COMPLEMENTS TO LI'S CRITERION FOR THE RIEMANN HYPOTHESIS

ENRICO BOMBIERI AND JEFFREY C. LAGARIAS

Abstract

In a recent paper Xian-Jin Li showed that the Riemann Hypothesis holds if and only if $\lambda_n = \sum_\rho \left[1-(1-1/\rho)^n\right]$ has $\lambda_n > 0$ for $n = 1, 2, 3, \ldots$ where ρ runs over the complex zeros of the Riemann zeta function. We show that Li's criterion follows as a consequence of a general set of inequalities for an arbitrary multiset of complex numbers ρ and therefore is not specific to zeta functions. We also give an arithmetic formula for the numbers λ_n in Li's paper, via the Guinand-Weil explicit formula and relate the conjectural positivity of λ_n to Weil's criterion for the Riemann Hypothesis.

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§1. **Introduction.** In a recent paper Xian-Jin Li [3] obtained an interesting criterion for the validity of the Riemann Hypothesis. His criterion can be stated in terms of the Riemann ξ-function

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

and the sequence

$$\lambda_n = \frac{1}{(n-1)!}\frac{d^n}{ds^n}\left[s^{n-1}\log\xi(s)\right]\Big|_{s=1}$$

for $n = 1, 2, 3, \ldots$, in the form that *the Riemann Hypothesis is equivalent to the statement that* $\lambda_n > 0$ *for every positive integer* n. He also showed that an identical result applies to the Riemann Hypothesis for the Dedekind zeta function of a number field.

The number λ_n can be written in terms of the complex zeros ρ of the Riemann zeta function (or Dedekind zeta function) as

$$\lambda_n = \sum_\rho \left[1 - \left(1 - \frac{1}{\rho}\right)^n\right]$$

where the sum* over ρ is understood as

$$\sum_\rho = \lim_{T\to\infty} \sum_{|\Im(\rho)|\le T} .$$

* We denote by $\Re(z)$ and $\Im(z)$ the real and imaginary part of the complex number z

Li's proof uses the positivity of certain numbers a_j and b_m (see [3, formulae (1.6) and (3.5)] obtained from an appropriate integral representation of the zeta function and one may wonder to what extent Li's criterion is specific for zeta functions and whether there is an arithmetic interpretation of it.

In this paper, Theorem 1 proves a general criterion for a multiset* of complex numbers to lie in the half-plane $\Re(s) \le \frac{1}{2}$. Then the Corollary of Theorem 1 is a criterion for a multiset which is invariant by complex conjugation and the *functional equation map* $s \mapsto 1-s$ to lie on the critical line $\Re(s) = \frac{1}{2}$. In the special case of the multiset of non-trivial zeros of a zeta function, we recover Li's theorem. These results show that Li's criterion is a consequence of a general set of inequalities that is not specific to zeta functions.

Theorem 2 and the final considerations in this paper give an arithmetic formula for the numbers λ_n in Li's paper via the Guinand-Weil explicit formula. Their conjectural positivity is then related to Weil's well-known criterion for the Riemann Hypothesis.

§2. Li's criterion. In this and the following sections, convergence of a sum over a multiset $\mathcal{R}$ of complex numbers ρ is understood as the existence of the limit

$$\sum_{\mathcal{R}} = \lim_{T\to\infty} \sum_{|\Im(\rho)|\le T} \tag{$*$}$$

where in the sum each ρ occurs according to its multiplicity. In this case, we say that the sum is $*$*-convergent.*

In what follows, for $\rho = 0$ the quantity $(1-1/\rho)^{-n} = (\rho/(\rho-1))^n$ is interpreted as 0 if n is a positive integer and as ∞ if n is a negative integer. We have

Lemma 1. *Let $\mathcal{R}$ be a multiset of complex numbers and suppose that $0 \notin \mathcal{R}$ if n is a positive integer, $1 \notin \mathcal{R}$ if n is a negative integer, and*

$$\sum_{\mathcal{R}} \frac{1+|\Re(\rho)|}{(1+|\rho|)^2} < +\infty. \tag{2.1}$$

Then for all integers n the sum

$$\sum_{\mathcal{R}} \Re\big[1-(1-1/\rho)^n\big]$$

converges absolutely.

