Well-Spaced Labellings of Points in Rectangular Grids

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Abstract

We describe methods to label the $M_1 \times M_2$ grid with the integers 1 to $M_1 M_2$ so that any K consecutively labelled cells are relatively far apart in the grid, in the Manhattan metric. Constructions of such labellings are given which are nearly optimal in a range of conditions. Such labellings can be used in addressing schemes for storing data on two-dimensional arrays that include randomly located "blobs" of defective cells. The data can be precoded using block error-correcting codes before storage, and the usefulness of well-spaced points is to decrease the probability of "burst" errors which cannot be corrected. Possible applications include the storage of speech or music on low-quality memory chips, and in "holographic memories" to store bit-mapped data.

More generally, we present a general family of mappings of the integers 1 to $M_1 M_2 \dots M_d$ onto the d-dimensional grid of size $M_1 \times M_2 \times \dots \times M_d$, called mixed radix vector mappings. These mappings give labellings whenever they are one-to-one. We give a sufficient condition for these mappings to be one-to-one which is easy to verify in many cases.

Key Words: radix expansions, error correction, packings

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1. Introduction

Let $[N_1, N_2]$ denote the set of consecutive integers $\{N_1, N_1+1, \ldots, N_2\}$. The *d-dimensional rectangular grid* $G(M_1, M_2, \ldots, M_d)$ is the set of lattice points

$$\begin{aligned} G(M_1, \ldots, M_d) &= [0, M_1 - 1] \times [0, M_2 - 1] \times \ldots \times [0, M_d - 1] \\ &= \{(m_1, \ldots, m_d) \in \mathbb{Z}^d : 0 \le m_j \le M_j - 1\} \ . \end{aligned} \tag{1.1}$$

A *labelling* of $G(M_1, M_2, \ldots, M_d)$ is a one-to-one mapping

$$\phi : [1, \prod_{j=1}^{d} M_j] \to G(M_1, M_2, \ldots, M_d) \ . \tag{1.2}$$

A labelling ϕ is clearly onto, hence it has a well-defined inverse ϕ^{-1}. We think of a labelling as giving an access ordering to the $\prod_{j=1}^{d} M_j$ cells in the grid $G(M_1, M_2, \ldots, M_d)$. We consider the general problem of finding labellings with the property that points with labels that are close are spaced far apart in the grid. We measure distance in the grid using the L^1-*norm*,

$$||(m_1, \ldots, m_d) - (m_1', \ldots, m_d')|| := \sum_{i=1}^{d} |m_i - m_i'| . \tag{1.3}$$

In the two-dimensional case there are a number of applications for such well-spaced labellings, which motivated this work. One application consists of schemes to address memory locations on chips which have memory arrays containing randomly distributed defects consisting of variable-sized "blobs" of defective memory cells. Such labellings are useful to minimize the probability of consecutively accessed locations being defective. The data can be pre-encoded using a block error-correcting code before storing it in such an array, and the usefulness of well-spaced points is to decrease the probability of "burst" errors, which block coding cannot handle. A second application concerns the design of proposed "holographic memories" to store bit-mapped data, in situations where scanning errors introduce "blocks" of blank data, cf. Bruckstein, Holt and Netravali [3]. In both these applications one wants labelling schemes ϕ that are simple to calculate and to invert.

In §2 we give a general construction of mappings $\phi : [1, \prod_{j=1}^{d} M_j] \rightarrow G(M_1, \ldots, M_d)$, *called mixed radix vector mappings*, which are mappings

$$\phi(n) = \sum_{j=1}^{r} k_j \mathbf{y}_j \ , \tag{1.4}$$

in which the vectors $\mathbf{y}_1, \ldots, \mathbf{y}_r \in \mathbb{Z}^d$ are a given set of r vectors, together with a mapping

$$n \longmapsto (k_1, \ldots, k_r) \ , \tag{1.5}$$

which is the mixed radix expansion of n to base $\mathbf{B} = (B_1, B_2, \ldots, B_r)$, as defined in §2. Such mappings are not necessarily one-to-one. We give a sufficient condition for such a map to be one-to-one which applies in many situations (Theorem 2.1). Suitable choices of the parameters $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r\}$ and $(B_1, B_2, \ldots, B_r)$ yield well-spaced labellings, particularly in the two-dimensional case.

In this paper we study the two-dimensional case, in §3 and §4. In §3 we consider the problem of constructing labellings that have every two consecutively labelled points far apart. We show that every admissible labelling has two consecutively labelled points at L^1-distance at most $\lfloor \frac{M_1+M_2}{2} \rfloor$. We find mixed radix vector labellings that achieve separation of consecutive labels by $\lfloor \frac{M_1+M_2}{2} \rfloor - 2$. (Theorem 3.1).

In §4 we consider the problem of constructing labellings that well separate k consecutive labels, for $k \geq 3$. We prove upper bounds for the attainable separation of $\frac{3M_2}{k}$ if $M_2 \geq \frac{k}{3} M_1$, and $\left(\frac{8M_1M_2}{k}\right)^{1/2}$ if $M_1 \leq M_2 \leq \frac{k}{3} M_1$. (Theorem 4.1). These bounds are shown to be attainable within a multiplicative factor of 4 by mixed radix labellings for many values of M_1, M_2 and k. (Theorem 4.2)

This paper was motivated by the two-dimensional case and does not consider labellings in dimension $d \geq 3$, except for the criterion of §2 for mixed radix vector mappings to be one-to-one on d-dimensional grids. We leave it as open problems to obtain good upper bounds on the maximum attainable L^1-distance between members of a labelling at distance at most m, and to determine whether suitable mixed radix vector mappings give nearly optimal well-spaced labellings of points for various d-dimensional rectangular grids when $d \geq 3$. For the upper bound we note a connection with a generalization of the unsolved problem D1 "Spreading Points in a Square" in Croft, Falconer and Guy [4, pp. 108-110]. The problem of packing

k points in an d-dimensional cube, so as to maximize the minimal distance between any two of the points (using L^1-norm) certainly gives an upper bound (after rescaling) on how much separation is possible in every consecutive k-tuple of a labelling of a d-dimensional cubical grid. The results of this paper suggest that for d-dimensional grids which are roughly cubical there exist well-spaced labellings with spacing within a multiplicative constant (depending on the dimension d) of such an upper bound.

