Counting Dyadic Equipartitions of the Unit Square [1]

Jeffrey C. Lagarias

AT&T Labs – Research, Florham Park, New Jersey 07932-0971, USA

Joel H. Spencer

Dept. of Computer Science, New York University, New York, New York 10012-1110, USA

Jade P. Vinson

Dept. of Mathematics, Princeton University, Princeton, New Jersey 08544, USA

Abstract

A dyadic interval is an interval of the form $\left[\frac{j}{2^k}, \frac{j+1}{2^k}\right]$, where j and k are integers, and a dyadic rectangle is a rectangle with sides parallel to the axes whose projections on the axes are dyadic intervals. Let u_n count the number of ways of partitioning the unit square into 2^n dyadic rectangles, each of area 2^{-n}. One has $u_0 = 1$, $u_1 = 2$ and $u_n = 2u_{n-1}^2 - u_{n-2}^4$. This paper determines an asymptotic formula for a solution to this nonlinear recurrence for generic real initial conditions. For almost all real initial conditions there are real constants ω and β (depending on u_0, u_1) with $\omega > 0$ such that for all sufficiently large n one has the exact formula

$$u_n = \omega^{2^n} g(\beta\lambda^n),$$

where $\lambda = 2\sqrt{5}-4 \approx 0.472$, and $g(z) = \sum_{j=0}^{\infty} c_j z^j$, in which $c_0 = \frac{-1+\sqrt{5}}{2}$, $c_1 = \frac{2-\sqrt{5}}{2}$, all coefficients c_j lie in the field $\mathbb{Q}(\sqrt{5})$, and the power series converges for $|z| < 0.16$. These results apply to the initial conditions $u_0 = 1$, $u_1 = 2$ with $\omega \approx 1.845$ and $\beta \approx 0.480$. The exact formula for u_n then holds for all $n \geq 2$. The proofs are based on an analysis of the holomorphic dynamics of iterating the rational function $R(z) = 2 - \frac{1}{z^2}$.

Key words: asymptotic enumeration, holomorphic dynamics

2000 MR Subject Classification: Primary 05A16, Secondary 30D05

[1] Dedicated to Daniel Kleitman: mentor and colleague

1 Introduction

We call an interval *dyadic* if it is of the form $\left[\frac{j}{2^k}, \frac{j+1}{2^k}\right]$, where j and k are integers. A *dyadic rectangle* is a rectangle whose sides are parallel to the axes and whose projections to the axes are dyadic intervals. A dyadic equipartition of order n is a dissection of the unit square of it into 2^n dyadic rectangles, each of area 2^{-n}. Let u_n denote the number of dyadic equipartitions of order n. In §2 we show that $u_1 = 1$, $u_2 = 2$ and that u_n satisfies the nonlinear recurrence

$$u_n = 2u_{n-1}^2 - u_{n-2}^4. \tag{1}$$

The first few values of u_n are

$$1, 2, 7, 82, 11047, 198860242, 64197955389505447, \ldots$$

In this paper we determine the asymptotic behavior of u_n, as follows.

Theorem 1 *There are constants ω and β such that number of dyadic equipartitions u_n of the unit square satisfies for all $n \geq 2$,*

$$u_n = \omega^{2^n} g(\beta\lambda^n) \tag{2}$$

where $\lambda = -4 + 2\sqrt{5} \approx 0.472136$, and

$$g(z) = c_0 + \sum_{k=1}^{\infty} c_k z^k \ , \tag{3}$$

where $g(z)$ is analytic in a neighborhood of $z = 0$, with $c_0 = \frac{-1+\sqrt{5}}{2}$, $c_1 = \frac{2-\sqrt{5}}{2}$, and the coefficients c_k all lie in the field $\mathbb{Q}(\sqrt{5})$.

The coefficients c_k remain the same in describing the asymptotics of most solutions to the nonlinear recurrence (1) with real initial conditions, while the values of ω and β are specific to the initial conditions, see Theorem 8. We show that

$$\omega \in [1.8445475709350505, 1.8445475709350507].$$

and

$$\beta \in [0.479835559, 0.479835561].$$

In §5 we show that the radius of convergence of the power series (3) is at least 0.16, see Theorem 12.

Theorem 1 is obtained by an analysis of the solutions to the nonlinear recurrence (1) with arbitrary real initial conditions. These solutions are closely tied to iteration of the rational function $R(z) = 2 - \frac{1}{z^2}$, on the Riemann sphere, particularly on the real axis, which we study in §3 - §5. For this rational function nearly all forward orbits approach an attracting fixed point $z = \phi := \frac{1+\sqrt{5}}{2}$, and this includes the orbit containing the initial conditions $(u_0, u_1) = (1, 2)$. An interesting feature is that the main result provides not only an asymptotic expansion for the u_n but gives an exact formula for all values of u_n whenever $z = \frac{u_n}{u_{n-1}^2}$ is sufficiently close to the attracting fixed point.

Theorem 1 relates to results obtained by Boros and Furedi [4], who studied the number r_m of distinct dissections of a square into rectangles each of area $\frac{1}{m}$. (They actually treated a more general dissection problem, where the rectangles need not all have the same area.) We clearly have $u_n \le r_{2^n}$. Boros and Furedi show that

$$\frac{1}{2^{m-1}} M_m \;\le\; r_m \;\le\; M_m, \tag{4}$$

where

$$M_m := \frac{2}{m(m-1)^2} \sum_{k=1}^{m} \binom{m+1}{k-1} \binom{m+1}{k} \binom{m+1}{k+1}.$$

They showed that M_m satisfies

$$M_m = (1 + o(1)) \frac{32}{\pi\sqrt{3}} \frac{8^m}{m^4},$$

and these asymptotics give

$$(4 - o(1))^{2^n} \;\le\; r_{2^n} \;\le\; 8^{2^n}.$$

We define the *configurational entropy* of a set of cardinality v_n to be

$$H := \limsup_{n \to \infty} \frac{1}{2^n} \log v_n,$$

with the scaling factor $\frac{1}{2^n}$ chosen because there are 2^n pieces in the partition. The results above show that the set of rectangular equipartitions of the square of area 2^{-n} has positive configurational entropy, no larger than $\log 8$. On the other hand Theorem 1 shows that the subset of dyadic equipartitions of area 2^{-n} has positive configurational entropy, at least $\log 1.844$.

Our motivation for studying dyadic equipartitions of the square arose as follows. Coffman et al. [6] studied packings of random axis-parallel rectangles inside the unit square. Here one draws n rectangles, each of which is a product of independent random subintervals of the unit interval, and studies the expected size $\mathbf{E}C_n$ of the maximum cardinality C_n of disjoint rectangles in the set. An answer $\mathbf{E}C_n = \Theta(n^{\frac{1}{2}})$ was obtained, with the upper bound based on a reduction to the study of packings using random dyadic rectangles [2]. In doing this it proved worthwhile to study packings obtained using dyadic rectangles of a particular fixed area. This motivated the question of studying tilings of the square using rectangles all having the same area.