Moreover if $\sum_{\mathcal{R}} 1/\rho$ is $*$-convergent then $\lambda_n = \sum_\rho \big[1-(1-1/\rho)^n\big]$ is also $*$-convergent.

Remark. If $\mathcal{R}$ is invariant by complex conjugation, $0 \notin \mathcal{R}$ and (2.1) holds, then $\sum 1/\rho$ is $*$-convergent to a real value.

* A multiset is a set whose elements have positive integral multiplicities assigned to them

Proof: We leave the proof of these assertions to the reader, as an easy exercise.

We are now ready to state our first result.

Theorem 1. *Let $\mathcal{R}$ be a multiset of complex numbers ρ such that*

(i) $1 \notin \mathcal{R}$*;*
(ii) $\sum_\rho (1 + |\Re(\rho)|)/(1+|\rho|)^2 < +\infty$.

Then the following conditions are equivalent:

(a) $\Re(\rho) \le 1/2$ *for every* ρ*;*
(b) $\sum_\rho \Re\big[1 - (1-1/\rho)^{-n}\big] \ge 0$ *for* $n = 1, 2, 3, \ldots$*;*
(c) for every fixed $\varepsilon > 0$ *there is a constant* $c(\varepsilon)$ *such that*

$$\sum_\rho \Re\big[1-(1-1/\rho)^{-n}\big] \ge -c(\varepsilon)e^{\varepsilon n}, \quad n = 1, 2, 3, \ldots.$$

Remarks. From a formal point of view if

$$f(z) = \prod_{\rho \in \mathcal{R}} \left(1 - \frac{z}{\rho}\right) \tag{2.3}$$

and if

$$\lambda_n = \sum_{\rho \in \mathcal{R}} \left[1 - (1-1/\rho)^n\right]$$

then one has

$$\frac{\mathrm{d}}{\mathrm{d}z} \log f\left(\frac{1}{1-z}\right) = \frac{1}{(1-z)^2} \frac{f'}{f}\left(\frac{1}{1-z}\right) = \sum_{n=0}^{\infty} \lambda_{-n-1} z^n. \tag{2.4}$$

By means of this formula, one could attack Theorem 1 by methods of conformal mapping and complex function theory. In fact, there is a function-theoretic interpretation of Theorem 1 when $f(z)$ is well defined. The mapping $w = 1/(1-z)$ is a conformal mapping of the open unit disk $|z| < 1$ onto the open half-plane $\Re(w) > \frac{1}{2}$, hence condition (a) asserts that $\frac{\mathrm{d}}{\mathrm{d}z} \log f(1/(1-z))$ is holomorphic in the unit disk, which is equivalent to $\limsup |\lambda_{-n-1}|^{1/n} \le 1$. Conditions (b) and (c) assert that this growth condition can be replaced by one-sided conditions on $\Re(\lambda_{-n-1})$. In this paper we shall follow a direct approach to proving Theorem 1 which applies even if $f(z)$ is not well defined.

Proof: We have $\rho \ne 1$ and

$$|1 - 1/\rho|^{-2} = 1 + (2\beta - 1)/|1-\rho|^2$$

where $\beta = \Re(\rho)$, therefore (a) implies $|1-1/\rho|^{-1} \le 1$ for every ρ. *A fortiori* we have $\Re\big[1-(1-1/\rho)^{-n}\big] \ge 0$ for every positive integer n, hence (a) implies (b).

It is clear that (b) implies (c).

Now suppose that (a) does not hold, so that $\Re(\rho) > \frac{1}{2}$ for at least one $\rho \in \mathcal{R}$.

We use again the fact that $|1-1/\rho|^{-2} = 1+(2\beta-1)/|1-\rho|^2$. Since $(2\beta-1)/|1-\rho|^2$ tends to zero, the maximum over ρ of this quantity is attained and there are finitely many elements $\rho_k \in \mathcal{R}$, $k = 1, \ldots, K$, such that $|1-1/\rho|^{-1} = 1+t = \max$. Note that $t > 0$ because $\beta > \frac{1}{2}$ for at least one ρ.