We also note a relation to *magic squares*, and more generally *magic d-cubes.* These are a special kind of labelling of the d-cube $G(M, M, \ldots, M)$ by integers in which all the row sums are required to be equal, cf. [1]–[6]. One of the standard constructions of magic squares and magic d-cubes using the integers from 1 to M^d is actually a mixed radix vector labelling, see [1], [2].

2. Mixed Radix Vector Mappings

We present a general construction of d-dimensional mappings

$$\phi : [1, \prod_{j=1}^{d} M_j] \to G(M_1, M_2, \ldots, M_d)$$

which we call mixed radix vector mappings, or MRV mappings for short. Whenever such a mapping ϕ is one-to-one then it gives an admissible labelling of

$$G(M_1, M_2, \ldots, M_d) = [0, M_1 - 1] \times [0, M_2 - 1] \times \ldots \times [0, M_d - 1] \ .$$

A mixed radix expansion is specified by an r-dimensional vector $\mathbf{B} := (B_1, B_2, \ldots, B_r)$ of positive integers, with all $B_i > 1$, called the *mixed base.* The *mixed radix expansion to base* $\mathbf{B}$ of an integer m satisfying $0 \le m < \prod_{i=1}^{r} B_i$ is

$$m = \sum_{i=1}^{r} d_i \prod_{j=1}^{i-1} B_j \ , \text{ with } \ 0 \le d_i < B_i \quad \text{for} \quad 1 \le i \le r \ , \tag{2.1}$$

To apply this to the grid $G(M_1, \ldots, M_d)$ we need

$$\prod_{i=1}^{r} B_i \ge \prod_{j=1}^{d} M_j \ . \tag{2.2}$$

The general construction is as follows. A *mixed radix vector mapping* or *MRV mapping* of $G(M_1, \ldots, M_d)$ is specified by a set of $r \ge 1$ nonzero vectors $\mathbf{y}_1, \ldots, \mathbf{y}_r \in \mathbb{Z}^d$ and by a mixed radix base

$$\mathbf{B} = (B_1, B_2, \ldots, B_r) \ ,$$

such that $\prod_{i=1}^{r} B_i \geq \prod_{j=1}^{d} M_j$. For $1 \leq m \leq \prod_{j=1}^{d} M_j$ take the mixed radix expansion

$$m - 1 = d_r \prod_{i=1}^{r} B_i + d_{r-1} \prod_{i=1}^{r-1} B_i + \ldots + d_2 B_1 + d_1$$

with $0 \leq d_i < B_i$, and associate to it the integer vector

$$\mathbf{v}(m) := \sum_{i=1}^{r} d_i \mathbf{y}_i = (v_1(m), v_2(m), \ldots, v_d(m)) \ . \tag{2.3}$$

The *MRV* map $\phi : [1, \prod_{j=1}^{d} M_j] \to G(M_1, M_2, \ldots, M_d)$ is given by

$$\phi(m) = (v_1(m) (\text{mod } M_1), \ldots, v_d(m) (\text{mod } M_d)) \ . \tag{2.4}$$

The integer r is called the *rank* of the MRV mapping. It bears no relation to the dimension d of the image space, and can be smaller or larger than d.

We are interested in the special case of MRV mappings ϕ that are one-to-one. We call these *MRV labellings.* There does not seem to be a simple general criterion for a MRV mapping to be one-to-one, but we give a sufficient condition in Theorem 2.1 below.

To formulate this sufficient condition we need a definition. Given a set $\mathcal{Y} = \{\mathbf{y}_1, \ldots, \mathbf{y}_r\}$ of vectors in $\mathbb{Z}^d$, and a grid $G = G(M_1, \ldots, M_d)$. Given $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r$ set

$$\mathbf{w}(\mathbf{m}) = (w_1, w_2, \ldots, w_m) := m_1 \mathbf{y}_1 + \ldots + m_r \mathbf{y}_r \ . \tag{2.5}$$

We associate to $\mathcal{Y}$ and the grid G the r-dimensional integer lattice

$$\Lambda := \Lambda_{\mathcal{Y},G} = \{\mathbf{m} \in \mathbb{Z}^r : w_j \equiv 0 \pmod{M_j} \ \text{ for } \ 1 \leq j \leq d\} \ . \tag{2.6}$$

We call Λ the *embedding lattice* of $(\mathcal{Y}, G)$, since it describes how the vectors $\mathcal{Y}$ fall into G. The embedding lattice satisfies

$$(M_1 M_2 \cdots M_d \mathbb{Z})^d \subseteq \Lambda \subseteq \mathbb{Z}^d \ . \tag{2.7}$$

and the index $[\mathbb{Z}^d : \Lambda] := \#(\mathbb{Z}^d / \Lambda)$ always divides $\prod_{j=1}^{d} M_j$. We say that $\mathcal{Y}$ is *nondegenerate* for the grid $G(M_1, \ldots, M_d)$ if $[\mathbb{Z}^d : \Lambda] = \prod_{j=1}^{d} M_j$.

Theorem 2.1. *Suppose that a mixed radix vector map ϕ of $G(M_1, M_2, \ldots, M_d)$ is specified by the set of vectors $\mathcal{Y} = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r\}$ and the mixed radix base $\mathbf{B} = (B_1, B_2, \ldots, B_r)$.*

(i). *If each nonzero vector $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ in the embedding lattice $\Lambda_{\mathcal{Y},G}$ satisfies:*

$$(*) \quad \textit{There is some coordinate } i \textit{ with } \ |m_i| \geq B_i \ , \tag{2.8}$$

then the map ϕ is one-to-one.

