The Lerch Zeta Function: Analytic Continuation

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Topics Covered

- Part I. History: Lerch Zeta Function and Lerch Transcendent
- Part II. Basic Properties
- Part III. Multi-valued Analytic Continuation
- Part IV. Consequences: Other Properties
- Part V. Lerch Transcendent

Credits

- J. C. Lagarias and W.-C. Winnie Li , The Lerch Zeta Function I. Zeta Integrals, Forum Math, in press.
  `arXiv:1005.4712`
  J. C. Lagarias and W.-C. Winnie Li , The Lerch Zeta Function II. Analytic Continuation, Forum Math, in press.
  `arXiv:1005.4967`
  J. C. Lagarias and W.-C. Winnie Li , The Lerch Zeta Function III. Polylogarithms and Special Values, preprint.

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Part I. Lerch Zeta Function: History and Objectives

- The Lerch zeta function is:

$$\zeta(s,a,c) := \sum_{n=0}^{\infty} \frac{e^{2\pi i n a}}{(n+c)^s}$$

- The Lerch transcendent is:

$$\Phi(s,z,c) = \sum_{n=0}^{\infty} \frac{z^n}{(n+c)^s}$$

- Thus

$$\zeta(s,a,c) = \Phi(s, e^{2\pi i a}, c).$$

Special Cases-1

- Hurwitz zeta function (1882)

$$\zeta(s,0,c)=\zeta(s,c):=\sum_{n=0}^{\infty}\frac{1}{(n+c)^s}.$$

- Periodic zeta function (Apostol (1951))

$$e^{2\pi ia}\zeta(s,a,1)=F(a,s):=\sum_{n=1}^{\infty}\frac{e^{2\pi ina}}{n^s}.$$

Special Cases-2

- Fractional Polylogarithm

$$z\,\Phi(s,z,1) = Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$$

- Riemann zeta function

$$\zeta(s,0,1) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

History-1

- Lipschitz (1857) studies general Euler integrals including the Lerch zeta function

- Hurwitz (1882) studied Hurwitz zeta function.

- Lerch (1883) derived a three-term functional equation. (Lerch's Transformation Formula)

$$\begin{aligned}\zeta(1-s,a,c) &= (2\pi)^{-s}\Gamma(s)\Big(e^{\frac{\pi i s}{2}}e^{-2\pi i a c}\zeta(s,1-c,a) \\ &+ e^{-\frac{\pi i s}{2}}e^{2\pi i c(1-a)}\zeta(s,c,1-a)\Big).\end{aligned}$$

History-2

- de Jonquiere (1889) studied the

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{x^n}{n^s},$$

  sometimes called the fractional polylogarithm, getting integral representations and a functional equation.

- Barnes (1906) gave contour integral representations and method for analytic continuation of functions like the Lerch zeta function.

History-3

- Further work on functional equation: Apostol (1951), Berndt (1972), Weil 1976.

- Much work on value distribution: Garunkštis (1996), (1997), (1999), Laurinčikas (1997), (1998), (2000), Laurinčikas and Matsumoto (2000). Work up to 2002 summarized in L. & G. book on the Lerch zeta function.

- Other books: Kanemitsu, Tsukada, *Vistas of Special Functions*, Chakraborty, Kanemitsu, Tsukada, *Vistas ... II* (2007, 2010).

Objective 1: Analytic Continuation

- Objective 1. Analytic continuation of Lerch zeta function and Lerch transcendent in three complex variables.

- Relevant Work: Kanemitsu, Katsurada, Yoshimoto (2000) gave a single-valued analytic continuation of Lerch transcendent in three complex variables: they continued it to various large simply-connected domain(s) in $\mathbb{C}^3$.

- We obtain a continuation to a multivalued function on a maximal domain of holomorphy in 3 complex variables.

Objective 2: Extra Structures

- Objective 2. Determine effect of analytic continuation on other structures: difference equations (non-local), linear PDE (local), and functional equation.

- Behavior at special values: $s \in \mathbb{Z}$.

- Behavior near singular values $a, c \in \mathbb{Z}$; these are "singularities" of the three-variable analytic continuation.

Objectives: Singular Strata

- The values $a, c \in \mathbb{Z}$ give (non-isolated) singularities of this function of three complex variables. What is the behavior of the function approaching the singular strata?

- The Hurwitz zeta function and Periodic zeta function lie on "singular strata" of real codimension 2. The Riemann zeta function lies on a "singular stratum" of real codimension 4.