For any other ρ we have $|1-1/\rho|^{-1} \le 1+t-\delta$ for a fixed small $\delta > 0$. Let ϕ_k be the argument of $1-1/\rho_k$. Then

$$1-(1-1/\rho_k)^{-n} = 1-(1+t)^n e^{-in\phi_k}.$$

For $\rho \ne \rho_k$ we have $|1-1/\rho|^{-n} = O\big((1+t-\delta)^n\big)$ and also

$$\Re\big[1-(1-1/\rho)^{-n}\big] = O\big((n\,|\Re(\rho)|+n^2)/|\rho|^2\big)$$

as soon as $|\rho| > n$, as one easily verifies using

$$1-\left(\frac{\rho}{\rho-1}\right)^n = 1-e^{n(\frac{1}{\rho}+\frac{1}{2\rho^2}+\ldots)}.$$

Hence the sum over $|\rho| > n$ is $O(n^2)$ because $\sum(1+|\Re(\rho)|)/(1+|\rho|)^2$ is convergent. The number of elements in the sequence with $|\rho| \le n$ is $O(n^2)$, again because $\sum(1+|\Re(\rho)|)/(1+|\rho|)^2$ is convergent. Hence elements other than ρ_k contribute at most $O(n^2(1+t-\delta)^n)$ to $\sum \Re\big[1-(1-1/\rho)^{-n}\big]$, while the remaining elements ρ_k contribute

$$K-(1+t)^n\sum_{k=1}^{K}\cos(n\phi_k).$$

We have shown that

$$\sum_\rho \Re\big[1-(1-1/\rho)^{-n}\big] = K-(1+t)^n\sum_{k=1}^{K}\cos(n\phi_k) + O\big(n^2(1+t-\delta)^n\big).$$

By Dirichlet's theorem on simultaneous Diophantine approximation we can make the sum of cosines arbitrarily close to K, making it plain that $\sum_\rho \Re\big[1-(1-1/\rho)^{-n}\big]$ is infinitely often negative and exponentially large in absolute value as n tends to ∞. Thus the negation of (a) implies the negation of (c), hence (c) implies (a), concluding the proof.

Corollary 1. (Li's Criterion) *Let $\mathcal{R}$ be a multiset of complex numbers ρ such that*

(i) $0, 1 \notin \mathcal{R}$;
(ii) if $\rho \in \mathcal{R}$ then $1-\rho$ and $\overline{\rho}$ are in $\mathcal{R}$, with the same multiplicity as ρ;
(iii) $\sum_\rho (1+|\Re(\rho)|)/(1+|\rho|)^2 < +\infty$.

Then the following conditions are equivalent:

(a) $\Re(\rho) = 1/2$ *for every* ρ;
(b) $\lambda_n = \sum_\rho \big[1-(1-1/\rho)^n\big] \ge 0$ *for* $n = 1, 2, 3, \ldots$;

(c) for every fixed $\varepsilon > 0$ there is a constant $c(\varepsilon)$ such that

$$\sum_{\rho} \left[1 - (1 - 1/\rho)^n\right] \geq -c(\varepsilon)e^{\varepsilon n}, \quad n = 1, 2, 3, \dots .$$

Proof: Conditions (ii) and (iii) ensure that the sums in (b) and (c) are $*$-convergent and real and $\lambda_n = \lambda_{-n}$ for $n = 1, 2, 3, \dots$. The proof is completed by applying Theorem 1 to $\mathcal{R}$ and to $1 - \mathcal{R}$.

Remark. Condition (iii) can be relaxed to $\sum 1/(1 + |\rho|)^2 < +\infty$; we leave the details to the reader.

§3. An arithmetic interpretation. In a well-known paper A. Weil [4] obtained a general formulation of the so-called explicit formula of the theory of prime numbers. An equivalent formula was obtained earlier for the case of the Riemann zeta function by A.P. Guinand [2], under the assumption of the Riemann Hypothesis*. Both papers [2] and [4] are formulated using the Fourier transform. As a historical note, we mention here the fact that there is an earlier paper [1] by Guinand, in which he states an explicit formula [1, (3.5), p.36] using the Mellin transform, with an indication of proof. In this paper Guinand does not assume the validity of the Riemann Hypothesis but says (referring to a formal proof of his summation formula) "This cannot readily be done without a series of involved assumptions, so I shall only derive a formal result which will indicate the type of formula to be expected. Any particular case of the result can be investigated separately." However it should be noted that Guinand's point of view, namely summation formulae arising from self-reciprocal transforms, is quite different from Weil's, where the emphasis is on the arithmetic.