(ii). *If $\Pi_{i=1}^{r} B_i = \Pi_{j=1}^{d} M_j$ then $(*)$ is a necessary and sufficient condition for ϕ to be one-to-one.*

Proof. (i). We argue by contradiction. Suppose that ϕ is not one-to-one, so that there exist $1 \le m_1 < m_2 \le M_1 M_2 \dots M_d$ such that $\phi(m_1) = \phi(m_2)$. Let the mixed radix expansions be

$$m_\ell - 1 = \sum_{i=1}^{r} d_i^{(\ell)} \prod_{k=1}^{i-1} B_k \ , \ \text{for} \ \ \ell = 1, 2 \ . \tag{2.9}$$

and set $\mathbf{d}^{(\ell)} \equiv (d_1^{(\ell)}, \dots, d_r^{(\ell)})$ for $\ell = 1, 2$. Now $\phi(m_1) = \phi(m_2)$ means that the vector $\mathbf{w} = (w_1, \dots, w_d)$ given by

$$\mathbf{w} = \sum_{i=1}^{r} (d_i^{(1)} - d_i^{(2)}) \mathbf{y}_i$$

satisfies $w_j \equiv 0 \pmod{M_j}$ for $1 \le j \le d$. Thus $\mathbf{d}^{(1)} - \mathbf{d}^{(2)} \in \Lambda_{\mathcal{Y},G}$. However the mixed radix expansion property gives

$$-B_i < d_i^{(1)} - d_i^{(2)} < B_i \ , \quad 1 \le i \le r \ ,$$

because $0 \le d_i^{(\ell)} < B_i$. Since $\mathbf{d}^{(1)} \ne \mathbf{d}^{(2)}$, this contradicts property (2.8).

(ii). We must show that if $\prod_{i=1}^{r} B_i = \prod_{j=1}^{d} M_j$ then any one-to-one MRV labelling map ϕ satisfies condition (*), given as (2.8). Indeed $\{\phi(m) : 1 \le m \le M_1 M_2 \dots M_d\}$ enumerates the set of $M_1 \dots M_d = B_1 \dots B_r$ vectors $\mathbf{w} \pmod{\mathbf{M}}$, given by

$$\mathbf{w} = d_1 \mathbf{v}_1 + d_2 \mathbf{v}_2 + \dots + d_r \mathbf{v}_r \ , \quad \text{with} \quad 0 \le d_i < B_i, \ 1 \le i \le r, \tag{2.10}$$

and

$$\mathbf{w} \pmod{\mathbf{M}} := (w_1 \pmod{M_1}, \ w_2 \pmod{M_2}, \dots, w_d (\text{mod} \ \ M_d)) \ .$$

By hypothesis these residues are all distinct, and this implies that $[\mathbb{Z}^r : \Lambda_{\mathcal{Y},G}] \ge M_1 M_2 \dots M_d$. Thus

$$[\mathbb{Z}^r : \Lambda] = M_1 M_2 \dots M_d \ , \tag{2.11}$$

and $\mathcal{Y}$ is nondegenerate. It follows that

$$\mathcal{W} = \{\mathbf{w} : \mathbf{w} \ \text{ given by (2.10)}\} \ ,$$

is a fundamental domain for the quotient lattice $\mathbb{Z}^r / \Lambda_{\mathcal{Y},G}$. In particular, given any nonzero vector

$$\mathbf{e} = (e_1, \dots, e_r) \in \mathbb{Z}^r \ \text{ with } \ -B_i < e_i < B_i \ \text{ for } \ \ 1 \le i \le r \ , \tag{2.12}$$

set

$$\mathbf{w}' = e_1\mathbf{v}_1 + \ldots + e_r\mathbf{v}_r$$

then we can find two distinct vectors $\mathbf{w}_1, \mathbf{w}_2$ of the form (2.10) such that

$$\mathbf{w}' = \mathbf{w}_1 - \mathbf{w}_2 \ .$$

Since the $\mathbf{w}_i$ are all distinct (mod M) we conclude that $\mathbf{w}' \not\equiv \mathbf{0}$ (mod $\mathbf{M}$), hence $\mathbf{e} \notin \Lambda_{\mathcal{Y},G}$. Thus the embedding lattice $\Lambda_{\mathcal{Y},G}$ contains no vector of the form (2.12) except the zero vector, which verifies (2.8). $\square$

As an example, consider the rectangular grid $G(6,6)$, and take the MRV mapping given by $\mathcal{Y} = \{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4\}$ with

$$\mathbf{y}_1 = (3,0), \ \mathbf{y}_2 = (0,3), \ \mathbf{y}_3 = (1,0), \ \text{ and } \ \mathbf{y}_4 = (0,1) \ ,$$

and with radix vector $\mathbf{B} = (2,2,3,3)$. We have $r = 4$ and $\Pi_{i=1}^4 B_i = \Pi_{i=1}^2 M_i = 36$. The mixed radix expansion is

$$m - 1 = 12d_4 + 4d_3 + 2d_2 + d_1 \ , \quad 0 \le d_1 < B_i \ .$$

and

$$\phi(m) = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3 + d_4\mathbf{v}_4 \ .$$

The embedding lattice Λ in $\mathbb{Z}^4$ is

$$\Lambda = (2\mathbb{Z} \times 2\mathbb{Z} \times 6\mathbb{Z} \times 6\mathbb{Z}) + \{(0,0,0,0), \ (1,0,3,0), \ (0,1,0,3), \ (1,1,3,3)\} \ .$$

This lattice satisfies the criterion (2.8), hence Theorem 2.1 shows that ϕ is one-to-one. The resulting labelling is pictured in Figure 2.1, using matrix notation, so that rows are numbered downward from 0 to 5, and columns are numbered across, from 0 to 5.

$$\begin{bmatrix} 1 & 5 & 9 & 2 & 6 & 10 \\ 13 & 17 & 21 & 14 & 18 & 22 \\ 25 & 29 & 33 & 26 & 30 & 34 \\ 3 & 7 & 11 & 4 & 8 & 12 \\ 15 & 19 & 23 & 16 & 20 & 24 \\ 27 & 31 & 35 & 28 & 32 & 36 \end{bmatrix}$$

We end our discussion of the d-dimensional case by commenting on the problem of computing the inverse of the map ϕ when it is one-to-one. This map may be hard to invert in the general case. In the special case that $\Pi_{i=1}^r B_i = \Pi_{j=1}^d M_j$, the inversion problem reduces to solving a system of linear Diophantine equations, and for this there are polynomial-time algorithms. In the two-dimensional case it appears that there exist good MRV labellings of this sort having rank $r \le 4$.