Let $R_{n,p}$ denote a random subfamily of the family of all dyadic rectangles of area 2^{-n}, in which each such rectangle is placed in the subfamily with independent probability p. Let $f(n,p)$ denote the probability that a dyadic equipartition may be created from the rectangles in $R_{n,p}$. We have the:

Open Question: Does there exist a probability $p < 1$ such that $\lim_{n\to\infty} f(n,p) = 1$?

While we have not resolved this question we note that the expected number of such equipartitions is equal to

$$u_n p^{2^n} = (p\omega)^{2^n} g(\beta\lambda^n) \sim c_0 (p\omega)^{2^n}.$$

When $p < \omega^{-1} \approx 0.54213836\cdots$ the expected number of equipartitions goes to zero and hence $f(n,p) \to 0$. When $p > \omega^{-1}$ the expected number of equipartitions goes to infinity but, a priori, this fact does not imply the almost sure existence of at least one equipartition.

The proof techniques of this paper apply in principle to other nonlinear recurrences which reduce to the iteration of a rational map, although the precise details will depend on the map. For example, they apply to the recurrence

$$v_n = 3v_{n-1}^2 - 2v_{n-2}^4, \tag{5}$$

with initial conditions $v_0 = 1$, $v_1 = 2$ which occurs [3] in Irving and Leather [7, p. 657]. The sequence v_n gives a lower bound for the maximal possible number of stable marriage arrangements in some set of 2^n men and 2^n women having suitable preference orderings. For the recurrence (5) the corresponding function $g(z)$ (analogous to (3)) describing the asymptotics of its solutions has coefficients c_k that lie in the field $\mathbb{Q}(\sqrt{3})$.

2 Dyadic Equipartition Recurrence

Let u_n be the number of ways of dividing the unit square into dyadic rectangles of size 2^{-n}. Then $u_0 = 1$, $u_1 = 2$, and $u_2 = 7$, as shown in Figure 1.

[2] Dyadic rectangles are called canonical rectangles in [6].

[3] The recurrence in Irving and Leather's paper is $g(m) = 3g(\frac{m}{2})^2 - 2g(\frac{m}{4})^4$, where $m = 2^n$, with initial conditions $g(1) = 1$ and $g(2) = 2$. We set $v_n = g(2^n)$.

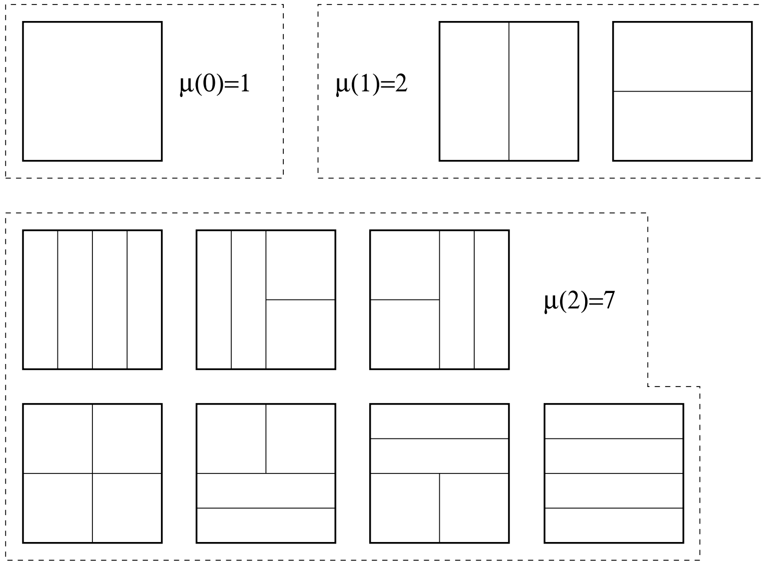

Fig. 1. Dyadic equipartitions of the unit square for $n = 0, 1, 2$.

Lemma 2 *The number u_n of dyadic equipartitions of the unit square satisfies $u_0 = 1$, $u_1 = 2$ and*

$$u_n = 2u_{n-1}^2 - u_{n-2}^4. \tag{6}$$

Proof. Suppose that some dyadic rectangle R_x straddles the vertical line $x = \frac{1}{2}$, i.e. contains part of the vertical line $x = \frac{1}{2}$ in its interior. Then its projection to the x-axis must be $[0, 1]$. If a dyadic rectangle R_y straddles the line $y = \frac{1}{2}$, then its projection to the y-axis is $[0, 1]$. The two rectangles R_x and R_y intersect. Since we have a partition, $R_x = R_y = [0, 1] \times [0, 1]$. This is only possible for $n = 0$.

For $n \geq 1$, either no rectangle straddles $x = 1/2$, or no rectangle straddles $y = 1/2$, or both. In the first case, $x = 1/2$ divides the unit square into left and right halves. There are u_{n-1} ways to partition the left half, and u_{n-1} ways to partition the right half, for a total of u_{n-1}^2. Similarly, when $y = \frac{1}{2}$ divides the square there are u_{n-1}^2 possibilities. Since we double counted the case when both $x = \frac{1}{2}$ and $y = \frac{1}{2}$ divide the unit square, we must subtract u_{n-2}^4. Thus $u_n = 2u_{n-1}^2 - u_{n-2}^4$. $\square$

Remark 3 This proof works only in two dimensions. One can define dyadic equipartitions of the unit n-cube for all $n \geq 3$ in a similar way, but one no longer has the property that such a partition always splits the n-cube in half along some line $x_k = \frac{1}{2}$.

3 Iterating a Rational Function

Set $v_n = \frac{u_n}{u_{n-1}^2}$. Then v_n satisfies the first-order nonlinear recurrence

$$v_n = 2 - \frac{1}{v_{n-1}^2}, \tag{7}$$

as is seen by dividing (1) by u_{n-1}^2. We have

$$u_n = u_{n-1}^2 v_n,$$

which when iterated yields

$$u_n = u_0^{2^n} \prod_{k=0}^{n-1} v_{n-k}^{2^k}. \tag{8}$$

The recurrence (7) for v_n iterates the rational function

$$R(z) = 2 - \frac{1}{z^2}, \tag{9}$$

so that we can apply the well-developed theory of iteration of rational maps on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, see Beardon[1], Blanchard [2] or Carlson and Gamelin [5]. Recall that given a rational map $R(z)$ the Riemann sphere partitions as $\hat{\mathbb{C}} = F(R) \cup J(R)$, in which the Fatou set $F(R)$ is an open set on which the forward dynamics of iterating $R(z)$ are simple, and the Julia set $J(R)$ is a closed set on which the corresponding dynamics are chaotic. The Fatou set $F(R)$ and Julia set $J(R)$ are both forward invariant and backward invariant under the map $R(z)$.