- There also is analytic continuation in the s-variable on the singular strata (in many cases, perhaps all cases).

Part II. Basic Structures

Important structures of the Lerch zeta function include:

1. Functional Equation(s).

2. Differential-Difference Equations

3. Linear Partial Differential Equation

4. Integral Representations

Four Term Functional Equation-1

- Defn. Let a and c be real with $0 < a < 1$ and $0 < c < 1$. Set
$$L^{\pm}(s,a,c) := \zeta(s,a,c) \pm e^{-2\pi i a}\zeta(s,1-a,1-c).$$
Formally:
$$L^{+}(s,a,c) = \sum_{-\infty}^{\infty} \frac{e^{2\pi i n a}}{|n+c|^{s}}.$$

- Defn. The completed function
$$\hat{L}^{+}(s,a,c) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\, L^{+}(s,a,c)$$
and the completed function
$$\hat{L}^{-}(s,a,c) := \pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2})\, L^{-}(s,a,c).$$

Four Term Functional Equation-2

- Theorem (Weil (1976)) Let $0 < a, c < 1$ be real. Then:

  (1) The completed functions $\hat{L}^{+}(s,a,c)$ and $\hat{L}^{-}(s,a,c)$ extend to entire functions of s and satisfy the functional equations

  $$\hat{L}^{+}(s,a,c) = e^{-2\pi iac}\hat{L}^{+}(1-s,1-c,a)$$

  and

  $$\hat{L}^{-}(s,a,c) = i\,e^{-2\pi iac}\hat{L}^{-}(1-s,1-c,a).$$

  (2) These results extend to $a = 0, 1$ and/or $c = 0, 1$. If $a = 0, 1$ then $\hat{L}^{+}(s,a,c)$ is a meromorphic function of s, with simple poles at $s = 0, 1$. In all other cases these functions are entire functions of s.

Functional Equation Zeta Integrals

- Part I of our papers obtains a generalized functional equation for Lerch-like zeta integrals containing a test function. (This is in the spirit of Tate's thesis.)

- These equations relate a integral with test function $\phi(x)$ at point s to integral with Fourier transform $\hat{f}(\xi)$ of test function at point $1-s$.

- The self-dual test function $f_0(x) = e^{-\pi x^2}$ yields the function $\hat{L}^{+}(s, a, c)$. The eigenfuctions $f_n(x)$ of the oscillator representation yield similar functional equations: Here $f_1(x)$ yields $\frac{1}{\sqrt{2\pi}}\hat{L}^{-}(s, a, c)$.

Differential-Difference Equations

- The Lerch zeta function satisfies two differential-difference equations.

- (Raising operator)
$$\frac{\partial}{\partial c}\zeta(s,a,c) = -s\zeta(s+1,a,c).$$

- Lowering operator)
$$\left(\frac{1}{2\pi i}\frac{\partial}{\partial a} + c\right)\zeta(s,a,c) = \zeta(s-1,a,c)$$

- These operators are non-local in the s-variable.

Linear Partial Differential Equation

- The Lerch zeta function satisfies a linear PDE:

$$(\frac{1}{2\pi i}\frac{\partial}{\partial a} + c)\frac{\partial}{\partial c}\zeta(s,a,c) = -s\,\zeta(s,a,c).$$

- The (formally) skew-adjoint operator

$$\Delta_L := \frac{1}{2\pi i}\frac{\partial}{\partial a}\frac{\partial}{\partial c} + c\frac{\partial}{\partial c} + \frac{1}{2}I$$

has

$$\Delta_L \zeta(s,a,c) = -(s - \frac{1}{2})\zeta(s,a,c).$$

Integral Representations

- The Lerch zeta function has two different integral representations, generalizing integral representations in Riemann's original paper.