3. Two-Dimensional Case: Separating Consecutive Points

We consider the problem of labelling the $M_1 M_2$ cells in an $M_1 \times M_2$ rectangular grid in such a way that consecutively labelled cells are far apart. The rectangular grid is

$$G(M_1, M_2) := \{(i,j) : 0 \le i \le M_1 - 1,\ 0 \le j \le M_2 - 1\} .$$

Recall that we measure distance in the grid using the Manhattan metric or (L^1-norm):

$$||(i,j) - (i',\ j')||_1 := |i - i'| + |j - j'| .$$

An *admissible labelling* of the $M_1 \times M_2$ grid $G(M_1, M_2)$ is a bijection $\phi : [1, M_1 M_2] \to G(M_1, M_2)$. The *2-spacing* $s_2(\phi)$ of a labelling is

$$s_2(\phi) = \min\{||\phi(m) - \phi(m+1)||_1 : 1 \le m \le M_1 M_2 - 1\} . \tag{3.1}$$

The *circular 2-spacing* $w_2^*(\phi)$ is the minimum Manhattan distance between consecutive points in G_{M_1,M_2}, viewing G_{M_1,M_2} as a torus. That is,

$$s_2^*(\phi) = \min\{||\phi(m) - \phi(m+1)||_T : 1 \le m \le M_1 M_2 - 1\} , \tag{3.2}$$

in which $|| \cdot ||_T$ is the torus Manhattan metric

$$||(i,j) - (i',\ j')||_T := |i - i' \mod{}^* M_1| + |j - j' \mod{}^* M_2| , \tag{3.3}$$

where $r \bmod{}^* M$ denotes the least absolute value residue mod M, i.e. that residue which lies in the range $[-M/2,\ (M-1)/2]$. Clearly

$$s_2^*(\phi) \le s_2(\phi) . \tag{3.4}$$

We start with an easy upper bound.

Theorem 3.1. *For any labelling ϕ,*

$$s_2(\phi) \le \begin{cases} \lfloor \dfrac{M_1 + M_2}{2} \rfloor, & \text{if } M_1 \text{ or } M_2 \text{ is even} \\[2ex] \dfrac{M_1 + M_2}{2} - 1, & \text{if } M_1, M_2 \text{ are both odd} . \end{cases} \tag{3.5}$$

Proof. The "central cell" in the grid is $\mathbf{v} := \left(\lfloor \frac{M_1+1}{2} \rfloor, \lfloor \frac{M_2+1}{2} \rfloor\right)$. All cells in the grid are at Manhattan distance at most $\lfloor \frac{M_1+M_2}{2} \rfloor$ from $\mathbf{v}$ if M_1 or M_2 is even, and at distance at most $\frac{M_1+M_2}{2} - 1$ from $\mathbf{v}$ if $M_1,\ M_2$ are both odd. The "extreme" cell is (M_1, M_2).

Now let $\phi(\mathbf{v}) = k$. At least one of the values $k-1$ and $k+1$ falls inside the range $[1, M_1M_2]$. If k' denotes this value, then

$$s_2(\phi) \leq ||\phi(k) - \phi(k')||_1 = || \left(\lfloor \frac{M_1+1}{2} \rfloor, \lfloor \frac{M_2+1}{2} \rfloor \right) - (i,j)||_1 \ ,$$

which gives (3.5). $\square$

Theorem 3.2. *For all $M_1,\ M_2 \geq 2$ there exists a mixed radix vector labelling ϕ such that*

$$s_2^*(\phi) = \begin{cases} \lfloor \frac{M_1+M_2}{2} \rfloor - 1 & \textit{if } M_1 \textit{ or } M_2 \textit{ is odd} \\ \lfloor \frac{M_1+M_2}{2} \rfloor - 2 & \textit{if } M_1 \textit{ and } M_2 \textit{ are even} \ . \end{cases} \tag{3.6}$$

Proof. We exhibit mixed radix vector labellings having the desired property. We use different constructions according to the parity of M_1 and M_2.

Case 1. *M_1 and M_2 are both odd.*

Set

$L =$ least common multiple of M_1 and M_2,

$G =$ greatest common divisor of M_1 and M_2

and note that $M_1M_2 = LG$. We use the MRV mapping ϕ of rank 2 given by

$$\mathbf{y}_1 = \left(\frac{M_1-1}{2}, \frac{M_2+1}{2} \right), \quad \text{and} \quad \mathbf{y}_2 = (1,0) \ ,$$

with mixed radix $\mathbf{B} = (L, G)$. If $m = iL + j$ with $0 \leq i < G$ and $0 \leq j < L$, then

$$\phi(m) = \left(i + j\left(\frac{M_1-1}{2}\right) \pmod{M_1} \ , \ j\left(\frac{M_2+1}{2}\right) \pmod{M_2} \right) \ .$$

We prove that ϕ is an MRV labelling. Suppose that $\phi(m) = \phi(m')$ with associated radix expansions (i,j) and (i',j'). Then

$$j\left(\frac{M_2-1}{2}\right) \equiv j'\left(\frac{M_2-1}{2}\right) \pmod{M_2} \ .$$

Since M_2 is odd, 2 is invertible (mod M_2), hence this gives $j \equiv j' \pmod{M_2}$, so $j = j'$. Now

$$i + j\left(\frac{M_1-1}{2}\right) = i + j'\left(\frac{M_1-1}{2}\right) \equiv i' + j'\left(\frac{M_1-1}{2}\right) \pmod{M_1}$$

hence $i \equiv i' \pmod{M_1}$ so $i = i'$, and ϕ is one-to-one.