For the rational function (9) the real axis is a forward-invariant set, and we will mainly be concerned with the behavior iterates of $R(z)$ on the real axis $\hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, and in particular the open interval $(1, \infty)$, which is also forward-invariant under iteration by $R(z)$. However we state results for the complex case whenever this is convenient.

Lemma 4 *The point $\phi = \frac{1+\sqrt{5}}{2}$ is an attracting fixed point of $R(z) = 2 - \frac{1}{z^2}$, with*

$$\lambda := R'(\phi) = \frac{2}{\phi^3} = -4 + 2\sqrt{5} \approx 0.472136\,. \tag{10}$$

On the real axis the segment $(1, \infty)$ is invariant under R, and all points on it *monotonically approach ϕ under iteration. The Fatou set $F(R)$ contains the open interval $(1,\infty)$, the point ∞ and the entire imaginary axis.*

Proof. Certainly $R(\phi) = \phi$ and $R'(z) = \frac{2}{z^3}$ has $|R'(\phi)| < 1$, so ϕ is an attracting fixed point. The point $z_1 = 1$ is a repelling fixed point of $R(z)$. One checks for real $x > 1$ that

$$x < R(x) < \phi \quad \text{if} \quad 1 < x < \phi \ ,$$

and

$$\phi < R(x) < x \quad \text{if} \quad \phi < x < \infty \ .$$

It is easy to show that $R^{(n)}(x) \to \phi$ for all $x \in (1,\infty)$. This implies that $(1,\infty)$ lies in the Fatou set $F(R)$. Since $R(\infty) = 2$, it is in the Fatou set. Finally the imaginary axis is mapped to the real interval $(2,\infty]$ by $R(\cdot)$, so it is also in the Fatou set. $\square$

Theorem 5 *Let $R(z) = 2 - \frac{1}{z^2}$. The Julia set $J(R)$ lies on the real line, satisfies $J(R) = -J(R)$, and is contained in the region $[-1, -\frac{1}{\sqrt{3}}] \cup [\frac{1}{\sqrt{3}}, 1]$. It is a topological Cantor set which has zero Lebesgue measure in $\mathbb{R}$.*

Proof. The two critical points of $R(z)$ are $\{0,\infty\}$, and Lemma 4 shows they are both in the Fatou set, and in the immediate attracting basin of the attracting fixed point $z = \phi$, which is the connected component of the Fatou set containing ϕ. (All points in the immediate attracting basin of an attractive fixed point approach that point under forward iteration, cf. Beardon [1, p. 104].) Theorem 9.8.1 of Beardon [1] states that any rational map R' of degree $d \geq 2$ which has an attracting fixed point whose immediate attracting basin contains all the critical points of R' has a Julia set $J(R')$ that is a topological Cantor set (perfect disconnected set). Theorem 9.8.2 of Beardon [1] states that under the same hypotheses the Julia set $J(R')$ contains points whose forward orbit is dense in $J(R')$. Both these theorems apply to R. Thus $J(R)$ is a topological Cantor set. It follows that $F(R)$ is connected, so that the immediate attracting basin of ϕ is $F(R)$, and all points in $F(R)$ approach ϕ under forward iteration. The second result above shows that $J(R)$ contains a point z_0 whose forward orbit is dense in $J(R)$.

We now show that the Julia set is contained in the closed annulus

$$\mathcal{A} := \{z \ : \ \frac{1}{\sqrt{3}} \leq |z| \leq 1.\}.$$

The region $\mathcal{S}_0 := \{z \ : \ |z| > 1\} \cup \{\infty\}$ is closed under forward iteration. It lies entirely in $F(R)$, for if it contained $z_1 \in J$ then the forward orbit of z_0 would get arbitrarily close to z_1, hence would contain some point in $\mathcal{S}_0$, whence all subsequent iterates lie in $\mathcal{S}_0$ and cannot have the repelling fixed point $\frac{1-\sqrt{5}}{2} \in J$ as a limit point, a contradiction. The region $\mathcal{S}_1 = \{z \ : \ |z| < \frac{1}{\sqrt{3}}\} \subset F(R)$ since $R(\mathcal{S}_1) \subset \mathcal{S}_0 \subset F(R)$ and $F(R)$ is bi-invariant under iteration of $R(.)$. Thus $J \subset \mathcal{A}$. We have $J(R) = -J(R)$ because the Julia set is closed under backwards iteration. Given $z \in J(R)$, so is $R^{-1}(R(z)) = \{z, -z\}$.

To show that the Julia set J is real, we argue by contradiction. If J were not real, then any point z_0 having a dense forward orbit in J is not real, for the forward iterates of a real point remain on the real axis. We claim that any point in J with $0 < |\Im(z)| \le \frac{\sqrt{2}}{2}$ has the property that both its preimages z' under R have $0 < |\Im(z')| < |\Im(z)|$. To see this, write $z = a + bi$, with $\frac{1}{3} \le a^2 + b^2 \le 1$ and $0 < |b| \le \frac{\sqrt{2}}{2}$. Then

$$(z')^2 = \frac{1}{2-z} = \frac{2-a+bi}{(2-a)^2+b^2}.$$

Set $re^{i\theta} = 2-a+bi$ and note that $r^2 = (2-a)^2+b^2 > 1$, so $r > 1$ and $b = r\sin\theta$ with $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$, using the bound on b and the fact that $2 - a > 0$. Now

$$z' = \pm \frac{e^{i\theta/2}}{r^{1/2}},$$

which gives

$$0 < |\Im z'| = \frac{|\sin\theta/2|}{r^{1/2}} \le \frac{|\sin\theta|}{r^{1/2}} \le \frac{|b|}{r^{3/2}} < |b|,$$

which proves the claim. The claim implies that any z'' whose forward orbit contains a given point $z \in J$ with $|\Im(z)| < \frac{\sqrt{2}}{2}$ must have $|\Im(z'')| \le |\Im(z)|$. It follows that the the forward orbit of z_0 contains no point z_1 that has $|\Im(z_1)| \le \min(\frac{\sqrt{2}}{2}, |\Im(z_0)|)$, hence the forward orbit of z_0 has no limit points on the real axis, and this contradicts the fact that the forward orbit of z_0 is dense in J. Thus J is real.

Now we know that $J \subset [-1, -\frac{1}{\sqrt{3}}] \cup [\frac{1}{\sqrt{3}}, 1]$, since it lies in the annulus $\mathcal{A}$. The four endpoints $\{1, -1, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\}$ lie in J, because $z = 1$ is a repelling fixed point and each of the other three points has $z = 1$ in its forward orbit.

To conclude, Theorem 10.2 of Brolin [3] states that if a rational function $R'(z)$ of degree $d \ge 2$ has an attractive fixed point that contains all of the critical points of $R'(z)$ in its immediate attracting basin, then the Julia set is totally disconnected and has zero two-dimensional Lebesgue measure, and if in addition the Julia set $J(R')$ lies on the real axis then it has zero one-dimensional Lebesgue measure on $\mathbb{R}$.. This applies to R, hence the Julia set $J(R)$ has zero one-dimensional Lebesgue measure on $\mathbb{R}$. $\square$

We study the growth rate of iterates u_n of the nonlinear recurrence (6). For a positive real number x, let $x^{1/2^n}$ denote the positive real 2^n-th root of x. We treat $x^{1/2^n}$ as undefined if $x \leq 0$.

