Polynomial Splitting Measures

and

Cohomology of the Pure Braid Group

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Topics Covered

- Part I. Factorization of Monic Polynomials: Probabilities
- Part II. Polynomial Splitting Measures
- Part III. $\mathbb{F}_1$-splitting measures
- Part IV. Splitting Measure Coefficients and Representation Theory
- Part V. Applications/Consequences

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Probabilistic Galois Theory over p-adic fields,
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• J. C. Lagarias and Benjamin L. Weiss,
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• J. C. Lagarias,
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Part I. Factoring Polynomials over $\mathbb{Q}$ and $\mathbb{Q}_p$

- **Question.** How do degree n monic polynomials factor over $\mathbb{Q}$?

- **Answer.** Almost all of them are irreducible.

- Hilbert Irrreducibility Theorem (1892). *Let $f(x)$ be a monic polynomial $f(X)$ in X which is irreducible in the ring $K[x]$ where K is the rational function field $K = \mathbb{Q}(a_1, ..., a_n)$. Then one can specialize the parameters $(a_0, ..., a_{n-1})$ to rational values $\mathbb{Q}^n$ so that the resulting polynomial is irreducible over $\mathbb{Q}[X]$.*

Factorization of Polynomials -2

• **Parametric Version.** Consider the generic polynomial

$$f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0.$$

Restrict the parameters a_i to be *integers* and bound their *height*).
Put them in a box $|a_i| \leq B$, then let $B \to \infty$.

• **Another improvement.** Control the Galois group in the Hilbert irreducibility theorem,

van der Waerden Theorem

Theorem (van der Waerden (1934)) Given the polynomial

$$f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0.$$

Consider the coefficients as integers in a box $-B < a_j \leq B$, drawn randomly (uniform distribution) Then as B goes to infinity:

(1) With probability one, the resulting polynomial is irreducible over $\mathbb{Q}$.

Moreover, can make the Galois group “maximal":

(2) With probability one, the splitting field (adjoining all roots) of $f(x)$ has Galois group the full symmetric group S_n.

Quantitative Version of van der Waerden theorem

- **Theorem** (Gallagher (1978)) Given the polynomial

$$f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$$

with all $|a_i| \leq B$, Then of the $(2B+1)^n$ such polynomials at most

$$\ll B^{n-\frac{1}{2}}$$

of them are “exceptional", either :

(1) $f(x)$ not irreducible over $\mathbb{Q}$, or,

(2) Galois group of splitting field of $f(x)$ is not S_n.

Factorization of polynomials: p-adic case

- **Motivating Question.** What is the analogue of Hilbert irreduciblity theorem over a p-adic field $\mathbb{Q}_p$?

- **Comment.** The answer must be *different*, because:

  All Galois groups over a p-adic field $\mathbb{Q}_p$ are *solvable*.

Factorization of polynomials: p-adic case-2

- **Answer 1.** The splitting probabilities of a monic polynomial of degree n with coefficients in $\mathbb{Z}_p$, the p*-adic integers*, depends on n and p. There is a positive probability of not being irreducible. There is a positive probability of splitting completely into linear factors, These depend on n and p. (They go to 0 as $p \to \infty$).

- **Answer 2.** The answer is nice if one restricts to polynomials in $\mathbb{Z}_p$ having polynomial discriminant *prime to* p.

  In that case it matches the distribution of factorizations of a random polynomial with coefficients in the finite field $\mathbb{F}_p$. This answer depends on p. It has a nice limit as $p \to \infty$.

Factorization of polynomials: p-adic case-3

- **Theorem.** (B. L Weiss (2013)) *As* $p \to \infty$*:*

  (1) A random monic polynomial $f(x)$ *of degree* n *with* $\mathbb{Z}_p$ *coefficients has with probability one a splitting field whose Galois group is* **cyclic.** *The order of this cyclic group is a random variable which equals the order of a random element of* S_n *drawn with the uniform distribution.*

  (2) The factorization type of the monic polynomial with probability one has the cycle structure of that of a randomly drawn element of S_n *(uniform distribution).*

- This answer is inherited from the finite residue field $\mathbb{F}_p$. The cyclic Galois extension of $\mathbb{Q}_p$ found is **unramified** with probability one.

Part II. Polynomial Splitting Measures

- **Problem.** Describe the the distribution of factorizations of monic random polynomials over $\mathbb{F}_p$.

- Factorizations are describable by the number of factors and the degree of each factor. Such data summarized by a partition λ of n, interpretable as a *conjugacy class* C_λ *of elements in* S_n.

Polynomial Splitting Measures-2

- The *p-splitting measures* $\nu_{n,p}(C_\lambda)$ describe the probability of factorization of a monic polynomial of degree n over $\mathbb{F}_p$ having splitting type λ, conditioned on the polynomial being **squarefree.**

- The **squarefree** condition means: the discriminant of $f(x)$ is not 0.

- **Proposition**. *The probability of a monic polynomial over $\mathbb{F}_p$ having discriminant* 0 *is* **exactly** $\frac{1}{p}$.

- The probability of being squarefree is then $1 - \frac{1}{p}$. As $p \to \infty$ this probability approaches 1. (As p varies, it is interpolatable by the Laurent polynomial $1 - \frac{1}{z}$ in the variable z, taking $z = p$.)