We restate Weil's formula as follows, in the special case of the Riemann zeta function and in terms of the Mellin transform. Its formulation in the general case does not involve additional difficulties, so we leave it as an exercise for the interested reader.

Consider functions $f(x)$ on the positive half-line $(0, \infty)$ such that:

(A) $f(x)$ is continuous and continuously differentiable everywhere except at finitely many points a_i, in which both $f(x)$ and $f'(x)$ have at most a discontinuity of the first kind, and in which one sets

$$f(a_i) = \frac{1}{2}\left[f(a_i + 0) + f(a_i - 0)\right];$$

(B) there is $\delta > 0$ such that $f(x) = O(x^{\delta})$ as $x \to 0+$ and $f(x) = O(x^{-1-\delta})$ as $x \to +\infty$.

Define an involution $f \to \widetilde{f}$ by the formula

$$\widetilde{f}(x) = \frac{1}{x} f\left(\frac{1}{x}\right)$$

* It appears that the Riemann Hypothesis is used in [2] only to obtain the validity of the Explicit Formula for a wider class of test functions. For example, one application of [2] is an exact formula for $N(T)$, the number of complex zeros of $\zeta(s)$ with $0 < \Im(s) \leq T$.

and the Mellin transform of f by

$$\widehat{f}(s) = \int_0^\infty f(x)\, x^{s-1}\, \mathrm{d}x,$$

where, by assumption (B), the integral is absolutely convergent for $-\delta < \Re(s) < 1+\delta$. The inverse Mellin transform formula is

$$f(x) = \frac{1}{2\pi i} \int_{\Re(s)=c} \widehat{f}(s)\, x^{-s}\, \mathrm{d}s,$$

valid for $-\delta < c < 1+\delta$.

With this notation, we have

Explicit Formula. *With $f(x)$ satisfying (A), (B) above, we have*

$$\sum_\rho \widehat{f}(\rho) = \int_0^\infty f(x)\, \mathrm{d}x + \int_0^\infty \widetilde{f}(x)\, \mathrm{d}x - \sum_{n=1}^\infty \Lambda(n) \left\{ f(n) + \widetilde{f}(n) \right\}$$
$$- (\log \pi + \gamma) f(1) - \int_1^\infty \left\{ f(x) + \widetilde{f}(x) - \frac{2}{x^2} f(1) \right\} \frac{x\, \mathrm{d}x}{x^2 - 1}$$

where the first sum ranges over all complex zeros of the Riemann zeta function and is understood as

$$\lim_{T\to\infty} \sum_{|\Im(\rho)|\le T} \widehat{f}(\rho).$$

Proof: This is obtained from Weil's result [4], p.261-262, taking $F(x) = e^{x/2} f(e^x)$ and making the change of variable from e^x to x. We obtain

$$\begin{aligned} \sum_\rho \widehat{f}(\rho) = \int_0^\infty f(x)\, \mathrm{d}x + \int_0^\infty \widetilde{f}(x)\, \mathrm{d}x - \sum_{n=1}^\infty \Lambda(n) \left\{ f(n) + \widetilde{f}(n) \right\} \\ - (\log 2\pi) f(1) - \mathrm{PF} \int_0^\infty f(x)\, \frac{\max(1,x)}{|x - 1/x|}\, \frac{\mathrm{d}x}{x} \end{aligned} \tag{3.1}$$

where PF is the limit

$$\mathrm{PF} \int_0^\infty a(x)\, \frac{\mathrm{d}x}{x} = \lim_{\lambda\to\infty} \left[\int_0^\infty \left(1 - \min(x^\lambda, x^{-\lambda})\right) a(x)\, \frac{\mathrm{d}x}{x} - 2a(1)\log\lambda \right]. \tag{3.2}$$