Lemma 6 *For real initial conditions $(u_0, u_1) \in \mathbb{R}^2$ of the recurrence $u_n = 2u_{n-1}^2 - u_{n-2}^4$, such that $v_1 = \frac{u_1}{u_0^2} \in F(R)$, the following limit exists and is nonzero:*

$$\omega(u_0, u_1) := \lim_{n\to\infty} u_n^{1/2^n} \ . \tag{11}$$

Proof. Suppose first that $1 < v_1 < \infty$. Then $1 < v_k < 2$ for $k \geq 2$, and

$$u_n = u_0^{2^n} v_1^{2^{n-1}} \cdots v_2^{2^{n-2}} v_{n-1}^2 v_n \ . \tag{12}$$

Thus $u_n > 0$ for $n \geq 1$, so $u_n^{1/2^n}$ is defined, and

$$u_n^{1/2^n} = |u_0| v_1^{1/2} v_2^{1/4} \cdots v_n^{1/2^n}, \quad \text{for} \quad n \geq 1\,. \tag{13}$$

The condition $1 < v_k < 2$ yields

$$v_k^{1/2^k} = 1 + O(2^{-k}) \quad \text{as} \quad k \to \infty\ .$$

Thus the infinite product $|u_0| \prod_{j=1}^{\infty} v_j^{1/2^j}$ converges to a positive quantity, namely

$$\omega(u_0, u_1) := \lim_{n\to\infty} |u_0| \prod_{j=1}^{n} v_j^{1/2^j} = |u_0| \prod_{j=1}^{\infty} v_j^{1/2^j} \ , \tag{14}$$

which is (11).

For the remaining cases, note that the real axis is invariant under $R(\cdot)$. Since $v_1 \in \mathcal{F}_R$, the iterates $R^{(k)}(v_1)$ approach ϕ, and since they lie in $\mathbb{R}$ they eventually enter the region $(1, \infty)$. Suppose $v_m \in (1, \phi)$. The argument above applies to show that $\omega(u_m, u_{m+1})$ exists. now

$$\omega(u_m, u_{m+1})^{1/m} = \left(\lim_{n\to\infty} u_{m+n}^{1/2^n}\right)^{1/2^m} = \lim_{n\to\infty} u_{m+n}^{1/2^{m+n}} \tag{15}$$

which gives $\omega(u_0, v_1) = \omega(u_m, u_{m+1})^{1/2^m}$. $\square$

For general complex initial conditions of the recurrence (6) we have the following weaker result.

Lemma 7 *For each complex initial condition* $(u_0, u_1) \in \mathbb{C}^2$ *other than* $(0, 0)$*, the following limit exists and is nonzero:*

$$|\omega|(u_0, u_1) := \lim_{n\to\infty} |u_n|^{1/2^n} . \tag{16}$$

Proof. The proof of (13) yields more generally that

$$|u_n|^{1/2^n} = |u_0| \prod_{j=1}^{n} |v_j|^{1/2^j} ,$$

provided that no value $v_j = 0$ or ∞. The case where some $v_j = 0$ or ∞ puts $v_{j+2} \in (1, \infty)$ and Lemma 6 applies in this case. We claim that

$$|v_j|^{1/2^j} = 1 + O\left(\frac{1}{2^j}\right) \quad \text{as} \quad j \to \infty . \tag{17}$$

If $v_0 \in F(R)$ then $v_j \to \phi$ hence $1 \le |v_j| \le 2$ for all large j, so (16) holds. If $v_0 \in J(R)$ then all $v_j \in J(R)$, and there are constants $c_1 > c_0 > 0$ with

$$c_0 < |v| < c_1 \quad \text{if} \quad v \in J(R) ,$$

because $0, \infty$ lie in the open set $F(R)$. Thus (16) holds in this case. Now one has

$$\lim_{j\to\infty} |u_0| \prod_{j=1}^{n} |v_j|^{1/2^j} = |u_0| \prod_{j=1}^{\infty} |v_j|^{1/2^j} ,$$

provided no $v_j = 0$ or ∞. $\square$

4 Dynamics near the Attracting Fixed Point

We now restrict attention to real initial conditions $(u_0, u_1) \in \mathbb{R}^2$ such that $v_1 = \frac{u_1}{u_0^2}$ lies in the Fatou set $F(R)$. The detailed asymptotics of such u_n are determined by the approach of $v_n = R^{(n-1)}(v_1)$ to the attracting fixed point ϕ.

For $n \ge 1$ we have

$$u_n = u_0^{2^n} v_1^{2^{n-1}} \cdots v_{n-1}^2 v_n = \omega^{2^n} \gamma_n , \tag{18}$$

where $\omega = \omega(u_0, u_1)$ is given by (14), and

$$\gamma_n = \prod_{j=1}^{\infty} (v_{n+j})^{-\frac{1}{2^j}} . \tag{19}$$

Our object in this section is to show that there is a closed form expression for γ_n which is a (convergent) asymptotic expansion.

Theorem 8 *There is a power series*

$$g(z) = \sum_{n=0}^{\infty} c_n z^n \tag{20}$$

with $c_0 = \frac{-1+\sqrt{5}}{2}$, $c_1 = \frac{2-\sqrt{5}}{2}$, *and each* $c_n \in \mathbb{Q}(\sqrt{5})$, *which converges for* $|z| < B$ *with positive* B, *such that the following holds. For each* $(u_0, u_1) \in \mathbb{R}^2$ *with* $v_1 = \frac{u_1}{u_0} \in F(R)$, *there exists* $\beta = \beta(v_1) \in \mathbb{R}$ *and* $\omega = \omega(u_0, u_1) > 0$ *such that*

$$u_n = \omega^{2^n} g(\beta\lambda^n), \quad \textit{for all } n \textit{ with } \quad |\beta\lambda^n| \le B \ , \tag{21}$$

where

$$\omega = \omega(u_0, u_1) = \lim_{n\to\infty} u_n^{1/2^n} \ , \tag{22}$$

and $\lambda = -4 + 2\sqrt{5} \simeq 0.472136$.

In §5 we show that one can take $B = 0.16$. We also note that the theorem is valid more generally for complex initial conditions $(u_0, u_1) \in \mathbb{C}^2$ such that $v_1 \in F(R)$, provided that ω is properly defined; see Theorem 12.

The proof uses the analytic linearization of $R(z)$ in a neighborhood of its attracting fixed point $z = \phi$, and the quantity $\beta = \beta(v_1)$ is expressible in terms of the linearizing map, cf. (29). Basic properties of the linearizing map are summarized in the following lemma.