Polynomial Splitting Measure-3

• Let z denote the interpolation variable, i.e. $z = p$ recovers the splitting measure values. The interpolated measure is constant on conjugacy classes, and $\nu^*_{n,z}(C_\lambda)$ denotes the measure over a conjugacy class C_λ. That is

$$\nu^*_{n,z}(C_\lambda) = \textstyle\sum_{g \in C_\lambda} \nu^*_{n,z}(g).$$

• The splitting measure on a conjugacy class is a rational function of z:

$$\nu^*_{n,z}(C_\lambda) := \frac{1}{z^n - z^{n-1}} N_\lambda(z)$$

where $N_\lambda(z)$ is the *cycle polynomial* associated to the partition λ (to be defined).

• $z^n - z^{n-1}$ interpolates at $z = p$ the number of monic polynomials of degree n minus the number of such having discriminant 0 over $\mathbb{F}_p$.

Necklace Polynomial and Cycle Polynomial

• For $j \geq 1$, the j**-th necklace polynomial** $M_j(z) \in \frac{1}{j}\mathbb{Z}[z]$ is

$$M_j(z) := \frac{1}{j}\sum_{d|j} \mu(d) z^{j/d},$$

where $\mu(d)$ is the Möbius function. (At $z = p$ it counts irreducible monic polynomials over $\mathbb{F}_p$)

• Given a partition λ of n, the **cycle polynomial** $N_\lambda(z) \in \frac{|C_\lambda|}{n!}\mathbb{Z}[z]$ is

$$N_\lambda(z) := \prod_{j \geq 1} \binom{M_j(z)}{m_j(\lambda)},$$

where $\binom{\alpha}{m} := \frac{1}{m!}\prod_{k=0}^{m-1}(\alpha - k)$ is extended binomial coefficient.
Notation: $z_\lambda = \frac{n!}{|C_\lambda|}$.

Counting Polynomial Factorizations

• For a partition $\lambda = (1^{m_1} 2^{m_2} \cdots n^{m_n})$ of n are counting the number of factorizations of $f(x)$ into products of m_j distinct irreducibles of degree j over $\mathbb{F}_p$ up to ordering. The value $\binom{M_j(z)}{m_j(\lambda)}$ for $z = p$ counts this number.

• We normalize by the number of squarefree monic polynomials, which is $z^n - z^{n-1} = z^{n-1}(z-1)$ evaluated at $z = p$.

• Thus

$$\nu^*_{n,z}(C_\lambda) := \frac{1}{z^{n-1}(z-1)} \prod_{j \geq 1} \binom{M_j(z)}{m_j(\lambda)}$$

.

Polynomial splitting measures-4

- **Proposition.** For $n \geq 2$ the z-splitting measures are rational functions of z. Moreover they are Laurent polynomials in z for each conjugacy class of S_n: their only poles as rational functions are at $z = 0$. (The $(z-1)$ factor in denominator cancels for every λ).

- At $z = \infty$ the z-splitting distribution on S_n is the uniform measure:

- The uniform measure is the probability distribution of the Chebotarev density theorem. It assigns total mass
$$\frac{|C|}{|G|} = \frac{|C_\lambda|}{|S_n|} = \frac{|C_\lambda|}{n!} = \frac{1}{z_\lambda}.$$
to each conjugacy class C_λ of S_n.

z-Splitting Measure when $n = 4$

λ	$\lvert C_\lambda \rvert$	z_λ	$\nu^*_{4,z}(C_\lambda)$
$[1,1,1,1]$	1	24	$\frac{1}{24}\left(1 - \frac{5}{z} + \frac{6}{z^2}\right)$
$[2,1,1]$	6	4	$\frac{1}{4}\left(1 - \frac{1}{z}\right)$
$[2,2]$	3	8	$\frac{1}{8}\left(1 - \frac{1}{z} - \frac{2}{z^2}\right)$
$[3,1]$	8	3	$\frac{1}{3}\left(1 + \frac{1}{z}\right)$
$[4]$	6	4	$\frac{1}{4}\left(1 + \frac{1}{z}\right)$

Values of the z-splitting measures $\nu^*_{4,z}(C_\lambda)$ on partitions λ of $n = 4$.

z-Splitting Measure when $n = 5$

λ	$\lvert C_\lambda \rvert$	z_λ	$\nu^*_{5,z}(C_\lambda)$
$[1,1,1,1,1]$	1	120	$\frac{1}{120}\left(1 - \frac{9}{z} + \frac{26}{z^2} - \frac{24}{z^3}\right)$
$[2,1,1,1]$	10	12	$\frac{1}{12}\left(1 - \frac{3}{z} + \frac{2}{z^2}\right)$
$[2,2,1]$	15	8	$\frac{1}{8}\left(1 - \frac{1}{z} - \frac{2}{z^2}\right)$
$[3,1,1]$	20	6	$\frac{1}{6}\left(1 + \frac{0}{z} - \frac{1}{z^2}\right)$
$[3,2]$	20	6	$\frac{1}{6}\left(1 + \frac{0}{z} - \frac{1}{z^2}\right)$
$[4,1]$	30	4	$\frac{1}{4}\left(1 + \frac{1}{z}\right)$
$[5]$	24	5	$\frac{1}{5}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3}\right)$

Values of the z-splitting measures $\nu^*_{5,z}(C_\lambda)$ on partitions λ of $n = 5$.

Integer Monic Polynomial Splitting Model

• An S_n**-number field** is a degree n (non-Galois) number field over $\mathbb{Q}$ whose Galois closure has Galois group S_n.