We now show that

$$s_2^*(\phi) \geq \lfloor \frac{M_1+M_2}{2} \rfloor - 1 \ . \tag{3.7}$$

To see this, note that incrementing m by 1 increments j by 1 (mod L), while

$$L\left(\frac{M_1-1}{2},\ \frac{M_2-1}{2}\right) \equiv (0,0) \pmod{\mathbf{M}} \ .$$

Thus we always add the vector $\left(\frac{M_1-1}{2},\ \frac{M_2-1}{2}\right)$, which has torus Manhattan length $\frac{M_1+M_2}{2}-1$. At certain steps the vector (1,0) is also added, which gives the vector $\left(\frac{M_1+1}{2},\ \frac{M_2-1}{2}\right)$, which also has torus Manhattan length $\frac{M_1+M_2}{2}-1$. Thus (3.10) follows.

Case 2. *One of $M_1,\ M_2$ is even and the other is odd.*

We treat the case M_1 is even and M_2 is odd, the other case being similar. For $1 \le K \le M_1M_2$ set $L^* = 2M_2$ and consider the MRV mapping ϕ with

$$\mathbf{y}_1 = \left(\frac{M_1}{2},\ \frac{M_2-1}{2}\right) \quad \text{and} \quad \mathbf{y}_2 = (1,0) \quad ,$$

and mixed radix base $\mathbf{B} := \left(2M_2,\ \frac{M_1}{2}\right)$. We will show that this is the desired MRV labelling. We have

$$\phi(m) = \left(i + j\frac{M_1}{2} \pmod{M_1}\ ,\ \ j\left(\frac{M_2-1}{2}\right) \pmod{M_2}\right) \ , \tag{3.8}$$

where

$$m-1 := iL^* + j\ , \quad \text{with} \ \ 0 \le j \le L^*-1\ , \quad 0 \le i < \frac{M_1}{2}\ . \tag{3.9}$$

To see that ϕ is one-to-one, observe that the second coordinate of $\phi(m)$ determines j (mod M_2), since M_2 is odd. Then the first coordinate determines i(mod $\frac{M_1}{2}$), and also j(mod 2), which determines j (mod $2M_2$). Now m is reconstructible by (3.9).

Next, since $\phi(m+1)-\phi(m)$ is one of $\left(\frac{M_1}{2},\ \frac{M_2-1}{2}\right)$ by $\left(\frac{M_1+2}{2},\ \frac{M_2-1}{2}\right)$, we obtain

$$s_2^*(\phi) \ge \frac{M_1+M_2-3}{2} = \lfloor\frac{M_1+M_2}{2}\rfloor - 1\ .$$

as desired.

Case 3. *M_1 and M_2 are both even.*

This is the most complicated case. A rank three MRV labelling is required. Define L^* to be the least $\ell > 0$ such that

$$\ell\left(\frac{M_1}{2}-1\right) \ \equiv \ 0 \pmod{M_1}\ ,$$

$$\ell\left(\frac{M_2}{2}-1\right) \ \equiv \ 0 \pmod{M_2}\ .$$

Then L^* is given by

$$L^* = \begin{cases} \frac{1}{2}\ l.c.m.\ (M_1, M_2) & \text{if } \dfrac{M_1 M_2}{4} \text{ is odd} \\ \\ l.c.m.\ (M_1, M_2) & \text{if } \dfrac{M_1 M_2}{4} \text{ is even .} \end{cases}$$

Set $G^* = g.c.d.\ (M_1, M_2)$ so that $M_1 M_2 = H^* G^* L^*$, with

$$H^* = \begin{cases} 2 & \text{if } \dfrac{M_1 M_2}{4} \text{ is odd ,} \\ \\ 1 & \text{if } \dfrac{M_1 M_2}{4} \text{ is even .} \end{cases}$$

We use the MRV mapping ϕ defined by

$$\mathbf{y}_1 = \left(\frac{M_1 - 2}{2}, \frac{M_2 - 2}{2} \right) \ , \quad \mathbf{y}_2 = (0, 1) \ \text{ and } \quad \mathbf{y}_3 = (1, 0) \ ,$$

with mixed radix base $\mathbf{B} = (B_1, B_2, B_3) := (L^*, G^*, H^*)$. In the case that $H^* = 1$ this simplifies to a MRV mapping on two generators, since $\mathbf{y}_3$ can be omitted. If

$$m = \ell G^* L^* + i L^* + j$$

with

$$0 \le j \le L^* - 1; \ 0 \le i \le G^* - 1; \ 0 \le \ell \le H^* - 1 \ ,$$

then

$$\phi(m) := (\ell + j \left(\frac{M_1 - 2}{2} \right) \ (\text{mod } M_1), \ i + j \left(\frac{M_2 - 2}{2} \ (\text{mod } M_2) \right) \tag{3.10}$$

We omit a proof that ϕ is one-to-one, which can be derived from (3.10) by an argument similar to that in Case 2.

To establish the bound

$$s_2^*(\phi^*) \ge \frac{M_1 + M_2}{2} - 2 \ ,$$

use the fact that each step from $\phi(m)$ to $\phi(m+1)$ is one of

$$\left(\frac{M_1 - 2}{2}, \frac{M_2 - 2}{2} \right) \ , \quad \left(\frac{M_1}{2}, \frac{M_2 - 2}{2} \right) \ , \text{ or } \left(\frac{M_1}{2} \ , \ \frac{M_2}{2} \right) \ . \quad \square$$

Examples.