Lemma 9 *Let* $F(z) = \sum_{k=1}^{\infty} \lambda_k z^k$ *be a formal power series with* $\lambda_1 \neq 0$ *or a root of unity. There is a unique formal power series*

$$\sigma(z) := z + \sum_{k=1}^{\infty} a_k z^k \tag{23}$$

such that

$$\sigma \circ F(z) = \lambda_1 \sigma(z) \ . \tag{24}$$

Furthermore, for $k \ge 2$,

$$a_k \in \mathbb{Q}(\lambda_1, \lambda_2, \ldots, \lambda_{k-1}) \ . \tag{25}$$

If $0 < |\lambda_1| < 1$ and the power series for $F(z)$ has a positive radius of convergence, then the power series for $\sigma(z)$ also has a positive radius of convergence.

Proof. The recursion (24) is

$$\sum_{k=1}^{\infty} a_k \left(\sum_{n=1}^{\infty} \lambda_n z^n \right)^k = \lambda_1 \sum_{k=1}^{\infty} a_k z^k \ .$$

Since the power z^j appears only on the left side in terms $1 \le k \le j$, equating terms for z^j gives a relation

$$(\lambda_1^j - \lambda_1) a_j = \sum_{k=1}^{j-1} a_k P_{j_k}(\lambda_1, \lambda_2, \ldots, \lambda_{j-k}) \tag{26}$$

where each P_{jk} is a (multivariate) polynomial with integer coefficients. By hypothesis $\lambda_1^j - \lambda_1 \neq 0$ for all $j \geq 1$, so taking $a_1 = 1$ the coefficients a_j are uniquely determined recursively by (26). By induction on $k \geq 1$ they satisfy $a_k \in \mathbb{Q}(\lambda_1, \lambda_2, \ldots, \lambda_{k-1})$.

The assertion on positive radius of convergence of the power series when $0 < |\lambda| < 1$ is a theorem of Koenigs [8], proofs of which appear in [1, Theorem 6.3.2] and [5, Theorem 2.1]. $\square$

When $F(z)$ is analytic and has an attracting fixed point at $z = 0$, with $F'(0) \neq 0$, the power series (23) converges in a small disk around $z = 0$ and has a well-defined inverse

$$\sigma^{-1}(z) = z + \sum_{k=2}^{\infty} \tilde{a}_k z^k.$$

The power series for $\sigma^{-1}(z)$ has a positive radius of convergence and its coefficients also satisfy $\tilde{a}_k \in \mathbb{Q}(\lambda_1, \lambda_2, \ldots, \lambda_{k-1})$ for $k \geq 2$. In this case we can rewrite (24) as a conjugacy

$$\sigma \circ F \circ \sigma^{-1}(z) = \lambda_1 z \ , \tag{27}$$

and this is valid in some disk $|z| < B$, with B positive.

Proof of Theorem 8. We apply Lemma 9 to $R(z)$ by first making a linear conjugacy that moves the fixed point $z = \phi$ to the origin. Let $T(z) = z + \phi$ and set

$$\tilde{R}(z) := T^{-1} \circ R \circ T(z) \ . \tag{28}$$

Here $\tilde{R}(z)$ has the fixed point $z = 0$ with

$$\lambda = \tilde{R}'(0) = R'(\phi) = \frac{2}{\phi^3} = -4 + 2\sqrt{5}\ .$$

Since $0 < |\lambda| < 1$, Lemma 9 shows that there exists a conjugacy map $\sigma(z)$ with $\sigma(\tilde{R}(z)) = \lambda\sigma(z)$. Equivalently, $\sigma^{-1}(\lambda z) = \tilde{R}(\sigma^{-1}(z))$. Since the power series coefficients of $\tilde{R}(z)$ are in $\mathbb{Q}(\sqrt{5})$, so are the power series coefficients of $\sigma(z)$ and $\sigma^{-1}(z)$.

Now let $f(z) = \sigma^{-1}(z) + \phi$, so that

$$f(\lambda z) = R(f(z))\ . \tag{29}$$

The power series coefficients of $f(z)$ are all in $\mathbb{Q}(\sqrt{5})$:

$$\begin{aligned} f(z) = &\frac{1+\sqrt{5}}{2} + z + \frac{15+9\sqrt{5}}{20}z^2 + \frac{359+155\sqrt{5}}{220}z^3 \\ &+\frac{718370+322497\sqrt{5}}{239800}z^4 + \ldots \end{aligned}$$

Since $f'(0) \neq 0$, f is invertible in a neighborhood of $f(0) = \phi$. The coefficients of f^{-1} are also in $\mathbb{Q}(\sqrt{5})$. The first few terms are:

$$\begin{aligned} f^{-1}(z) = &(z-\phi) + \frac{-15-9\sqrt{5}}{20}(z-\phi)^2 + \frac{167+71\sqrt{5}}{110}(z-\phi)^3 \\ &+\frac{-132636-293785\sqrt{5}}{119900}(z-\phi)^4 + \ldots \end{aligned}$$

The relation $f(\lambda z) = R(f(z))$ may be replaced by a conjugacy relation:

$$\begin{aligned} R(z) &= f(\lambda f^{-1}(z)) \\ R^n(z) &= f(\lambda^n f^{-1}(z))\ . \end{aligned}$$

The conjugacy relation is valid whenever z is sufficiently close to ϕ. Recalling the definition of γ_n from equation 19, we can use the conjugacy relation to express v_{n+j} in terms of f, f^{-1}, and v_n:

$$\begin{aligned} \log(\gamma_n) &= \sum_{j=1}^{\infty} \frac{-1}{2^j} \log(v_{n+j}) \\ &= \sum_{j=1}^{\infty} \frac{-1}{2^j} \log(R^j(v_n)) \end{aligned}$$

$$= \sum_{j=1}^{\infty} \frac{-1}{2^j} \log(f(\lambda^j f^{-1}(v_n))) \ .$$

This expression for $\log(\gamma_n)$ is valid when v_n is sufficiently close to ϕ. Since $v_1 \in F(R)$ by assumption, this is true when n is sufficiently large. Continuing, let $\beta = \lambda^{-n} f^{-1}(v_n)$. Then

$$\begin{aligned}
\log(\gamma_n) &= \sum_{j=1}^{\infty} \frac{-1}{2^j} \log(f(\lambda^{n+j}\beta)) \\
&= -\log(\phi) + \sum_{j=1}^{\infty} \frac{-1}{2^j} \log\left(1 + \left(\frac{f(\lambda^{n+j}\beta)}{\phi} - 1\right)\right) \\
&= -\log(\phi) + \sum_{j=1}^{\infty} \frac{-1}{2^j} \tilde{\Psi}(\lambda^{n+j}\beta) \ ,
\end{aligned}$$

where $\tilde{\Psi}$ is a convergent power series in a neighborhood of zero. Recall that $\log(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \cdots$ in a neighborhood of $z = 0$. Since the power series for $\left(\frac{f(\lambda^{n+j}\beta)}{\phi} - 1\right)$ has coefficients in $\mathbb{Q}(\sqrt{5})$ and the constant term is zero, the coefficients of $\tilde{\Psi}$ are also in $\mathbb{Q}(\sqrt{5})$. Let b_k be the k-th coefficient for $\tilde{\Psi}$ and define $\Psi(z) = \sum_{j=1}^{\infty} \frac{-1}{2^j} \tilde{\Psi}(\lambda^j z)$. Then:

$$\begin{aligned}
\Psi(z) &= \sum_{j=1}^{\infty} \frac{-1}{2^j} \sum_{k=1}^{\infty} b_k \lambda^{jk} z^k \\
&= \sum_{k=1}^{\infty} b_k z^k \sum_{j=1}^{\infty} \frac{-1}{2^j} \lambda^{jk} \\
&= \sum_{k=1}^{\infty} b_k \left(\frac{-\lambda^k}{2 - \lambda^k}\right) z^k
\end{aligned}$$