• **Random Monic Polynomial Model.** *Take monic integer polynomials $f(X)$ of degree n with all coefficients in a box $|a_i| \le B$. Take a prime p, and condition on $Disc(f(x)$ being prime to p. Then study how the prime ideal (p) factorizes in the number field K_f obtained by adjoining one root of $f(X)$ to $\mathbb{Q}$, as parameter $B \to \infty$.*

• **Theorem** (B.L. Weiss- L (2015)) *With probability one, as $B \to \infty$.*

(1) The field K_f is an S_n-number field.

(2) For fixed p, the limiting splitting distribution of (p) over all K_f (as $f(x)$ varies) is given by the z-splitting distribution at $z = p$.

Bhargava Random S_n-Number Field Model

• **Random S_n-number field model** *For a fixed B let $S(B)$ run over the finite set of all S_n-number fields having discriminant $|Disc(K)|$ bounded by B.Then for a given prime p consider the subset of all such fields having $(Disc(K)$ prime to p.*

• **Conjecture** (Bhargava (2007)) *As $B \to \infty$:*
(1) The limiting fraction of prime to p discriminants exists as $B \to \infty$ and is a rational function of p.

(2) Consider the fraction of this subset that has a given splitting type of the rational prime ideal (p), which is described by a partition of n. Then the limiting fraction exists as $B \to \infty$ and is the uniform distribution on S_n.

• Bhargava (2007) proved his conjecture for $n \leq 5$. It is open for $n \geq 6$.

Comparison of Random Polynomial and Random Field Models

• The z-splitting distribution deviates greatly from the uniform distribution for $z = p$ small compared to n. Some probabilities become 0. For example for $p = 2$ all partitions λ containing 1^3 have probability $\nu^*_{n,2}(C_\lambda) = 0$, because no monic polynomial can have 3 distinct linear factors over $\mathbb{F}_2$.

• The Bhargava conjecture (uniform) distribution is the limit as $p \to \infty$ of the Random Monic Polynomial Model distribution.

• What features of the two models account for the mismatch between the predictions of the models? *We currently do not have a satisfactory answer.*

• This mismatch motivates further study of the z-splitting measures.

Part III. $\mathbb{F}_1$-Splitting Distribution

- **Problem.** Study the splitting measure distribution at $z = 1$. (The extreme "bad case")

- The value $z = 1$ is the only integer point (excluding $z = 0$) where the distribution is a **signed** measure.

- (Counting points over $\mathbb{F}_1$= "field with one element") The $z = 1$ probabilities are the "*ghost of a departed quantity*" since the probability of squarefreeness at $p = 1$ is $1 - \frac{1}{p} = 0$!
  ("Quantum" case: allow negative probabilities.)

$\mathbb{F}_1$ Splitting Measure-2

Theorem. *The $z = 1$-splitting (signed) measures $\nu^*_{n,1}$ have the following properties:*

(1) The support of the measure $\nu^*_{n,1}$ is exactly the set of conjugacy classes $[\lambda]$ such that λ is one of:

(i) Rectangular partitions $\lambda = [b^a]$ for $ab = n$.

(ii) Almost-rectangular partitions $\lambda = [d^c, 1]$ for $cd = n - 1$.

[**Note**: These partitions label the conjugacy classes of S_n that comprise the **Springer regular elements** of S_n.]

$\mathbb{F}_1$ Splitting Measure-3

Theorem. *The $z = 1$-splitting (signed) measures $\nu^*_{n,1}$ have the following properties: (dontinued)*

(2) The measure $\nu^*_{n,1}$ is a sum of two signed measures on S_n,

$$\nu^*_{n,1} = \omega_n + \omega^*_{n-1},$$

which are characterized for all $n \geq 1$ by the following two properties:

(P1) ω_n is supported on the rectangular partitions $[b^a]$ of S_n,

(P2) ω^*_{n-1} is supported on the almost-rectangular partitions of S_n, those of the form $[d^c, 1]$, and is obtained from the measure ω_{n-1} on S_{n-1} by a simple recipe.

$\mathbb{F}_1$ Splitting Measure-4

Theorem. *The $z = 1$-splitting measures $\nu^*_{n,1}$ have the following properties.*

(2) (continued) For $\lambda \vdash n$,

$$\omega^*_{n-1}(C_\lambda) := \begin{cases} \omega_{n-1}(C_{\lambda'}) & \textit{if } \lambda = [\lambda', 1] \textit{ with } \lambda' \vdash n-1, \\ 0 & \textit{otherwise.} \end{cases}$$

(3) The supports of ω_n and ω^*_{n-1} overlap on the identity conjugacy class $\lambda = [1^n]$, viewing $[1^n]$ as being both rectangular and almost-rectangular.

Here for $n \geq 2$,

$$\nu^*_{n,1}(C_{[1^n]}) = \frac{(-1)^n}{n(n-1)}.$$

Auxiliary splitting measure ω_n

Theorem *(Auxiliary measures ω_n)* [**Number Theory Properties**]

(1) The measure ω_n for each $n \geq 1$ and each partition $\lambda \vdash n$, is:

$$\omega_n(C_\lambda) = \begin{cases} (-1)^{a+1}\frac{\phi(b)}{n} & \text{if } \lambda = [b^a] \text{ for the factorization } n = ab, \\ 0 & \text{otherwise.} \end{cases}$$

where $\phi(n)$ is Euler's totient function.