As a Case 1 example, take $M_1 = 5$, $M_2 = 5$, in which case $L = 5$, $G = 5$. Here $\left(\frac{M_1-1}{2}, \ \frac{M_2-1}{2}\right) = (2, 2)$ and the labelling is given below. (We use matrix notation, so that rows are numbered 0 to 4,reading down, and columns are numbered 0 to 4, reading across.)

$$\begin{bmatrix} 1 & 24 & 17 & 15 & 8 \\ 6 & 4 & 22 & 20 & 13 \\ 11 & 9 & 2 & 25 & 18 \\ 16 & 14 & 7 & 5 & 23 \\ 21 & 19 & 12 & 10 & 3 \end{bmatrix}$$

In this example

$$s_2^*(\phi) = 4 = \lfloor \frac{M_1 + M_2}{2} \rfloor - 1 \ .$$

Note that $\phi(5)$ and $\phi(6)$ are at distance 4 in the torus Manhattan metric.

As a Case 3 example, for $M_1 = 4,\ M_2 = 6$, we have $L^* = 12$, $G^* = 2$ and $M^* = 1$, and obtain the labelling (in matrix form):

$$\begin{bmatrix} 1 & 13 & 5 & 17 & 9 & 21 \\ 10 & 22 & 2 & 14 & 6 & 18 \\ 7 & 19 & 11 & 23 & 3 & 15 \\ 4 & 16 & 8 & 20 & 12 & 24 \end{bmatrix} .$$

4. Two Dimensional Case: Separating Several Points

We consider the problem of finding labellings ϕ of $G(M_1, M_2)$ that well-separate all sequences of k consecutive points. We define the *k-spacing* of a labelling to be

$$s_k(\phi) := \min\{||\phi(m) - \phi(m+j)|| : 1 \le m < m+j \le M_1M_2,\ 1 \le j \le k\} \ . \qquad (4.1)$$

We define the *circular k-spacing* $s_k^*(\phi)$ analogously; to be

$$s_k^*(\phi) := \min\{||\phi(m) - \phi(m+j)||_T : 1 \le m < m+j \le M_1M_2 \ ,\ 1 \le j \le k\} \ , \qquad (4.2)$$

using the torus Manhattan distance measure (3.3). Note that $s_k^*(\phi) \le s_k(\phi)$.

How large can one make $s_k(\phi)$? We define

$$S_k(M_1, M_2) := \max\{s_k(\phi) : \phi \ \text{ a labelling for } G_{M,N}\} \ .$$

Upper bounds for $S_k(M_1, M_2)$ depend on the shape of the rectangular grid $G(M_1, M_2)$. Using symmetry considerations we may reduce to the case that $M_1 \le M_2$. If the grid is narrow in one direction, there is an upper bound proportional to $\frac{M_1+M_2}{k}$, while if M_1 and M_2 are within a constant ratio of one another, there is an upper bound proportional to $\frac{M_1+M_2}{\sqrt{k}}$, as the following result shows.

Theorem 4.1. *Suppose that $k \geq 3$, and that $M_1 \leq M_2$.*

(i). *If $M_2 \geq \frac{k}{3} M_1$, then*

$$S_k(M_1, M_2) \leq \frac{3M_2}{k} \ . \tag{4.3}$$

(ii). *If $M_1 \leq M_2 \leq \frac{k}{3} M_1$, then*

$$S_k(M_1, M_2) \leq \left(\frac{8M_1M_2}{k} \right)^{1/2} \ . \tag{4.4}$$

Proof. Let $V(\mathbf{x}; D)$ denote the set of cells within Manhattan distance at most D of a cell $\mathbf{x}$ in the square lattice $\mathbb{Z}^2$. The number of such cells is

$$1 + 4\binom{D+1}{2} = 2D^2 + 2D + 1 \ .$$

We obtain upper bounds for $S = S_k(M_1, M_2)$ using a packing argument, based on the geometric fact that none of the regions $V(\mathbf{x}_i; D) \cap G(M_1, M_2)$ can overlap, for $D = \lfloor \frac{1}{2} S \rfloor$. If we define

$$N(\mathbf{x}; D) := |V(\mathbf{x}; D) \cap G(M_1, M_2)| \ ,$$

then we have the bound

$$\sum_{i=1}^{k} N\left(\mathbf{x}_i, \lfloor \frac{1}{2} S \rfloor \right) \leq M_1 M_2 \ . \tag{4.5}$$

To bound S from above we need lower bounds for $N(\mathbf{x}; D)$.

Claim. (i). *If $D \leq M_1 \leq M_2$, then*

$$N(\mathbf{x}; D) \geq \binom{D+2}{2} = \frac{1}{2} D^2 + \frac{3}{2} D + 1 \ , \tag{4.6}$$

and equality occurs if $\mathbf{x}$ is a corner cell of $G(M_1, M_2)$.

(ii). *If $M_1 \leq D \leq M_2$, then*

$$N(\mathbf{x}; D) \geq (D - M_1)M_1 + \binom{M_1 + 1}{2} = DM_1 - \frac{1}{2} M_1^2 + \frac{1}{2} M_1 \ , \tag{4.7}$$

and equality occurs if $\mathbf{x}$ is a corner cell of $G(M_1, M_2)$.

The claim follows by inspection of the possible ways that the region $N(\mathbf{x}, D)$ may intersect the boundary of $G(M_1, M_2)$. Figure 4.1 illustrates two cases for $D \leq M_1$; and Figure 4.2 illustrates cases with $M_1 \leq D \leq M_2$. The distinction below $D \leq M_1$ and $D \geq M_1$ arises because for $D \geq M_1$ two opposite sides of $N(\mathbf{x}, D)$ always lie on the boundary of the rectangle $G(M_1, M_2)$. In all cases the corner cell $\mathbf{x}$ minimizes the number of points in $V(\mathbf{x}, D) \cap G(M_1, M_2)$.