- Riemann's formulas are:

$$\int_0^\infty \frac{e^{-t}}{1-e^{-t}} t^{s-1} dt = \Gamma(s)\zeta(s)$$

  and, formally,

$$\int_0^\infty \vartheta(0; it^2) t^{s-1} dt \text{ ``}={}\text{''}\ \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})\zeta(s),$$

  where

$$\vartheta(0;\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}.$$

Integral Representations

- The generalizations to Lerch zeta function are

$$\int_0^\infty \frac{e^{-ct}}{1-e^{2\pi ia}e^{-t}} t^{s-1} dt = \Gamma(s)\zeta(s,a,c)$$

and

$$\int_0^\infty e^{\pi c^2 t^2} \vartheta(a+ict^2, it^2) t^{s-1} dt = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s,a,c).$$

where

$$\vartheta(z;\tau) = \sum_{n\in\mathbb{Z}} e^{\pi in^2\tau} e^{2\pi inz}.$$

Part III. Analytic Continuation for Lerch Zeta Function

- Theorem. $\zeta(s, a, c)$ analytically continues to a multivalued function over the domain

$$\mathcal{M} = (s \in \mathbb{C}) \times (a \in \mathbb{C} \smallsetminus \mathbb{Z}) \times (c \in \mathbb{C} \smallsetminus \mathbb{Z}).$$

  It becomes single-valued on the maximal abelian cover of $\mathcal{M}$.

- The monodromy functions giving the multivaluedness are computable. For fixed s, they are built out of the functions

$$\phi_n(s, a, c) \ := \ e^{2\pi i n a}(c - n)^{-s}, \qquad n \in \mathbb{Z}.$$

$$\psi_{n'}(s, a, c) \ := \ e^{2\pi c(a - n')}(a - n')^{s-1} \qquad n' \in \mathbb{Z}.$$

Analytic Continuation-Features

- The manifold $\mathcal{M}$ is invariant under the symmetries of the functional equation: $(s, a, c) \mapsto (1 - s, 1 - c, a)$.

- Fact. The four term functional equation extends to the maximal abelian cover by analytic continuation. It expresses a non-local symmetry of the function.

Lerch Analytic Continuation: Proof

- Step 1. The first integral representation defines $\zeta(s, a, c)$ on the simply connected region

$$\{0 < Re(a) < 1\} \times \{0 < Re(c) < 1\} \times \{0 < Re(s) < 1\}.$$

  Call it the fundamental polycylinder.

- Step 2a. Weil's four term functional equation extends to fundamental polycylinder by analytic continuation. It leaves this polycylinder invariant.

- Step 2b. Extend to entire function of s on fundamental polycylinder in (a, c)-variables, together with the four-term functional equation.

Lerch Analytic Continuation: Proof -2

- Step 3. Integrate single loops around $a = n, c = n'$ integers, using contour integral version of first integral representation to get initial monodromy functions

  Here monodromy functions are difference (functions) between a function and the same function traversed around a closed path. They are labelled by elements of $\pi_1(\mathcal{M})$.

- Step 4. The monodromy functions themselves are multivalued, but in a simple way: Each is multivalued around a single value $c = n$ (resp. $a = n'$). They can therefore be labelled with the place they are multivalued. (This gives functions $\phi_n, \psi_{n'}$)

Lerch Analytic Continuation: Proof -3

- Step. 5. Iterate to the full homotopy group in (a, c)-variables by induction on generators; use fact that a-loop homotopy commutes with c-loop homotopy.

- Step. 6. Explicitly calculate that the monodromy functions all vanish on the commutator subgroup $[\pi_1(\mathcal{M}), \pi_1(\mathcal{M})]$ of $\pi_1(\mathcal{M})$. This gives single-valuedness on the maximal abelian covering of $\mathcal{M}$.

Exact Form of Monodromy Functions-1

- At points $c = m \in \mathbb{Z}$,

$$M_{[Y_m]}(Z) = c_1(s) e^{2\pi i m a}(c-m)^{-s}$$

in which

$$c_1(s) = 0 \qquad \text{for} \quad m \geq 1,$$

$$c_1(s) = e^{2\pi i s} - 1 \quad \text{for} \quad m \leq 0.$$

Also

$$M_{[Y_m]^{-1}}(Z) = -e^{2\pi i s} M_{[Y_m]}(Z).$$

$$M_{[Y_m]^{\pm k}}(Z) = \frac{e^{\pm 2\pi i k s} - 1}{e^{\pm 2\pi i s} - 1} M_{[Y_m]^{\pm 1}}(Z).$$

Exact Form of Monodromy Functions-2

- At points $a = m \in \mathbb{Z}$,

$$M_{[X_m]}(Z) = c_2(s) e^{2\pi i c(a-m)} (a-m)^{s-1}$$

where

$$c_2(s) = -\frac{(2\pi)^s e^{\frac{\pi i s}{2}}}{\Gamma(s)}.$$

Also

$$M_{[X_m]^{-1}}(Z) = -e^{2\pi i s} M_{[X_m]}(Z)$$

$$M_{[X_m]^{\pm k}}(Z) = \frac{e^{\mp 2\pi i k s} - 1}{e^{\mp 2\pi i s} - 1} M_{[X_m]^{\pm 1}}(Z).$$