(2) The measure ω_n is supported on exactly $d(n)$ conjugacy classes, where $d(n)$ counts the number of positive divisors of n. It is a nonnegative measure for odd n and is a strictly signed measure for even n.

Auxiliary Splitting Measure- ω_n- 2

Theorem *(Structure of auxiliary measures ω_n)* **[Support properties]** *The measures ω_n on S_n have the following properties.*

(1) For all n the absolute value measure $|\omega_n|$ has total mass 1, so is a probability measure.

(2) For $n = 2m + 1$ the measure ω_{2m+1} is nonnegative and has total mass 1, so is a probability measure. Its support is on even permutations, and restricted to the alternating group A_n is a probability measure.

(3) For $n = 2m$ the measure ω_{2m} is a signed measure having total signed mass 0. It is nonnegative on odd permutations and nonpositive on even permutations. The measure $-2\omega_{2m}|_{A_{2m}}$ restricted to the alternating group A_{2m} is a probability measure.

Auxiliary Splitting Measure- ω_n- 3

Theorem *(Structure of auxiliary measures ω_n)*
[Multiplicative properties]
The measures ω_n on S_n have the following properties.

(4) (Conjugacy Class Multiplicativity) *The family of all measures ω_n has an internal product structure compatible with multiplication of integers. Set $n = \prod_i {p_i}^{e_i}$.*
Then for any factorization $ab = n$, there holds

$$\omega_n(C_{[b^a]}) = \prod_i \omega_{{p_i}^{e_i}}(C_{[(b_i)^{a_i}]}), \tag{1}$$

in which $b_i = p_i^{e_{i,2}}$ (resp. $a_i = p_i^{e_{i,1}}$) represent the maximal power of p_i dividing b (resp. a), so that $e_{i,1} + e_{i,2} = e_i$. We allow some or all values $e_{i,j} = 0$ for $j = 1, 2$, so that values $b = 1$ (resp. $a_i = 1$) are allowed.

1-Splitting Measure for $N = 4$

$n=4$	$C_{[1^4]}$	$C_{[2^2]}$	$C_{[4]}$	$C_{[3,1]}$	Other
$\lvert C_\lambda \rvert$	1	3	6	8	6
ω_4	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	0	0
ω_3^*	$\frac{1}{3}$	0	0	$\frac{2}{3}$	0
$\nu_{4,1}^*$	$\frac{1}{12}$	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{2}{3}$	0

TABLE: Symmetric group S_4.

1-Splitting Measure for $N = 5$

$n = 5$	$C_{[1^5]}$	$C_{[5]}$	$C_{[2^2,1]}$	$C_{[4,1]}$	Other
$\lvert C_\lambda \rvert$	1	24	15	30	50
ω_5	$\frac{1}{5}$	$\frac{4}{5}$	0	0	0
ω_4^*	$-\frac{1}{4}$	0	$-\frac{1}{4}$	$\frac{1}{2}$	0
$\nu_{5,1}^*$	$-\frac{1}{20}$	$\frac{4}{5}$	$-\frac{1}{4}$	$\frac{1}{2}$	0

TABLE A: Symmetric group S_5.

1-Splitting Measure for $N = 8$

$n = 8$	$C_{[1^8]}$	$C_{[2^4]}$	$C_{[4^2]}$	$C_{[8]}$	$C_{[7,1]}$	Other
$\|C_\lambda\|$	1	105	1260	5040	5760	28154
ω_8	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{4}$	$\frac{1}{2}$	0	0
ω_7^*	$\frac{1}{7}$	0	0	0	$\frac{6}{7}$	0
$\nu_{8,1}^*$	$\frac{1}{56}$	$-\frac{1}{8}$	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{6}{7}$	0

TABLE A: Symmetric group S_8.

1-Splitting Measure for $N = 9$

$n = 9$	$C_{[1^9]}$	$C_{[3^3]}$	$C_{[9]}$	$C_{[2^4,1]}$	$C_{[4^2,1]}$	$C_{[8,1]}$	Other
$\lvert C_\lambda \rvert$	1	2240	40320	945	11340	45360	262674
ω_9	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{3}$	0	0	0	0
ω_8^*	$-\frac{1}{8}$	0	0	$-\frac{1}{8}$	$-\frac{1}{4}$	$\frac{1}{2}$	0
$\nu_{9,1}^*$	$-\frac{1}{72}$	$\frac{2}{9}$	$\frac{2}{3}$	$-\frac{1}{8}$	$-\frac{1}{4}$	$\frac{1}{2}$	0

TABLE : Symmetric group S_9.

Observations on ω_n

- There is interesting number theoretic structure in the $\mathbb{F}_1$ distribution. It is visible in the rectangular conjugacy class support of ω_n.

- How does this number theory structure relate to the representation theory of S_n? The representation structure of S_{n+1} and S_n are quite different, and the measure relates conjugacy classes on different S_k with $k|n$ (or $k|(n-1)$) preserving rectangles.

Observations-2

- **Motivating Question.** *Is there a representation-theoretic interpretation of these measures, viewing them not as conjugacy class functions but as characters of some (rational) representation of S_n?*

- The measure is a class function, so it is automatically a rational linear combination of irreducible characters. But is it something understandable?

- **Answer:** *Yes.*

  Nice answer at $z = 1$, and also at $z = -1$.