We note that when $M_1 \le M_2$ the bound (4.7) of the claim is actually valid for all $D \le M_2$, because it is weaker than the bound (4.6) when $D \le M_1$. However (4.6) can be applied only when we know that $D = \lfloor \frac{1}{2}S \rfloor \le M_1$. Note that $D = \lfloor \frac{1}{2}S \rfloor \le \frac{1}{2}(M_1 + M_2) \le M_2$.

We obtain upper bounds for S by substituting the bounds of the claim into (4.5). Substituting (4.7) in (4.5) yields

$$k\left(\lfloor \frac{1}{2}S \rfloor M_1 - \frac{1}{2}M_1^2 + \frac{1}{2}M_1\right) \le M_1 M_2 \ ,$$

hence

$$\frac{1}{2}S \le \lfloor \frac{1}{2}S \rfloor + \frac{1}{2} \le \frac{M_2}{k} + \frac{1}{2}\frac{M_1}{k} \tag{4.8}$$

since $M_1 \le M_2$, this yields

$$S \le \frac{3M_2}{k} \ . \tag{4.9}$$

This bound is universally valid, and gives (i) as a special case.

Now if $M_2 \le \frac{k}{3}M_1$, then (4.9) gives $S \le M_1$. In this case we may legitimately apply the bound (4.6) in (4.5) to obtain

$$k\left(\frac{1}{2}\left(\lfloor \frac{1}{2}S \rfloor\right)^2 + \frac{3}{2}\lfloor \frac{1}{2}S \rfloor + 1\right) \le M_1 M_2 \ .$$

Thus

$$\left(\frac{1}{2}S + 1\right)^2 \le \left(\lfloor \frac{1}{2}S \rfloor + \frac{3}{2}\right)^2 = \lfloor \frac{1}{2}S \rfloor^2 + 3\lfloor \frac{1}{2}S \rfloor + \frac{9}{4} \le \frac{2M_1M_2}{k} + \frac{1}{4} \ .$$

which yields

$$(S+2)^2 \le \frac{8M_1M_2}{k} + 1 \ , \tag{4.10}$$

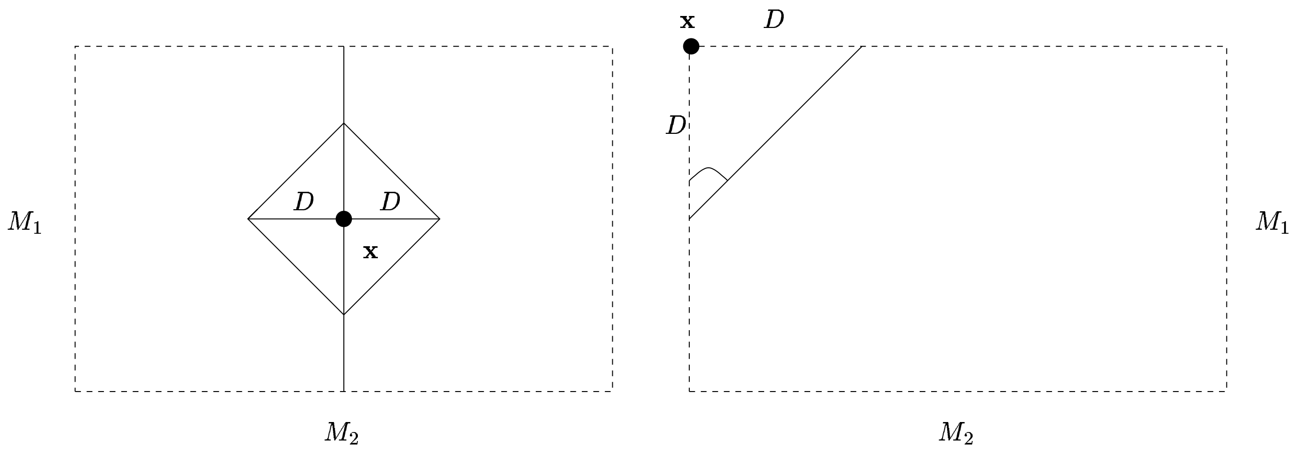
from which the bound of (ii) follows. □

The upper bounds of Theorem 4.1 are conservative and can potentially be improved by a multiplicative constant, which varies between $\sqrt{2}$ and 2, for most values of M_1 and M_2. This occurs because "nearly all" cells $\mathbf{x} \in G(M_1, M_2)$ have $N(\mathbf{x}, D)$ at least twice as large as the value of $N(\mathbf{x}, D)$ of a corner cell, but $N(\mathbf{x}, D)$ is never more than four times as large the value of $N(\mathbf{x}, D)$ for a corner cell.

We show that there exist MRV labellings that achieve k-spacings within a multiplicative constant of $\frac{1}{3}$ of these upper bound range (i) of Theorem 4.1.

Theorem 4.2. *Suppose that $k \ge 3$ and consider a grid $G(M_1, M_2)$ with $M_2 \ge M_1 \ge 3$. If $M_2 \ge \frac{k}{3}M_1$, then there exists a multiple radix vector labelling ϕ with*

$$s_k^*(\phi) \ge \lfloor \frac{M_2}{k} \rfloor - 2 \ . \tag{4.11}$$



(a) $\mathbf{x}$ = center cell ($D \le \frac{1}{2}M_1$)   (b) $\mathbf{x}$ = corner cell

Figure 4.1: Regions $V(\mathbf{x}, D) \cap G(M_1, M_2)$ for $D \le M_1 \le M_2$.