The interchange of order of summation is justified by the absolute convergence near $z = 0$. Note that the coefficients of $\Psi(z)$ are in $\mathbb{Q}(\sqrt{5})$ because those of $\tilde{\Psi}(z)$ are and because $\lambda \in \mathbb{Q}(\sqrt{5})$. Thus, $\log(\gamma_n) = -\log(\phi) + \Psi(\lambda^n\beta)$. Exponentiating gives

$$\gamma_n = \frac{1}{\phi} \exp(\Psi(\lambda^n\beta)) \equiv g(\lambda^n\beta), \tag{30}$$

where g is analytic in a neighborhood of the origin and has coefficients in $\mathbb{Q}(\sqrt{5})$. Choose B so that the disk of radius B is contained in this neighborhood. Finally, pick n sufficiently large that $|\lambda^n\beta| < B$ and $\gamma_n = g(\lambda^n\beta)$. We can use equation 18 to express $\omega > 0$ in terms of u_n and γ_n:

$$\omega = \left(\frac{u_n}{\gamma_n}\right)^{\frac{1}{2^n}}$$

□

The proof of Theorem 8 gives a method for computing the coefficients of $g(z)$. The first few terms are:

$$g(z) = \frac{-1+\sqrt{5}}{2} + \frac{2-\sqrt{5}}{2} z + \frac{55-89\sqrt{5}}{2480} z^2 + \frac{-16921-7031\sqrt{5}}{927520} z^3 + \frac{-6778805950-3031957499\sqrt{5}}{564136214400} z^4 + \cdots$$

To conclude this section, we observe that the nonlinear recurrence $u_n = 2u_{n-1}^2 - u_{n-2}^4$ can be reformulated as a discrete dynamical system which iterates the map

$$F(x,y) := (y, 2y^2 - x^4) \ , \tag{31}$$

because

$$F(u_{n-1}, u_n) = (u_n, u_{n+1}) \ . \tag{32}$$

We take the domain of F to be $\mathbb{R}^2$, although one could more generally consider $\mathbb{C}^2$. The results above can be interpreted as giving an analytic conjugacy of the map F to a simple form on part of its phase space.

Theorem 10 *For all c_0 with $0 < c_0 < 1$ the map F keeps the domain*

$$\Omega_{c_0} = \left\{(x,y) \in \mathbb{R}^2 : x > 0, y > 0, \left|\frac{y}{x^2} - \phi\right| < c_0\right\} \tag{33}$$

invariant. For small enough c_0 there is an invertible real-analytic conjugacy map $\Phi(x,y) = (\omega, \beta)$ whose range has the form

$$\Gamma = \{(\omega,\beta) \in \mathbb{R}^2 : \omega > 0 \text{ and } \beta_- < \beta < \beta_+\} \ , \tag{34}$$

where $\beta_- < 0 < \beta_+$ depend on c_0, and

$$\Phi \circ F \circ \Phi^{-1}(\omega, \beta) = (\omega^2, \lambda\beta) \ , \tag{35}$$

where $\lambda = -4 + 2\sqrt{5}$.

Proof. The invariance of Ω_{c_0} follows from the proof of Lemma 4. We take

$$\Phi(x,y) = \left(\frac{x}{g\left(\lambda^{-1} f^{-1}\left(\frac{y}{x^2}\right)\right)}, \lambda^{-1} f^{-1}\left(\frac{y}{x^2}\right) \right) , \tag{36}$$

where $0 < x < \infty$, the function $g(y)$ is that given in Theorem 8 and

$$f(y) := T \circ \sigma_1 \circ T^{-1}(y) = \phi\ \sigma_1 \left(\frac{y}{\phi} - 1 \right) + \phi\ . \tag{37}$$

The inverse map is

$$\Phi^{-1}(\omega, \beta) = \left(\omega g(\beta), \omega^2 g(\lambda\beta) \right) \tag{38}$$

provided that c_0 is small enough. The form of (36) shows that all $x > 0$ are permitted, and the form of the range (34) is easily checked. □

Remark 11 Theorem 10 shows that on part of its domain the dynamics of the two-dimensional map F is conjugate to a product of two one-dimensional mappings. For small enough c_0 the proof above produced a convergent power series expansion for the real-analytic conjugacy map Φ. One can extend this map by backwards iteration to a larger domain on which it still has a real-analytic inverse, but it does not then have a globally convergent power series on the whole domain.

5 Numerical Bounds

To complete the results, it remains to obtain rigorous numerical bounds for the values, derivatives, and radii of convergence of the power series defined in the previous section. We let $B(z_0; r) := \{z : |z - z_0| \le r\}$.

Theorem 12 *Associated to the nonlinear recurrence* $u_n = 2u_{n-1}^2 - u_{n-2}^4$ *there are power series* f, f^{-1}, *and* g *of the form:*

$$\begin{aligned} f(z) = &\quad \phi + z + \frac{15 + 9\sqrt{5}}{20} z^2 + \cdots \\ f^{-1}(z) = &\ (z - \phi) + \frac{-15 - 19\sqrt{5}}{20} (z - \phi)^2 + \cdots \end{aligned}$$

$$g(z) = \phi^{-1} + \frac{2-\sqrt{5}}{2} z + \frac{55 - 89\sqrt{5}}{2480} z^2 + \cdots,$$

where $\phi = \frac{1+\sqrt{5}}{2}$, which have the following properties.

(i). The coefficients of these power series are all in $Q[\sqrt{5}]$, and the power series for f, f^{-1} and g converge in $B(0; 0.08)$, $B(\phi; 0.03)$, and $B(0; 0.16)$ respectively.