Representation-Theoretic Interpretation

• We must rescale the measure by the factor $n!$. Here $n!$ is the minimal multiplier making its values on group elements integers.

• The discovery is: this rescaled measure is the character of a **virtual representation**, i.e. an integer linear combination of irreducible representations. Secondly, the multiplicities of the irreducible representations are “small".

Representation Interpretation-2

Theorem ((Representation theory interpretation of $n!|\omega_n|$)

(1) For all $n \geq 1$ the class function $n!|\omega_n|(g)$ for $g \in S_n$ is the character of the induced representation

$$\rho_n^+ := Ind_{C_n}^{S_n}(\chi_{\mathsf{triv}}),$$

from the cyclic group C_n generated by an n-cycle of S_n, carrying the trivial representation χ_{triv}.

(2) The representation ρ_n^+ is of degree $(n-1)!$ and is a (highly) reducible representation, for $n \geq 3$.

(3) The trivial representation occurs in ρ_n^+ with multiplicity 1. The sign representation χ_{sgn} occurs in ρ_n^+ if and only if n is odd, and then occurs with multiplicity 1.

Representation Interpretation-2

Theorem. (Representation theory interpretation of $|\omega_n|$)
(1) If $n = 2m$ is even then the class function $-n!\omega_n$ is the character of the induced representation

$$\rho_{2m}^{-} := Ind_{C_{2m}}^{S_{2m}}(\chi_{\mathsf{sgn}})$$

from the cyclic group C_{2m} of a $2m$-cycle in S_{2m}, carrying on it the sign representation χ_{sgn}.

(2) The representation ρ_{2m}^{-} is of degree $(2m-1)!$ and is a (highly) reducible representation for $m \geq 2$.

(3) The trivial representation χ_{triv} of S_{2m} occurs in ρ_{2m}^{-} with multiplicity 0 and the sign representation χ_{sgn} occurs with multiplicity 1.

Representation Interpretation-3

Theorem. (Relation of ω^*_{n-1} and ω_{n-1})

(1) For each $n \geq 2$ *the class function* $(-1)^n\, n!\, \omega^*_{n-1}(g)$ *for* $g \in S_n$ *is the character* $\psi_n^{(L)}$ *of the induced representation*

$$\rho_n^{(L)} := \mathrm{Ind}_{C_{n-1}}^{S_n}((\chi_{\mathrm{sgn}})^n),$$

with cyclic group $C_{n-1} \subset S_n$ *generated by an* $(n-1)$*-cycle that holds the symbol* n *fixed. The representation* $\rho_n^{(L)}$ *is of degree* $n \cdot (n-2)!$.

(2) The representation $\rho_n^{(L)}$ *is also given as the induced representation*

$$\rho_n^{(L)} = \mathrm{Ind}_{S_{n-1}}^{S_n}(\rho^{\epsilon}_{n-1}),$$

in which the representation ρ^{ϵ}_{n-1} *with* $\epsilon = (-1)^n$ *is the representation of* S_{n-1} *having character* $(-1)^n(n-1)!\, \omega_{n-1}$, *viewing* S_{n-1} *as the subset of* S_n *of all permutations holding the symbol* n *fixed.*

Representation Interpretation-4

Theorem. (1-Splitting measure as Character of Virtual Repn.)

The class function $(-1)^{n-1}n!\nu_{n,1}(g)$ *for* $g \in S_n$ *of the rescaled* 1*-splitting measure* $\nu^*_{n,1}(\cdot)$ *is the character of a virtual representation* $\rho_{n,1}$ *of* S_n*.*

(1) For even $n = 2m$*, we have*

$$\rho_{2m,1} = (\rho^-_{2m})^{-1} \oplus \rho^{(L)}_{2m-1}.$$

(2) For odd $n = 2m+1$ *we have*

$$\rho_{2m+1,1} = \rho^+_{2m+1} \oplus (\rho^{(L)}_{2m})^{-1}.$$

z-splitting measure for $z = -1$: conjugacy class data

Theorem. ((-1)-splitting measure, Conjugacy Class support)

(1) For $n \geq 2$ for $z = -1$ the splitting measure $\nu^*_{n,-1}$ is a nonnegative measure supported on the conjugacy classes C_λ with $\lambda = [1^n]$, the identity class, and $\lambda = [2, 1^{n-2}]$, the class of a 2-cycle.

(2) It has equal mass $\nu_{n,-1}(C_\lambda) = \frac{1}{2}$ on these two classes.

z-splitting measure for $z = -1$: character of a representation.

Theorem. ((-1)-splitting measure, representation theory interpretation)

(1) For $n \geq 2$ the rescaled (-1)-splitting measure $n!\nu^*_{n,-1}$ is the character of a representation $\tilde{\rho}_n$ of S_n.

(2) This representation is realized as a permutation representation, given as the induced representation $\mathrm{Ind}_{C_2}^{S_n}(\chi_{\mathrm{triv}})$ where $C_2 = \{e, (12)\}$ is a group given by a 2-cycle.

Part IV: Splitting Measure Coefficients and Representation Theory

- Recall that the z**-splitting measures**

$$\nu^*_{n,z}(C_\lambda) = \sum_{k=0}^{n-1} \alpha_n^k(C_\lambda)\frac{1}{z^k}$$

  on conjugacy classes C_λ of S_n are Laurent polynomials in z.