By splitting the integral over $(0,\infty)$ as a sum of two integrals over $(0,1)$ and $(1,\infty)$ and making the change of variable from x to $1/x$ in the integral over $(0,1)$, we see that

$$\int_0^\infty \left(1 - \min(x^\lambda, x^{-\lambda})\right) f(x)\, \frac{\max(1,x)}{|x - 1/x|}\, \frac{\mathrm{d}x}{x} = \int_1^\infty \left(1 - x^{-\lambda}\right) \left[f(x) + \widetilde{f}(x) \right] \frac{x\, \mathrm{d}x}{x^2 - 1}. \tag{3.3}$$

Now for $\lambda \to \infty$ we have the asymptotic formula

$$\int_1^\infty \left(1 - x^{-\lambda}\right) \frac{1}{x^2} \frac{x\,\mathrm{d}x}{x^2 - 1} = \frac{1}{2}\left[\log(\lambda/2) + \gamma\right] + o(1), \tag{3.4}$$

which may be verified as follows. An easy step shows that we may assume that λ is an even integer $2N$. Then the integral is

$$\int_1^\infty \frac{1 - x^{-2N}}{1 - x^{-2}} \frac{\mathrm{d}x}{x^3} = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2N}$$

and the asymptotic formula follows from the definition of Euler's constant as

$$\gamma = \lim_{N\to\infty} \left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{N} - \log N\right).$$

From (3.2), (3.3) and (3.4) we deduce

$$\begin{aligned}
&\mathrm{PF}\int_0^\infty f(x)\, \frac{\max(1,x)}{|x - 1/x|}\, \frac{\mathrm{d}x}{x} \\
&\quad = (-\log 2 + \gamma) f(1) + \lim_{\lambda\to\infty} \int_1^\infty \left(1 - x^{-\lambda}\right) \left\{ f(x) + \widetilde{f}(x) - \frac{2}{x^2}\, f(1) \right\} \frac{x\,\mathrm{d}x}{x^2 - 1} \\
&\quad = (-\log 2 + \gamma) f(1) + \int_1^\infty \left\{ f(x) + \widetilde{f}(x) - \frac{2}{x^2}\, f(1) \right\} \frac{x\,\mathrm{d}x}{x^2 - 1}
\end{aligned}$$

because here we may interchange the limit and the integral. By (3.1), this completes the proof of the Explicit Formula.

Now we are able to give an arithmetic interpretation to Li's criterion.

Lemma 2. *For $n = 1, 2, 3, \ldots$ the inverse Mellin transform of $1 - (1 - 1/s)^n$ is*

$$g_n(x) = \begin{cases} P_n(\log x) & \text{if} \quad 0 < x < 1 \\ n/2 & \text{if} \quad x = 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$$

where $P_n(x)$ is the polynomial

$$P_n(x) = \sum_{j=1}^n \binom{n}{j} \frac{x^{j-1}}{(j-1)!}\,.$$

Proof: We have

$$\begin{aligned}
\sum_{j=1}^n \binom{n}{j} \frac{1}{(j-1)!} \int_0^1 (\log x)^{j-1} x^{s-1}\,\mathrm{d}x &= \sum_{j=1}^n \binom{n}{j} \frac{1}{(j-1)!} \frac{\mathrm{d}^{j-1}}{\mathrm{d}s^{j-1}} \int_0^1 x^{s-1}\,\mathrm{d}x \\
&= \sum_{j=1}^n \binom{n}{j} \frac{(-1)^{j-1}}{s^j} \\
&= 1 - (1 - 1/s)^n.
\end{aligned}$$

This completes the proof.

Theorem 2. *For $n = 1, 2, 3, \ldots$ we have*

$$\sum_{\rho}\left[1 - (1 - 1/\rho)^n\right] =$$

$$- \sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}\frac{1}{(j-1)!}\lim_{\varepsilon\to 0+}\left\{\sum_{m\le 1/\varepsilon}\frac{\Lambda(m)(\log m)^{j-1}}{m} - \frac{1}{j}(\log 1/\varepsilon)^j\right\}$$
$$+ 1 - (\log 4\pi + \gamma)\,\frac{n}{2} - \sum_{j=2}^{n}(-1)^{j-1}\binom{n}{j}(1 - 2^{-j})\,\zeta(j).$$

Remark. For $n = 1$ this gives the well-known evaluation*

$$\sum_{\rho}\frac{1}{\rho} = 1 + \frac{1}{2}\gamma - \frac{1}{2}\log 4\pi = 0.0230957^{+}.$$