(ii). For $(u_0, u_1) \in \mathbb{C}^2$ with $v_1 = \frac{u_1}{u_0^2} \in F(R)$ let $n_0 \geq 1$ be minimal such that $v_{n_0} = \frac{u_{n_0}}{u_{n_0-1}^2} \in B(\phi; 0.03)$. Set $\beta = \lambda^{-n_0} f^{-1}(v_{n_0})$, where $\lambda = 2\sqrt{5} - 4 \approx 0.472136$, and define ω by

$$\omega^{2^{n_0-1}} := \frac{u_{n_0-1}}{g(\beta\lambda^{n_0-1})}.$$

Then

$$u_n = \omega^{2^n} g(\beta\lambda^n) \qquad for \qquad n \geq n_0 - 1,$$

and

$$v_n = f(\beta\lambda^n) \qquad for \qquad n \geq n_0.$$

Remark 13 The quantity ω is only defined up to a choice of 2^{n_0-1}-th root of unity. When (u_0, u_1) are positive reals we can choose the root of unity so that ω is a positive real, and it will then agree with the definition in Lemma 6.

We proceed in a series of lemmas, beginning with a numerical version of lemma 9.

Lemma 14 *There is an analytic function f, defined in the ball $B(0; 0.08)$, so that $f(0) = \phi$, $f'(0) = 1$, and*

$$f(\lambda z) = R(f(z))$$

holds for all $z \in B(0; 0.08)$.

Proof. For $|z| < 2$ we choose the single-valued branch $S(z) := R^{-1}(z) = \sqrt{\frac{1}{2-z}}$, by the requirement that $S(\phi) = \phi$. Define $f_n(z) := S^n(\phi + \lambda^n z)$ for $n \geq 0$ and $z \in B(0; 0.08)$. We claim that in evaluating $f_n(z)$, S is evaluated only inside $B(0; 0.08)$. That is,

$$S^j(\phi + \lambda^n z) \in B(\phi; 0.1)$$

whenever $z \in B(\phi; 0.08)$ and $0 \leq j < n$.

We will prove uniform convergence for $z \in B(0; 0.08)$ of $f_n(z)$ to an analytic function $f(z)$. Since $f_n(\lambda z) = R(f_{n+1}(z))$, the limiting function f will satisfy $f(\lambda z) = R(f(z))$.

We use the following facts about S:

(1) $S(\phi) = \phi$ and $S'(\phi) = \lambda^{-1}$.
(2) $|S'(z)| < 4$ for $z \in B(\phi; 0.08)$.
(3) $|S''(z)| < 24$ for $z \in B(\phi; 0.08)$.

Now $f_{n+1}(z) = f_n(\tilde{z})$, where

$$\lambda^n \tilde{z} = S(\phi + \lambda^{n+1} z) - \phi.$$

Properties 1 and 3 imply that

$$|\lambda^n \tilde{z} - \lambda^n z| < 12\lambda^{2(n+1)}|z|^2.$$

Finally,

$$|f_{n+1}(z) - f_n(z)| = |f_n(\tilde{z}) - f_n(z)| = |S^n(\lambda^n \tilde{z}) - S^n(\lambda^n z)|.$$

Using property 2, $|\frac{\partial}{\partial z} S^n(z)| < 4^n$ for $z \in B(\phi, \lambda^n 0.08)$. Thus,

$$|f_{n+1}(z) - f_n(z)| < 4^n |\lambda^n \tilde{z} - \lambda^n z| < 4^n \cdot 12 \cdot \lambda^{2(n+1)}|z|^2.$$

Since $4\lambda^2 < 1$, the sum of such errors over all n is finite, establishing the uniform convergence of f_n to an analytic function f on $B(0; 0.04)$.

We now verify the claim that S is only evaluated inside $B(\phi; 0.08)$. Since all evaluation points are of the form $f_n(\lambda z)$ for some $z \in B(0; 0.08)$ and $\lambda < \frac{1}{2}$, it suffices to show that $|f_n(z) - (\phi + z)| < |z|$ for $n \geq 0$, $z \in B(\phi; 0.04)$. This is the case, because

$$\begin{aligned} |f_n(z) - (\phi + z)| &< \sum_{k=0}^{\infty} |f_{n+1}(z) - f_n(z)| < \sum_{k=0}^{\infty} 4^k \cdot 12 \cdot \lambda^{2(k+1)}|z|^2 \\ &< (0.04) 12\lambda^2 \frac{1}{1 - 4\lambda^2}|z| < (0.04) \cdot 24.7|z| < |z|, \end{aligned} \tag{39}$$

as required. □

We can use Lemma 9 to numerically solve for the first several coefficients in the power series for f at $z = 0$:

$$\begin{aligned} f(z) \approx\, & 1.618 + z + 1.756z^2 + 3.207z^3 + 6.003z^4 + 11.431z^5 + 22.045z^6 \\ & + 42.936z^7 + 84.269z^8 + 166.425z^9 + 330.352z^{10} + \cdots \end{aligned}$$

The radius of convergence of f appears to be about 0.48.

By conjugating with the function f, we linearize the function R near its attractive fixed point ϕ, i.e. $R(z) = f(\lambda f^{-1}(z))$. This makes iterating easy: $R^n(z) = f(\lambda^n f^{-1}(z))$. It is therefore crucial to know an explicit neighborhood on which f is invertible.

Lemma 15 *Suppose that h is analytic in $B(0;r)$ and that $Re(h'(z)) > \alpha > 0$ in this neighborhood. Then h^{-1} is well defined in $B(h(0);\alpha r)$ and takes values in $B(0;r)$.*

Proof. To see that h is 1-1 in $B(0;r)$ is easy. To see that the range of h contains $B(f(0);\alpha r)$, consider the curve $\gamma_\rho(\theta) = \rho e^{i\theta}$. For all θ, $|h(\gamma(\theta)) - h(0)| > \alpha\rho$ and the image of this curve wraps around the ball $B(h(0);\alpha\rho)$. Letting ρ vary between 0 and r, we sweep out the entire ball $B(0;\alpha r)$. $\square$

Applying Lemma 15 to the function f defined in Lemma 14, we obtain the following result:

Lemma 16 *The function $f^{-1}(z)$ is well defined, analytic and single-valued in $B(\phi;0.03)$, and takes values in $B(0;0.06)$.*

Proof. We seek a neighborhood of the origin in which $Re(f') > 0.5$. Recall that $f'(z) = \lim_{n\to\infty} f_n'(z)$, where $f_n(z) = S^n(\phi + \lambda^n z)$. Thus

$$\begin{aligned} f_n'(z) &= \frac{S'(S^{n-1}(\phi+\lambda^n z))}{S'(\phi)} \frac{S'(S^{n-2}(\phi+\lambda^n z))}{S'(\phi)} \cdots \frac{S'(\phi+\lambda^n z)}{S'(\phi)} \\ &= \exp\left(\sum_{j=0}^{n-1} \log(S'(S^j(\phi+\lambda^n z)) - \log(S'(\phi))\right). \end{aligned}$$