- **Consequence.** Each Laurent coefficient $\{\alpha_n^k(C_\lambda) : \lambda \vdash n\}$ defines a class function on S_n.

- **Question.** *Is there a "nice" representation-theoretic interpretation of these class functions?*

Splitting Measure Coefficients and Representation Theory-2

- **Evidence.** The $k = 0$ coefficient is the uniform measure, corresponds to the “trivial" representation.

  The $z = 1$ and $z = -1$ cases give “nice" representations which are simple integer linear combinations of the α_n^k as k varies.

- Positive answer (next slide):
  These measures are a constructed from conjugacy class data, dual to representation-theory data, so a positive answer is not obvious.

- As before, we rescale the splitting measures by the factor $n!$.

Splitting Measure Coefficients and Representation Theory-3

- **Answer.** (Hyde and L. (2016))The rescaled coefficients $n!(-1)^j \alpha_n^k(\cdot)$ are characters of the S_n-action on a piece of the k-th cohomology group of the pure braid group P_n, carrying its S_n-action. This piece comes from the k-th cohomology group of a quotient manifold Y_n of the A_n-braid hyperplane arrangment.

- **Elaboration of Answer.** The cycle polyomials $N_\lambda(z)$ viewed as a function of λ have coefficients interpretable in terms of pure braid group cohomology characters. (Lehrer (1987)) The division by $\frac{1}{z^n - z^{n-1}}$ is a unique new feature of the splitting measures that requires a splitting of the braid group cohomology into smaller pieces, explainable in terms of Y_n.

Pure Braid Cohomology Interpretation of Cycle Polynomials

Theorem. (Lehrer (1987))
(Character interpretation of cycle polynomial coefficients)

Let λ be a partition of n and $N_\lambda(z)$ be a cycle polynomial. Then

$$N_\lambda(z) = \frac{|C_\lambda|}{n!} \sum_{k=0}^{n} (-1)^k h_n^k(\lambda) z^{n-k}.$$

where h_n^k is the character of the kth cohomology of the pure braid group $H^k(P_n, \mathbb{Q})$, viewed as an S_n-representation.

- Lehrer's result is stated in a rather different form than above.

Cohomology Interpretation of Splitting Measure Coefficients

Theorem. (Hyde-L. (2016))
(Character interpretation of splitting measure coefficients)

For each $n \geq 1$ *and* $0 \leq k \leq n-1$ *there is an* S_n*-subrepresentation* A_n^k *of* $H^k(P_n, \mathbb{Q})$ *with character* χ_n^k *such that for each partition* λ *of* n*,*

$$\nu_{n,z}^*(C_\lambda) = \frac{|C_\lambda|}{n!} \sum_{k=0}^{n-1} \chi_n^k(\lambda)\Big(-\tfrac{1}{z}\Big)^k.$$

Thus the splitting measure coefficient $\alpha_n^k(C_\lambda)$ *is*

$$\alpha_n^k(C_\lambda) = |C_\lambda|\, \alpha_n^k(\lambda) = (-1)^k \frac{|C_\lambda|}{n!} \chi_n^k(\lambda).$$

Arnold Theorem

Theorem (Arnold (1969)) *The cohomology ring* $H^\bullet(P_n, \mathbb{Q})$ *of the pure braid group as an* S_n*-module is given by an isomorphism of graded* S_n*-algebras*

$$H^\bullet(P_n, \mathbb{Q}) \cong \Lambda^\bullet[\omega_{i,j}]/\langle R_{i,j,k} \rangle,$$

where $1 \leq i, j, k \leq n$ *are distinct,* $\omega_{i,j} = \omega_{j,i}$ *have degree* 1*, and*

$$R_{i,j,k} = \omega_{i,j} \wedge \omega_{j,k} + \omega_{j,k} \wedge \omega_{k,i} + \omega_{k,i} \wedge \omega_{i,j}.$$

An element $\sigma \in S_n$ *acts on* $\omega_{i,j}$ *by* $\sigma \cdot \omega_{i,j} = \omega_{\sigma(i),\sigma(j)}$*.*

- The **pure braid group** P_n is the subgroup of the n-strand braid group B_n that acts as the identity permutation on the braid strands.

Pure Braid Cohomology Splitting-1

• $H^\bullet(P_n, \mathbb{Q})$ is the cohomology ring of the complement in $\mathbb{C}^n$ of the A_n-braid arrangement of (complex) hyperplanes $\{H_{i,j} := (z_i = z_j) : 1 \leq i < j \leq n\}$.

• **Proposition.** The cup product with the S_n-equivariant 1-form $\omega = \sum_{1 \leq i < j \leq n} \omega_{i,j}$ leads to a splitting of cohomology as S_n-modules A_n^k:

$$H^k(P_n, \mathbb{Q}) \cong A_n^{k-1} \oplus A_n^k.$$

for certain S_n-modules A_n^k.