Proof: The function $g_n(x)$ does not satisfy condition (B), so we cannot apply the Explicit Formula directly. Hence for $0 < \varepsilon < 1$ we replace $g_n(x)$ by its truncation

$$g_{n,\varepsilon}(x) = \begin{cases} g_n(x) & \textit{if} \quad \varepsilon < x \le \infty \\ \frac{1}{2}g_n(\varepsilon) & \textit{if} \quad x = \varepsilon \\ 0 & \textit{if} \quad x < \varepsilon. \end{cases}$$

The function $g_{n,\varepsilon}(x)$ satisfies (A) and (B). Then the Explicit Formula yields

$$\begin{aligned}\sum_{\rho}\widehat{g}_{n,\varepsilon}(\rho) = \int_{\varepsilon}^{1} g_{n,\varepsilon}(x)\,\frac{\mathrm{d}x}{x} + \int_{\varepsilon}^{1} g_{n,\varepsilon}(x)\,\mathrm{d}x - \sum_{m\le 1/\varepsilon}\frac{\Lambda(m)}{m}\,g_{n,\varepsilon}\left(\frac{1}{m}\right) \\ - (\log\pi + \gamma)g_{n,\varepsilon}(1) - \int_{1}^{1/\varepsilon}\left[\frac{1}{x}g_{n,\varepsilon}\left(\frac{1}{x}\right) - \frac{2}{x^2}g_{n,\varepsilon}(1)\right]\frac{x\,\mathrm{d}x}{x^2-1}.\end{aligned} \tag{3.5}$$

We want to compute the limit of this formula for $\varepsilon \to 0$. For the right-hand side, we have

$$\begin{aligned}\int_{\varepsilon}^{1} g_{n,\varepsilon}(x)\,\frac{\mathrm{d}x}{x} &= \sum_{j=1}^{n}\binom{n}{j}\frac{1}{(j-1)!}\int_{\varepsilon}^{1}(\log x)^{j-1}\frac{\mathrm{d}x}{x} \\ &= -\sum_{j=1}^{n}\binom{n}{j}\frac{1}{j!}(\log\varepsilon)^j\end{aligned} \tag{3.6}$$

* H. Davenport, *Multiplicative Number Theory*, Markham 1967, pp.83-84

and

$$\begin{aligned}
\int_\varepsilon^1 g_{n,\varepsilon}(x)\,\mathrm{d}x &= \sum_{j=1}^n \binom{n}{j} \frac{1}{(j-1)!} \int_\varepsilon^1 (\log x)^{j-1}\,\mathrm{d}x \\
&= \sum_{j=1}^n \binom{n}{j} \frac{1}{(j-1)!} \left\{ (-1)^{j-1}(j-1)! - \int_0^\varepsilon (\log x)^{j-1}\,\mathrm{d}x \right\} \\
&= \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} + O\big(\varepsilon(\log 1/\varepsilon)^{n-1}\big) \\
&= 1 + O\big(\varepsilon(\log 1/\varepsilon)^{n-1}\big).
\end{aligned} \tag{3.7}$$

Since $g_{n,\varepsilon}(1) = n/2$, from (3.5), (3.6) and (3.7) we obtain that the limit of the right-hand side of (3.5) as $\varepsilon \to 0$ is

$$-\sum_{j=1}^n (-1)^{j-1} \binom{n}{j} \frac{1}{(j-1)!} \lim_{\varepsilon\to 0+} \left\{ \sum_{m \le 1/\varepsilon} \frac{\Lambda(m)(\log m)^{j-1}}{m} - \frac{1}{j}(\log 1/\varepsilon)^j \right\}$$
$$+ 1 - (\log \pi + \gamma)\frac{n}{2} - \int_0^1 [g_n(x) - nx] \frac{\mathrm{d}x}{1-x^2}.$$