Extended Lerch Analytic Continuation

- Theorem. $\zeta(s, a, c)$ analytically continues to a multivalued function over the (larger) domain

$$\mathcal{M}^\sharp = (s \in \mathbb{C}) \times (a \in \mathbb{C} \smallsetminus \mathbb{Z}) \times (c \in \mathbb{C} \smallsetminus \mathbb{Z}_{\leq 0}).$$

  Here the extra points $c = 1, 2, 3, ...$ are glued into $\mathcal{M}$. The extended function is single-valued on the maximal abelian cover of $\mathcal{M}^\sharp$.

- The manifold $\mathcal{M}^\sharp$ is not invariant under the four term Lerch functional equation. This is a broken symmetry.

Part IV. Consequences: Other Properties

We determine the effect of analytic continuation on the other properties

1. Functional Equation. This is inherited by analytic continuation on $\mathcal{M}$ but not on $\mathcal{M}^\sharp$.

2. Differential-Difference Equations. These equations lift to the maximal abelian cover of $\mathcal{M}$. However they are not inherited individually by the monodromy functions.

Consequences: Other Properties

(3) Linear PDE. This lifts to the maximal abelian cover. That is, this PDE is equivariant with respect to the covering map. The monodromy functions are all solutions to the PDE.

For fixed s the monodromy functions give an infinite dimensional vector space of solutions to this PDE.
(View this vector space as a direct sum.)

Consequences: Special Values

- Theorem. The monodromy functions vanish identically when $s = 0, -1, -2, -3, ...$ That is, for these values of s the value of the Lerch zeta function is well-defined on the manifold $\mathcal{M}$, without lifting to the maximal abelian cover.

- The functional equation takes $s \to 1 - s$. At nonnegative integer values of $s = 1, 2, ...$ there is nontrivial monodromy. However at these points the monodromy functions satisfy linear dependencies that cancel out of the functional equation.

Consequences: Special Values-2

- It is well known that at the special values $s = 0, -1, -2, \ldots$ the Lerch zeta function simplifies to a rational function of c and $e^{2\pi i a}$.

- p-adic interpolation is possible at the special values $s = 0, -1, -2, -3, \ldots$. This can be obtained from the analytic continuation as a continuous limit, taking $c \to 1+$, using the periodic zeta function. (It does not seem to be possible using the Hurwitz zeta function.)

Approaching Singular Strata

- There are (sometimes!) discontinuities in the Lerch zeta function's behavior approaching a singular stratum: these depend on the value of the s-variable.

  Observation. Only the real part of the s-variable matters. Three regimes:

  $$Re(s) < 0; \quad 0 \leq Re(s) \leq 1; \quad Re(s) > 1.$$

Part V. Lerch Transcendent

We determine the effect of analytic continuation on the Lerch transcendent

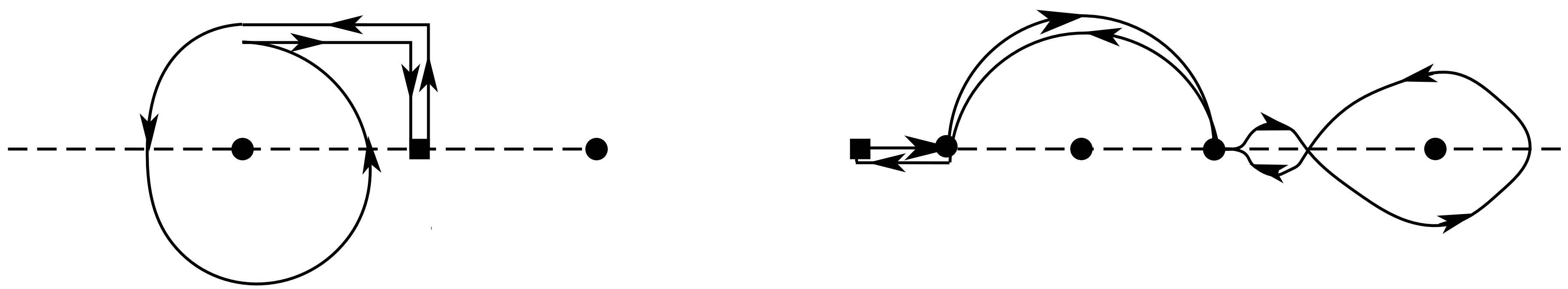
$$\Phi(s, z, c) = \sum_{n=0}^{\infty} \frac{z^n}{(n+c)^s}.$$

We make the change of variable $z = e^{2\pi i a}$ so that

$$a = \frac{1}{2\pi i} \log z.$$

This introduces extra multivaluedness: a is a multivalued function of z.