Proof. We use an MRV labelling ϕ that fills the rows of $G(M_1, M_2)$ one at a time. The hypotheses imply $M_2 \ge k$, so $\lfloor \frac{M_2}{k} \rfloor \ge 1$. Let ϕ be the MRV mapping of rank 3 that uses the vectors

$$\mathbf{y}_1 = \left(0, \lfloor \frac{M_2}{k} \rfloor\right) \ , \quad \mathbf{y}_2 = (0, 1) \ , \quad \mathbf{y}_3 = (1, 0) \ ,$$

and the base $\mathbf{B} = (B_1, B_2, B_3)$ given by

$$B_1 = \frac{M_2}{g.c.d.(M_2, \lfloor \frac{M_2}{k} \rfloor)} \ , \quad B_2 = g.c.d.(M_2, \lfloor \frac{M_2}{k} \rfloor) \ , \quad B_3 = M_1 \ .$$

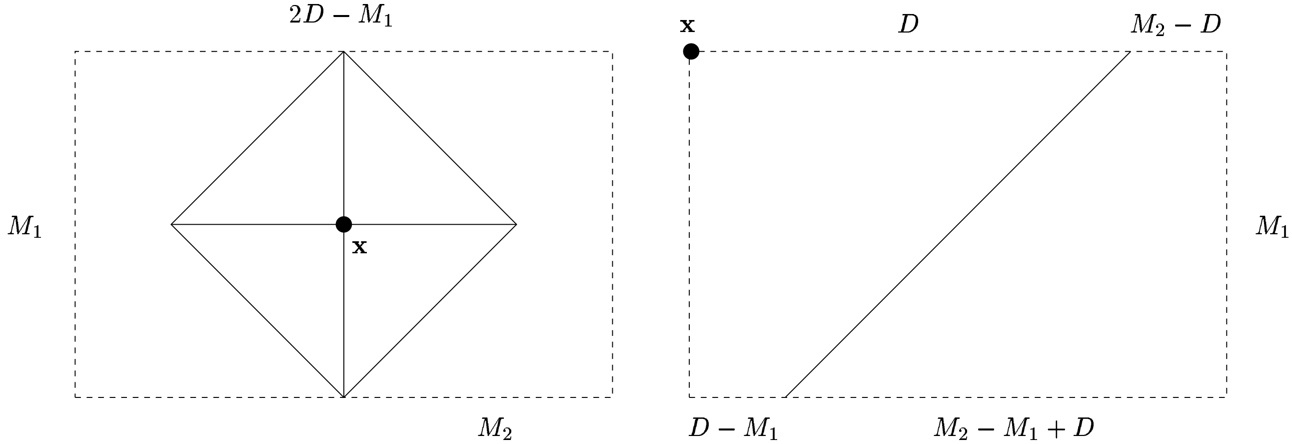
If $B_2 = 1$, then the rank drops to 2, in which case $\mathbf{y}_2$ is omitted. We have $B_1 B_2 B_3 = M_1 M_2$ and the lattice $\Lambda = \Lambda_{\mathcal{Y},G}$ is the set of vectors $(m_1, m_2, m_3) \in \mathbb{Z}^3$ such that

$$m_3 \equiv 0 \pmod{M_1} \ , \tag{4.12a}$$

$$m_1 \lfloor \frac{M_2}{k} \rfloor + m_2 \equiv 0 \pmod{M_2} \ . \tag{4.12b}$$

The criterion of Theorem 2.1(ii) for Λ is easily checked: either $|m_1| \ge B_1$ or $|m_2| \ge B_2$ follows from (4.12b) if $(m_1, m_2) \ne (0, 0)$ while (4.12a) gives $|m_3| \ge M_1 = B_3$ if $m_3 \ne 0$. Thus ϕ is one-to-one, hence is an MRV labelling.

Any k consecutive values of $\mathbf{y}_1$ have each pair of values separated by at least $||\mathbf{y}_1|| = \lfloor \frac{M_2}{k} \rfloor$ in their second coordinate, in the torus metric. Also the cycle of $\mathbf{y}_1$ values which occur before a repeated value is of length at least k, hence the updates by $\mathbf{y}_2$ and $\mathbf{y}_3$ occur at time intervals more than k units apart. It follows that each k consecutive vectors differ from a translate of consecutive multiples of $\mathbf{y}_1$ by at most $||\mathbf{y}_2|| + ||\mathbf{y}_3|| \le 2$. This yields (4.11). $\square$



(a) $\mathbf{x} =$ center cell

(b) $\mathbf{x} =$ corner cell

Figure 4.2: Regions $V(\mathbf{x}, D) \cap G(M_1, M_2)$ for $M_1 \leq D \leq M_2$.

Constructing MRV labellings to get within a multiplicative constant factor of the optimal spacing in the remaining upper bound range (ii) of Theorem 4.1 appears complicated, and a general proof would seem to require consideration of many different MRV mappings. We do not attempt it here. In the special case that $M_1 = M_2 = M$, $k = \ell^2$, and ℓ divides M, a good MRV labelling is easy to come by. It is of rank 4 with

$$\mathbf{y}_1 = \left(0, \frac{M}{\ell}\right) \ , \ \ \mathbf{y}_2 = \left(\frac{M}{\ell}, \ 0\right) \ , \ \ \mathbf{y}_3 = (0, 1) \ \text{ and } \ \mathbf{y}_4 = (1, 0) \ ,$$

with mixed radix $\mathbf{B} = (\ell, \ell, \frac{M}{\ell}, \frac{M}{\ell})$. In this example

$$s_k(\phi) \geq \left(\lfloor \frac{M_1 M_2}{k} \rfloor\right)^{1/2} - 2 \ .$$

A special case of this mapping appears in Figure 2.1, with $M = 6$, $k = 4$ and $\ell = 2$. It has

$$s_4(\phi) = 2 \geq \left(\lfloor \frac{36}{4} \rfloor\right)^{1/2} - 2 = 1 \ .$$

We conjecture that for $k \geq 3$ and $M_1 \leq M_2 \leq \frac{k}{3} M_1$ there exist MRV labellings that achieve

$$s_k(\phi) \geq c_0 \left(\lfloor \frac{M_1 M_2}{k} \rfloor\right)^{1/2} - 2 \ , \tag{4.13}$$

for some positive constant c_0, e.g. $c_0 = \frac{1}{2}$.

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