Here we have made the change of variable from x to $1/x$ in the last integral. We make the change of variable $x = e^{-t}$ and compute

$$\begin{aligned}
\int_0^1 [g_n(x) - nx] \frac{\mathrm{d}x}{1-x^2} &= \int_0^1 \left[(n - nx) + \sum_{j=2}^n \binom{n}{j} \frac{1}{(j-1)!} (\log x)^{j-1} \right] \frac{\mathrm{d}x}{1-x^2} \\
&= (\log 2)n + \sum_{j=2}^n (-1)^{j-1} \binom{n}{j} \frac{1}{(j-1)!} \int_0^\infty t^{j-1} \frac{e^{-t}}{1-e^{-2t}}\,\mathrm{d}t \\
&= (\log 2)n + \sum_{j=2}^n (-1)^{j-1} \binom{n}{j} \frac{1}{(j-1)!} \left(\sum_{m=0}^\infty \frac{(j-1)!}{(2m+1)^j} \right) \\
&= (\log 2)n + \sum_{j=2}^n (-1)^{j-1} \binom{n}{j} (1 - 2^{-j})\,\zeta(j),
\end{aligned}$$

obtaining the right-hand side of the formula of Theorem 2.

Finally, we note that

$$\lim_{\varepsilon\to 0+} \left(\sum_\rho \widehat{g}_{n,\varepsilon}(\rho) \right) = \sum_\rho \widehat{g}_n(\rho). \tag{3.8}$$

This follows from the Prime Number Theorem with error term. We have (without error term if $n = 1$)

$$\begin{aligned}
\widehat{g}_n(s) - \widehat{g}_{n,\varepsilon}(s) &= \int_0^\varepsilon P_n(\log x) x^{s-1}\,\mathrm{d}x \\
&= P_n(\log \varepsilon)\,\frac{\varepsilon^s}{s} + O\left(\frac{\varepsilon^{\Re(s)}}{|s|^2} (\log 1/\varepsilon)^{n-2} \right)
\end{aligned}$$

for $0 < \Re(s) < 1$ and $|s| \ge 1$. Thus to prove (3.8) we need to show that $\sum \varepsilon^\rho/\rho$ and $\sum \varepsilon^{\Re(\rho)}/|\rho|^2$ tend to 0, as $\varepsilon \to 0$, faster than any negative power of $\log 1/\varepsilon$.

By the De la Vallée-Poussin zero-free region we have

$$\frac{c}{\log(|\rho|+2)} \le \Re(\rho) \le 1 - \frac{c}{\log(|\rho|+2)}$$

for some constant $c > 0$, therefore

$$\begin{aligned}\sum_\rho \frac{\varepsilon^{\Re(\rho)}}{|\rho|^2} &\le \Big(\max_\rho \varepsilon^{c/\log(|\rho|+2)}|\rho|^{-1/2}\Big)\Big(\sum_\rho \frac{1}{|\rho|^{3/2}}\Big) \\ &= O\big(e^{-c'\sqrt{\log 1/\varepsilon}}\big)\end{aligned} \tag{3.9}$$

for some constant $c' > 0$. This quantity tends to 0, as $\varepsilon \to 0$, faster than any negative power of $\log 1/\varepsilon$.

In a similar way, one has

$$\sum_\rho \frac{\varepsilon^\rho}{\rho} = O\big(e^{-c'\sqrt{\log 1/\varepsilon}}\big),$$

possibly by readjusting the constant $c' > 0$. In fact,

$$\begin{aligned}\sum_\rho \frac{\varepsilon^\rho}{\rho} &= \sum_\rho \frac{\varepsilon^{1-\rho}}{1-\rho} \\ &= -\varepsilon \sum_\rho \frac{(1/\varepsilon)^\rho}{\rho} + O\Big(\sum_\rho \frac{\varepsilon^{\Re(\rho)}}{|\rho|^2}\Big) \\ &= \varepsilon\big\{\psi\big(\frac{1}{\varepsilon}\big) - \frac{1}{\varepsilon} - \log 2\pi - \frac{1}{2}\log(1-\varepsilon^2)\big\} + O\big(e^{-c'\sqrt{\log 1/\varepsilon}}\big) \\ &= O\big(e^{-c'\sqrt{\log 1/\varepsilon}}\big)\end{aligned}$$

by the classical explicit formula for $\psi(x)$, the Prime Number Theorem with error term* and by (3.9). This proves what we want.