Image under $z = e^{2\pi i a}$ of loop around $a = 0$, using basepoints $a = \frac{1}{2}$ and $z = -1$.



Polylogarithm

- The Lerch transcendent (essentially) specializes to the m-th order polylogarithm at $c = 1$, $s = k \in \mathbb{Z}_{>0}$.

$$L_m(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^k} = z\Phi(k, z, 1).$$

- The m-th order polylogarithm satisifes an $(m+1)$-st order linear ODE in the complex domain. This equation is Fuchsian on the Riemann sphere, i.e. it has regular singular points. These are located at $\{0, 1, \infty\}$.

- The point $c = 1$ is on a regular stratum. This uses the extended analytic continuation, which is not invariant under the functional equation.

Analytic Continuation for Lerch Transcendent

- Theorem. $\Phi(s,z,c)$ analytically continues to a multivalued function over the domain

$$\mathcal{N} = (s \in \mathbb{C}) \times (z \in \mathbb{P}^1 \setminus \{0,1,\infty\}) \times (c \in \mathbb{C} \setminus \mathbb{Z}).$$

  It becomes single-valued on a two-fold solvable cover of $\mathcal{N}$.

- The monodromy functions giving the multivaluedness are explicitly computable, but complicated.

Monodromy Functions for Lerch Transcendent

For fixed s, the monodromy functions are built out of the functions

$$\phi_n(s,z,c) \ := \ z^n(c-n)^{-s}, \qquad n \in \mathbb{Z}.$$

and

$$f_n(s,z,c) := \ e^{\pi i(s-1)} e^{2\pi i n c} z^{-c} (n - \frac{1}{2\pi i} Log\, z)^{s-1} \quad \text{if} \quad n \geq 1.$$

$$f_n(s,z,c) := e^{2\pi i n c} z^{-c} (\frac{1}{2\pi i} Log\, z - n)^{s-1} \qquad \text{if} \quad n \leq 0.$$

taking $z^{-c} = e^{-cLogz}$. where $Log\, z$ denotes a branch of the logarithm cut along the positive real axis.

Functional Equations: Lerch Transcendent

- The Lerch transcendent satisfies functional equations inherited from the Lerch zeta function. However they become multivalued!

Differential-Difference Equations: Lerch Transcendent

- The Lerch transcendent satisfies two differential-difference equations. These operators are non-local in the s-variable.

- (Raising operator)

$$\frac{\partial}{\partial c}\Phi(s,z,c) = -s\Phi(s+1,z,c).$$

- Lowering operator)

$$\left(z\frac{\partial}{\partial z}+c\right)\Phi(s,z,c) = \Phi(s-1,z,c)$$

Linear Partial Differential Equation: Lerch Transcendent

- As with Lerch zeta function, the Lerch transcendent satisfies a linear PDE:

$$\left(z\frac{\partial}{\partial z} + c\right) \frac{\partial}{\partial c}\Phi(s, z, c) = -s\,\Phi(s, a, c).$$

- The (formally) skew-adjoint operator

$$\tilde{\Delta}_L := \left(z\frac{\partial}{\partial z} + c\right)\frac{\partial}{\partial c} + c\frac{\partial}{\partial c} + \frac{1}{2}I$$

has

$$\tilde{\Delta}_L\Phi(s, z, c) = -(s - \frac{1}{2})\Phi(s, z, c).$$

Specialization to Polylogarithm-1

- For integer $s = k \geq 1$, and c as a parameter, the function $z\Phi(k, z, c)$ gives a deformation of the polylogarithm:

$$Li_k(z, c) := \sum_{m=0}^{\infty} z^m (m + c)^k.$$

- Viewing c as fixed, it satisfies the Fuchsian ODE $D_c\, Li_k(z, c) = 0$ where the differential operator is:

$$D_c := z^2 \frac{d}{dz}\left(\frac{1-z}{z}\right)\left(z\frac{d}{dz} + c - 1\right)^m.$$

- The singular stratum points are $c = 0, -1, -2, -3, \ldots$.