Cohomology Dimensions

- The Betti numbers

$$\dim\left(H^k(P_n, \mathbb{Q})\right) = \left[{n \atop n-k}\right],$$

are *unsigned Stirling numbers (of the first kind)*. They are given by the rising factorial identity

$$\prod_{k=0}^{n-1}(x+k) = \sum_{k=0}^{n-1}\left[{n \atop k}\right]x^k.$$

- The Betti numbers

$$\dim(A_n^k) = \sum_{j=0}^{k}(-1)^j\left[{n \atop n-k+j}\right].$$

Betti Number Table: Pure Braid Group Cohomology

$n \setminus k$	0	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0
3	1	3	2	0	0	0	0	0	0
4	1	6	11	6	0	0	0	0	0
5	1	10	35	50	24	0	0	0	0
6	1	15	85	225	274	120	0	0	0
7	1	21	175	735	1624	1764	720	0	0
8	1	28	322	1960	6769	13132	13068	5040	0
9	1	36	546	4536	22449	67284	118124	109584	40320

Betti numbers of pure braid group cohomology $H^k(P_n, \mathbb{Q})$.

Betti Number Table: A_n^k as Cohomology

$n \setminus k$	0	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0
3	1	2	0	0	0	0	0	0
4	1	5	6	0	0	0	0	0
5	1	9	26	24	0	0	0	0
6	1	14	71	154	120	0	0	0
7	1	20	155	580	1044	720	0	0
8	1	27	295	1665	5104	8028	5040	0
9	1	35	511	4025	18424	48860	69264	40320

dim(A_n^k)

A_n^k as Cohomology of Quotient Space Y_n

Theorem.

Let $Y_n = \mathsf{PConf}_n(\mathbb{C})/\mathbb{C}^\times$ be the quotient of pure configuration space by the free $\mathbb{C}^\times$ action. The symmetric group S_n acts on $\mathsf{PConf}_n(\mathbb{C})/\mathbb{C}^\times$ by permuting coordinates. Then for each $k \geq 0$ we have an isomorphism of S_n-modules

$$H^k(\mathsf{PConf}_n(\mathbb{C})/\mathbb{C}^\times, \mathbb{Q}) \cong A_n^k.$$

Part V: Consequences/Applications

The explanation of the splitting measure coefficients α_n^k in terms of braid group cohomology representations has several consequences:

(1) It gives a “nice" representation theory interpretation for splitting measures at values $z = \pm\frac{1}{m}$ for all integers $m \geq 1$.

(2) It gives (possibly new) information on the internal structure of the pure braid group cohomology ring.

(3) The individual α_n^k exhibit representation stability as n varies for k fixed.

(4) $\mathbb{F}_1$-splitting measure as S_n-equivariant Euler characteristicof Y_n

Consequence 1: z-Splitting measures at $z = \pm\frac{1}{m}$

- **Theorem.**

  (1) The z-splitting measures for $z = -\frac{1}{m}$ (rescaled by the factor $n!$) are the characters of (reducible) representation of S_n.

  (2) The z-splitting measures for $z = -\frac{1}{m}$ (rescaled by the factor $n!$) are the difference of two characters of (reducible) representation of S_n.

- This result explains that representation-theory structure exists at $z = \pm 1$. But it does not explain the *small support* and the *number theory internal structure* in the 1-splitting measure.

Reconstruct z-Splitting measures by interpolation

- **Dichotomy.** We can recover the z-splitting data for all n simultaneously by interpolation in two independent ways:

  (i) By interpolation of measure data at $z = p$ for all primes p.

  (ii) By interpolation of representation data at all $z = -\frac{1}{m}$, $m \geq 1$ (alternatively by data at all $z = \frac{1}{m}$, $m \geq 1$).

- **Problem.** Find a direct interpretation of the data at $z = \pm\frac{1}{m}$.

Consequence 2: Structure of Pure Braid Group Cohomology Ring

Theorem. (Hyde-L (2016))

(Pure Braid Cohomology =Twisted Regular Representation of S_n)

Let triv_n, sgn_n, *and* $\mathbb{Q}[S_n]$ *be the trivial, sign, and regular representations of* S_n *respectively. Then there is an isomorphism of* S_n*-representations,*

$$\bigoplus_{k=0}^{n} H^k(P_n, \mathbb{Q}) \otimes \mathsf{sgn}_n^{\otimes k} \cong \mathbb{Q}[S_n].$$

Here $\mathsf{sgn}_n^{\otimes k} \cong \mathsf{triv}_n$ *or* sgn_n *according to whether* k *is even or odd.*

• Here $\mathbb{Q}[S_n]$ is the regular representation of S_n. The proof uses $H^k(P_n, \mathbb{Q}) \simeq A_n^{k-1} \oplus A_n^k$.

Consequence 3: Representation Stability for A_n^k

Theorem. (Hersh-Reiner (2015))
(Representation stability for A_n^k)

For each fixed $k \geq 1$, the sequence of S_n-representations A_n^k with characters χ_n^k are representation stable, and stabilizes sharply at $n = 3k + 1$.

• Hersh-Reiner (2015) study a different cohomology, which one shows is isormorphic to A_n^k as an S_n-module.

Representation Stabilization of $H^1(P_n, \mathbb{Q})$

n	$\dim H^1$	$H^1(P_n, \mathbb{Q})$	$\dim A_n^1$	A_n^1
2	1	$[2]$	0	0
3	3	$[3] \oplus [2,1]$	2	$[2,1]$
4	6	$[4] \oplus [3,1] \oplus [2,2]$	5	$[3,1] \oplus [2,2]$
5	10	$[5] \oplus [4,1] \oplus [3,2]$	9	$[4,1] \oplus [3,2]$
n	$\begin{bmatrix} n \\ n-1 \end{bmatrix}$	$[n] \oplus [n-1,1]$ $\oplus [n-2,2]$	$\begin{bmatrix} n \\ n-1 \end{bmatrix} - 1$	$[n-1,1] \oplus [n-2,2]$

Irreducible decompositions for $H^1(P_n, \mathbb{Q})$ and A_n^1.