§4. Concluding remarks. Let $f(x)$ and $g(x)$ be complex valued functions satisfying conditions (A) and (B) of the preceding section. The multiplicative convolution of $f(x)$ and $g(x)$ is given by

$$(f * g)(x) = \int_0^\infty f(x/y)g(y)\,\frac{\mathrm{d}y}{y}\,.$$

The Mellin transform of $f * g$ is $\widehat{f}(s)\,\widehat{g}(s)$, whence, denoting by $\overline{f}$ the complex conjugate of f, the Mellin transform of $f * \widetilde{\overline{f}}$ is $\widehat{f}(s)\,\widehat{\overline{f}}(1-s)$, which is real and

* H. Davenport, *loc. cit.*, Ch.17,18

positive on the critical line $\Re(s) = \frac{1}{2}$. Thus the positivity of the Explicit Formula on functions of the type $f * \widetilde{\overline{f}}$ is a necessary condition for the validity of the Riemann Hypothesis. As shown by Weil, this is also a sufficient condition.

Li's criterion can be interpreted quite easily in this light. Let $g_n(x)$ be the function defined in the preceding section. We have the identity

$$\big[1-(1-1/s)^n\big]+\big[1-(1-1/(1-s))^n\big]=\big[1-(1-1/s)^n\big]\cdot\big[1-(1-1/(1-s))^n\big]$$

and taking the inverse Mellin transform we find

$$g_n(x)+\widetilde{g}_n(x)=(g_n*\widetilde{g}_n)(x).$$

Since the right-hand side of the Explicit Formula is invariant by changing $f(x)$ into $\widetilde{f}(x)$, the positivity in Li's criterion has the same meaning as in Weil's criterion. The interesting point is that Li's criterion requires an explicit and rather simple set of test functions for its verification.

The numbers

$$\eta_n=\frac{(-1)^n}{n!}\lim_{x\to\infty}\left\{\sum_{m\le x}\frac{\Lambda(m)(\log m)^n}{m}-\frac{1}{n+1}(\log x)^{n+1}\right\} \tag{4.1}$$

which appear in Theorem 2 are analogous to the Stieltjes constants

$$\gamma_n=\frac{(-1)^n}{n!}\lim_{x\to\infty}\left\{\sum_{m\le x}\frac{(\log m)^n}{m}-\frac{1}{n+1}(\log x)^{n+1}\right\} \tag{4.2}$$

which appear in the Laurent expansion*

$$\zeta(s+1)=\frac{1}{s}+\sum_{n=0}^{\infty}\gamma_n s^n.$$

In fact, we have

$$-\frac{\zeta'}{\zeta}(s+1)=\frac{1}{s}+\sum_{n=0}^{\infty}\eta_n s^n. \tag{4.3}$$

Hence

$$\log\big(s\,\zeta(s+1)\big)=-\sum_{n=0}^{\infty}\eta_n\,\frac{s^{n+1}}{n+1}$$

and also

$$\begin{aligned}\log\big(s\,\zeta(s+1)\big)&=\log\left(1+\sum_{n=0}^{\infty}\gamma_n s^{n+1}\right)\\&=\sum_{h=1}^{\infty}\frac{(-1)^{h-1}}{h}\left(\sum_{n=0}^{\infty}\gamma_n s^{n+1}\right)^h\\&=\gamma_0 s+\big(\gamma_1-\frac{1}{2}\gamma_0^2\big)s^2+\big(\gamma_2-\gamma_0\gamma_1+\frac{1}{3}\gamma_0^3\big)s^3+\dots.\end{aligned}$$

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* A. Erdélyi, W Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions*, McGraw-Hill 1953, vol. I, p.34, errata p.1

This expresses the constants η_n as polynomials in the constants γ_n and in view of Theorem 2 one obtains a formula for λ_n in terms of the constants $\log 4\pi$, γ_h for $h \geq 0$ and $\zeta(h)$ for $h = 2, 3, \ldots$. A direct proof of such a formula can also be given without recourse to the explicit formula, via power series expansions relating (2.4) to (4.3) using the change of variable $s = 1/(1-z)$.

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Enrico Bombieri
Institute for Advanced Study
School of Mathematics
Princeton, NJ 08540, USA
eb@math.ias.edu

Jeffrey C. Lagarias
Information Sciences Research
AT&T Labs
Florham Park, NJ 07932-0971, USA
jcl@research.att.com