Remark. The finite graph associated to $N = 3^{k+1} + 1$ has 2^k states! The symbolic dynamics depend on k!

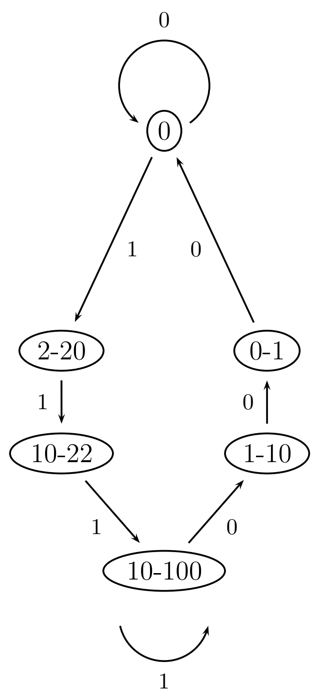
- The eigenvector for the maximal eigenvalue (Perron-Frobenius eigenvalue) of the adjacency matrix of this graph is explicitly describable. It has a self-similar structure, and has all entries in $\mathbb{Q}(\sqrt{5})$.

Graphs for $N = (20^k1)_3$

- Empirical Results. Take $N = 2 \cdot 3^{k+1} + 1 = (20^k1)_3$. For $1 \leq k \leq 4$, the graphs have exactly two strongly connected components.

- There is an outer component with about k states, whose Hausdorff dimension goes rapidly to 0 as k increases. (This is provable for all $k \geq 1$).

- There is also an strongly connected inner component, which appears to have exponentially many states, and whose Hausdorff dimension monotonically increases for small k, and eventually exceeds that of the outer component.



Graph: $N = 19 = (201)_3$

Graph for $N = 139 = (12011)_3$

- This value N=139 is a value of $N \equiv 1 \pmod 3$ where the associated set has Hausdorff dimension 0.

- The corresponding graph has 5 strongly connected components; each one separately has Perron-Frobenius eigenvalue 1, giving Hausdorff dimension 0!

General Graphs-Some Properties of $\mathcal{C}(1, N)$

- The states in the graph can be labelled with integers k satisfying $0 \le k \le \lfloor \frac{N}{6} \rfloor$ (if entering edge label is 0) and $\lfloor \frac{N}{3} \rfloor \le k \le \lfloor \frac{N}{2} \rfloor$ (if entering edge label is 1).

- The paths in the graph starting from given state k describe the symbolic dynamics of numbers in the intersection of shifted multiplicatively translated 3-adic Cantor sets

$$\mathcal{C}_k := \Sigma_{3,\bar{2}} \cap \frac{1}{N}\left(\Sigma_{3,\bar{2}} + k\right).$$

- The Hausdorff dimension of "shifted intersection set" is the maximal Hausdorff dimension of a strongly connected component of graph reachable from the state k.

Lower Bound for Hausdorff Dimension

- Theorem. (Lower Bound Theorem) For any any $k \geq 1$ there exist

$$N_1 < N_2 < \cdots < N_k, \quad \text{all } N_i \equiv 1 \pmod 3$$

such that

$$dim_H(\mathcal{C}(N_1, N_2, ..., N_k)) := dim_H(\bigcap_{i=1}^{k} \frac{1}{N_i}\Sigma_{3,\bar{2}}) \geq 0.35.$$

Thus: the maximal Hausdorff dimension of intersection of translates is uniformly bounded away from zero.

- Proof. Take suitable N_i of the form $3^j + 1$ for various large j. One can show the Hausdorff dimension of intersection remains large (large overlap of symbolic dynamics).

Conclusions: Part III

- (1) The graphs for $\mathcal{C}(1, N)$ exhibit a complicated structure depending on an irregular way on the ternary digits of N. Their Hausdorff dimensions vary irregularly.

- (2) It might still be true that

$$\alpha_k := \sup_{r_1 < r_2 < \cdots < r_k} dim_H \left(\mathcal{C}(2^{r_1}, ...2^{r_k})\right)$$

has $\alpha_k \to 0$ as $k \to \infty$. But ...

- (3) Lower bound theorem suggests: analyzing the special case where all $N_i = 2^{r_i}$ may not be easy!

Paul Erdős says: 

“As far as I can see there is no method at our disposal to attack this conjecture.”

(Ref. P. Erdős, Some unconventional problems in number theory, Math. Mag. **52** (1979), 67–70.)