(Here λ abbreviates the irreducible representation $\mathcal{S}^\lambda$.)

Representation Stabilization of $H^2(P_n, \mathbb{Q})$

n	$\dim H^2$	$H^2(P_n, \mathbb{Q})$
3	2	$[2,1]$
4	11	$2[3,1] \oplus [2,2] \oplus [2,1,1]$
5	35	$2[4,1] \oplus 2[3,2] \oplus 2[3,1,1] \oplus [2,2,1]$
6	85	$2[5,1] \oplus 2[4,2] \oplus 2[4,1,1] \oplus [3,3] \oplus 2[3,2,1]$
7	175	$2[6,1] \oplus 2[5,2] \oplus 2[5,1,1] \oplus [4,3] \oplus 2[4,2,1] \oplus [3,3,1]$
8	322	$2[7,1] \oplus 2[6,2] \oplus 2[6,1,1] \oplus [5,3] \oplus 2[5,2,1] \oplus [4,3,1]$
n	$\begin{bmatrix} n \\ n-2 \end{bmatrix}$	$2[n-1,1] \oplus 2[n-2,2] \oplus 2[n-2,1,1] \oplus [n-3,3]$ $\oplus 2[n-3,2,1] \oplus [n-4,3,1]$

Irreducible decomposition for $H^2(P_n, \mathbb{Q})$

Representation Stabilization of A_n^2

n	dim A_n^2	A_n^2
3	0	0
4	6	$[3,1] \oplus [2,1,1]$
5	26	$[4,1] \oplus [3,2] \oplus 2[3,1,1] \oplus [2,2,1]$
6	71	$[5,1] \oplus [4,2] \oplus 2[4,1,1] \oplus [3,3] \oplus 2[3,2,1]$
7	155	$[6,1] \oplus [5,2] \oplus 2[5,1,1] \oplus [4,3]$ $\oplus\, 2[4,2,1] \oplus [3,3,1]$
8	295	$[7,1] \oplus [6,2] \oplus 2[6,1,1] \oplus [5,3]$ $\oplus\, 2[5,2,1] \oplus [4,3,1]$
n	$\begin{bmatrix} n \\ n-2 \end{bmatrix} - \begin{bmatrix} n \\ n-1 \end{bmatrix} + 1$	$[n-1,1] \oplus [n-2,2] \oplus 2[n-2,1,1]$ $\oplus [n-3,3] \oplus 2[n-3,2,1] \oplus [n-4,3,1]$

Irreducible decomposition for A_n^2

Consequence 4: $\mathbb{F}_1$-splitting as an equivariant Euler Characteristic

• General result:

Proposition. *Let X be an (algebraic) variety defined over a ring of algebraic integers. Denote by $X_{\mathbb{C}}$ (resp. $X_{\mathbb{F}_q}$) the set of $\mathbb{C}$-points (resp. $\mathbb{F}_q$-points) of X. Suppose that there exists a polynomial $P(X) \in \mathbb{Z}[X]$ such that $|X_{\mathbb{F}_q}| = P(q)$ for infinitely many prime powers q. Then the Euler-Poincaré characteristic (with compact support) of $X_{\mathbb{C}}$ in $H^\bullet(X, \mathbb{C})$ is given by $\chi(X_{\mathbb{C}}) = P(1)$.*

• ("classical" result) Caldero and Chapoton [Comm. Math. Helv. **81** (2006), Lemma 3.5], Reinecke [IMRN 2006, ID 70456, 1–19] (Representations of quivers)

• Our framework has *Laurent polynomials*.

Equivariant Cohomology Interpretation

Theorem. (z*-splitting measures as equivariant Poincaré polynomials*)

Let $Y_n = \mathsf{PConf}_n(\mathbb{C})/\mathbb{C}^{\times}$. *Setting* $w = -\frac{1}{z}$, *then for each* $g \in S_n$ *the* z*-splitting measure is given by the scaled equivariant Poincaré polynomial*

$$\nu^*_{n,z}(g) = \frac{1}{n!} \sum_{k=0}^{n-1} Trace(g|H^k(Y_n, \mathbb{Q}))w^k,$$

attached to the complex manifold Y_n, *where* g *acts as a permutation of the coordinates.*

1-Splitting Measure as an Equivariant Euler Characteristic

- **Conclusion:** The rescaled $\mathbb{F}_1$-splitting measure is the value at $z = 1$ (so that $w = -1$), yielding:

  Corollary. *The rescaled* 1*-splitting measure* $\frac{n!}{|C_\lambda|}\nu^*_{n,1}(C_\lambda)$ *is the* equivariant Euler characteristic *of the space* $Y_n(\mathbb{C})$ *with respect to its* S_n*-action.*

Conclusion

- There is still some mystery in the 1-splitting measure, concerning its small support and number-theoretic values.

- The 1-splitting measure for each n combines stable and unstable cohomology. It is not entirely explained by representation stability. (Perhaps one can go to a stable limit and ask what that measure is.)

- Representation stability is now being extended outside the stable range to a theory of secondary stability of unstable cohomology (under development) by others. (Jeremy Miller and Jennifer Wilson)[arXiv, Nov 2016]

Thank you for your attention!