Specialization to Polylogarithm-2

- A basis of solutions for each regular stratum point is

$$\{Li_m(z,c), z^{1-c}(\log z)^{m-1}, z^{1-c}(\log z)^{m-2}, \cdots, z^{1-c}\}.$$

- The monodromy of the loop $[Z_0]$ on this basis is:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & e^{-2\pi ic} & e^{-2\pi ic}\frac{2\pi i}{1!} & \cdots & e^{-2\pi ic}\frac{(2\pi i)^{m-2}}{(m-2)!} & e^{-2\pi ic}\frac{(2\pi i)^{m-1}}{(m-1)!} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{-2\pi ic} & e^{-2\pi ic}\frac{2\pi i}{1!} \\ 0 & 0 & 0 & \cdots & 0 & e^{-2\pi ic} \end{pmatrix}.$$

The monodromy of the loop $[Z_1]$ is unipotent and independent of c.

Specialization to Polylogarithm-3

- A basis of solutions for each singular stratum point is

$$\{Li^*_m(z,c), z^{1-c}(\log z)^{m-1}, z^{1-c}(\log z)^{m-2}, \cdots, z^{1-c}\}$$

- The monodromy of the loop $[Z_0]$ in this basis is:

$$\begin{pmatrix} 1 & \frac{2\pi i}{1!} & \frac{(2\pi i)^2}{2!} & \cdots & \frac{(2\pi i)^{m-1}}{(m-1)!} & \frac{(2\pi i)^m}{m!} \\ 0 & 1 & \frac{2\pi i}{1!} & \cdots & \frac{(2\pi i)^{m-2}}{(m-2)!} & \frac{(2\pi i)^{m-1}}{(m-1)!} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{2\pi i}{1!} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

  It is unipotent. The monodromy of the loop $[Z_1]$ is unipotent and independent of c.

Specialization to Polylogarithm-4

Some observations:

- The monodromy representation (of π_1 of the Riemann sphere minus $0, 1, \infty$) is upper triangular, and is unipotent exactly when c is a positive integer (regular strata) or c is a negative integer (singular strata!).

- The equation makes sense on the singular strata. The equation remains Fuchsian on the singular strata. The monodromy representation continues to be unipotent, paralleling the positive integer case (polylogarithms at $c = 1$). However it takes a discontinuous jump at these points. (It appears unfixable by change of basis).

Summary-1

- We have found a number of interesting extra structures associated with the Lerch zeta function, and related these to analytic continuation.

- Observation. The analytic continuation of Lerch zeta fails at values a, c integers, which are the most interesting values where the Hurwitz and Riemann zeta functions appear. These are singular points. Understanding the behavior as the singular points are approached may shed new light on these functions.

Summary-2

- Does the Lindelöf hypothesis hold for the Lerch zeta function? (Question raised by Garunkštis-Steuding).

  Observation. If so, Lindelöf hypothesis would hold also for all the multivalued branches as well, because the monodromy functions are all of slow growth in the t-direction.

Part VI. Further extensions

- In Part IV (paper in preparation with Winnie Li) we study two-variable "Hecke operators"

$$T_m(F)(a,c) := \frac{1}{m}\sum_{j=0}^{m-1} F(\frac{a+k}{m}, mc)$$

- We show that formally these operators commute, and they also commute with the differential operator Δ_L. These operators are non-local, and dilate in the c-direction while contract and shift in the a-direction.

- For fixed s the LZ function is a simultaneous eigenfunction of these operators, with eigenvalue m^{-s} for T_m.

Two-variable Hecke operators-continued

- In part IV we show a generalization to the Lerch zeta function of Milnor's 1983 result characterizing the Hurwitz zeta function $\zeta(s,z)$ as a simultaneous eigenfunction of "Kubert operators":

$$T_m(F)(z) = \frac{1}{m}\sum_{j=0}^{m-1} F(\frac{z+k}{m})$$

- Our two-variable Hecke operators exchange the a and c-variables. This exchanges s and $1-s$, compared to Milnor's results.

Two-variable Hecke operators-3

- In part IV we also show the two-variable Hecke operators induce an action on the vector space of all monodromy functions of the analytically continued Lerch zeta functions.

- Observation. This action resembles part of the Bost-Connes Hecke algebra action in the C^*-dynamical system they found whose partition function is the Riemann zeta function.

Thank You!