From equation 39, $|S^j(\phi+\lambda^n z) - (\phi + \lambda^{n-j} z)| < |f_j(\lambda^{n-j} z) - (\phi + \lambda^{n-j})| < \lambda^{n-j}|z|$ for $z \in B(0;0.08)$ and $n \geq j$. We control the summands by observing that $\left|\frac{\partial}{\partial z}\log(S'(z))\right| = \left|\frac{S''(z)}{S'(z)}\right| = \left|\frac{-3}{2(2-z)}\right|$, which is less than 6 when $z \in B(\phi;0.08)$. Thus,

$$\begin{aligned} |\log(f_n'(z))| &< \sum_{j=0}^{n-1} 6 \cdot 2\lambda^{n-j}|z| \\ &< \sum_{k=1}^{\infty} 6 \cdot 2 \cdot \lambda^k |z| = \frac{12\lambda}{1-\lambda}|z| < 11|z|. \end{aligned}$$

When $|z| < 0.06$, this sum is less than $\log(2)$. Thus, $Re(f_n'(z)) > 0.5$ in this neighborhood. Since $f_n \to f$ uniformly in this neighborhood, the same bounds are true for f', concluding the proof. $\square$

The first several terms in the power series for $f^{-1}(z)$ for z near ϕ are:

$$\begin{aligned} f^{-1}(z) \approx\, & (z-\phi) - 1.756(z-\phi)^2 + 2.961(z-\phi)^3 - 4.924(z-\phi)^4 \\ & +8.131(z-\phi)^5 - 13.375(z-\phi)^6 + 21.940(z-\phi)^7 - 35.919(z-\phi)^8 \\ & +58.720(z-\phi)^9 - 95.885(z-\phi)^{10} + \cdots \end{aligned}$$

The radius of convergence of f^{-1} appears numerically to be exactly $\phi - 1$. It cannot be any larger because the point $z = 1$ is in the Julia set, and is at distance $\phi - 1$ from ϕ.

Proof of Theorem 12. We have constructed f and f^{-1} in the lemmas above. For a general initial condition $v_1 \in F(R)$ we will have $v_n \in B(\phi; 0.03)$ for all sufficiently large n. Let n_0 be the first such n. In this case we set $\beta = \lambda^{-n_0} f^{-1}(v_{n_0}) \in \lambda^{-n_0} B(0; 0.06)$. Then for $j \geq n_0$, $v_j = f(\lambda^j \beta)$.

We now shift attention to the γ_n. Recall that $\gamma_n = \prod_{j=1}^{\infty} (v_{n+j})^{\frac{-1}{2^j}}$. This is equal to $g(\beta\lambda^n)$, where

$$g(z) := \exp\left(\sum_{j=1}^{\infty} \frac{-1}{2^j} \log(f(\lambda^j z)) \right).$$

Since $\log(f(z))$ is analytic in $B(0; 0.08)$, $g(z)$ is analytic in $B(0; 0.16) \in \lambda^{-1} B(0; 0.08)$. The first several terms are:

$$\begin{aligned} g(z) \approx\, & 0.618 - 0.118z - 0.0581z^2 - 0.0352z^3 - 0.0240z^4 - 0.0177z^5 \\ & -0.0137z^6 - 0.0111z^7 - 0.00916z^8 - 0.00774z^9 - 0.00665z^{10} \end{aligned}$$

For $n \geq n_0 - 1$, γ_n is well defined. Thus $\omega^{2^{n_0-1}}$ is well defined, and

$$u_n = \omega^{2^n} \gamma_n \qquad for \qquad n \geq n_0 - 1,$$

which verifies (ii). $\square$

We now derive rigorous bounds for the values of β and ω, in the case of our given initial conditions $u_0 = 1$ and $u_1 = 2$.

Recall that f^{-1} is defined in $B(\phi; 0.03)$ and takes values in $B(0; 0.06)$. Inside the smaller domain $B(\phi; 0.005)$ we control the derivatives of f^{-1} using the

Cauchy integral formula:

$$\frac{1}{n!}\frac{\partial^n}{\partial z^n}f^{-1}(z) < \frac{2 \cdot 0.03}{(0.03 - 0.005)^n} = (0.06) \cdot 40^n.$$

Recall also that for n large enough that $v_n \in B(\phi; 0.03)$, we have $\beta = \lambda^{-n} f^{-1}(v_n)$. We calculate v_{10} using exact arithmetic and observe that $v_{10} = \phi + \tau$, where

$$\tau \in [0.000264219375415529, 0.000264219375415530].$$

Using the sixth derivative estimate,

$$|\lambda^{10} \cdot \beta - P_5(f^{-1}(v_{10}))| < (0.06) \cdot 40^6 \cdot \tau^6 < 8.4 \cdot 10^{-14},$$

where P_5 denotes the truncation of the power series after 5 terms. Thus we obtain an estimate for β:

$$\beta \in [0.479835559, 0.479835561].$$

In order to obtain bounds for ω we use bounds for the derivatives of g. Inside $B(0; 0.16)$, g is bounded by 1.0. Using the Cauchy integral formula inside $B(0, 0.06)$,

$$|\frac{1}{n!}\frac{\partial^n}{\partial z^n} g(z)| < 10^n.$$

Thus, bounds on $\lambda^{10}\beta$ imply bounds on $\gamma_{10} = g(\lambda^{10}\beta)$. Explicitly,

$$\gamma_{10} \in [0.61800281229753, 0.61800281229756].$$

The use of the equation $u_{10} = \omega^{2^{10}}\gamma_{10}$ gives the desired bounds for ω:

$$\omega \in [1.8445475709350505, 1.8445475709350507].$$

One can obtain more accurate estimates for β and ω by using more than five terms for f^{-1} and g, or by using u_{n-1} and u_n for $n > 10$.

To conclude, we can choose $n_0 = 3$ in Theorem 12 because $|\beta\lambda^2| < 0.08$. Theorem 1 is valid for $n \geq n_0 - 1 = 2$.

Acknowledgements

We are indebted to Andrew Odlyzko. Mario Szegedy and the referees for helpful comments and references, and to Ed Thurber for bringing the reference [7] to our attention.

References

[1] A. F. Beardon, *Iteration of Rational Functions,* Springer-Verlag: New York 1991.

[2] P. Blanchard, Complex analytic dynamics on the Riemann sphere, Bull. Amer. Math. Soc. **11** (1984), 85–141.

[3] H. Brolin, Invariant sets under iteration of rational functions, Ark. Mat. *6* (1965), 103–144.

[4] E. Boros and Z. Furedi, Rectangular Dissections of a Square, European J. Combin. **9** (1988), 271–280.

[5] L. Carlson and T. W. Gamelin, *Complex Dynamics,* Springer-Verlag: New York 1993

[6] E. G. Coffman, Jr., G. S. Leuker, J. Spencer and P. M. Winkler, Packing Random Rectangles, Probability Theory and Related Fields **120** (2001), 585–599.

[7] R. W. Irving and P. Leather, The complexity of counting stable marriages, SIAM J. Computing **15** (1986), 655-667.

[8] G. Koenigs, Recherches sur les intégrales de certaines équations fonctionelles, Ann. Sci. École Norm. Sup. **1** (1884), Supp. 3–41.