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Number Theory Zeta Functions and Dynamical Zeta Functions

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Abstract. We describe analogies between number theory zeta functions, dynamical zeta functions, and statistical mechanics zeta functions, with emphasis on multi-variable zeta functions. We mainly consider two-variable zeta functions $\zeta_f(z, s)$ in which the variable z is a "geometric variable" , while the variable s is an "arithmetic variable". The s-variable has a thermodynamic interpretation, in which s parametrizes a family of energy functions ϕ_s. We survey results on the analytic continuation and location of zeros and poles of two-variable zeta functions for four examples connected with number theory. These examples are: (1) the beta transformation $f(x) = \beta x \pmod 1$, (2) the Gauss continued fraction map $f(x) = 1/x \pmod 1$, (3) zeta functions of varieties over finite fields, and (4) Riemann zeta function. [Revised: Nov. 2002]

1. Introduction

Dynamical zeta functions were introduced by Artin and Mazur [**6**] in 1965, based on an analogy with the number theory zeta functions attached to a function field over a finite field. Later Ruelle associated zeta functions to statistical mechanics models in one dimension (lattice gases). In the past twenty years many parallels have been drawn between number theory zeta functions, dynamical zeta functions, and statistical mechanics zeta functions, starting with [**77**]. The object of this paper is to describe such parallels in the case of multi-variable zeta functions. Such zeta functions of several variables naturally arise in the thermodynamic formalism in statistical mechanics, where the extra variables represent thermodynamic quantities. We mainly consider two-variable zeta functions, and survey what is known about the dynamic and geometric information encoded in two-variable zeta functions in various specific cases of interest in number theory.

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This paper was motivated by a question concerning number theory zeta functions attached to function fields over finite fields. It is well known that there are two distinct treatments of the theory of zeta functions attached to algebraic function fields K over a finite field $\mathbb{F}_q$. The first is an "arithmetic theory" which expresses the zeta function of K as a Dirichlet series in an "arithmetic variable" s. This zeta function encodes the unique factorization of ideals in a ring of integers in K. The second is a "geometric theory" developed by Weil which counts points over $\bar{\mathbb{F}}_q$ on a nonsingular projective algebraic variety V attached to the function field, and which encodes information on the topology of this variety. In this case the zeta function is a power series in a "geometric variable" z, and is a rational function of z. These two types of zeta function are not identical, but are related by the change of variable $z = q^{-s}$. These approaches to zeta functions of function fields have different historical roots and appear to be independent theories. The fact that their associated zeta functions are related by a simple change of variable is a "coincidence" which I feel requires explanation.

This paper proposes such an explanation based on an analogy with (multi-variable) dynamical zeta functions. The explanation is that one may regard the function field zeta function as a function of two complex variables $\zeta(z, s)$ which happens to satisfy a special identity

$$\zeta(z, s) = \zeta(zq^{-s}, 0) \ . \tag{1.1}$$

This identity allows one to reduce this zeta function to a function of one complex variable, which may be chosen to be either z or s. Indeed $\zeta(1, s)$ gives the "arithmetic theory" zeta function and $\zeta(z, 0)$ gives the "geometric theory" zeta function. The two-variable zeta function $\zeta(z, s)$ is to be thought of as analogous to a statistical mechanics zeta function.

To describe the parallel identity for dynamical zeta functions which motivates (1.1), we recall that dynamical zeta functions also arose in two distinct contexts. The first of these defines a dynamical zeta function as a generating function for periodic points of iterated maps and, more generally, to periodic orbits of flows. This development was initiated by Artin and Mazur [**6**] and extended by Smale [**88**]. The Artin-Mazur zeta function contains a single variable z and was constructed by analogy with the "geometric theory" zeta function above. The second context came from the statistical mechanics of lattice gases, and was developed by Ruelle based on his thermodynamic formalism, see Ruelle [**77**], [**78**]. The statistical mechanics zeta function is attached to an energy function ϕ describing the physical system, and contains an additional "scaling variable" z, which is a generating function variable for combining finite systems of different sizes. When one considers a family of energy functions ϕ_s that depends on a thermodynamic parameter labelled s, one obtains a zeta function of two variables. More generally, when d thermodynamic parameters are introduced. one obtains zeta functions of $d + 1$ variables. These parameters describe macroscopically measurable quantities of the physical system. We consider here the case of zeta functions of two complex variables $\zeta_f(z,s)$, in which z is the "scaling variable", which can be used to take the thermodynamic limit as the size of the system becomes infinite, while the variable s corresponds to the thermodynamic quantity "inverse temperature".

Ruelle carried over his statistical mechanics construction to apply to dynamical systems $f:\Omega\to\Omega$ on a compact phase space ("generalized zeta functions"). In this way one obtains multivariable dynamical zeta functions attached to certain dynamical systems. The theory is particularly effective for "expanding maps". We consider the case of piecewise-C^1 expanding maps of the interval $f:[0,1]\to[0,1]$, with $|f'(x)|>1$. We show that in the case of a *homogeneously expanding function*, which is one for which $|f'(x)|=\beta$ is constant, the associated two-variable zeta function satisfies

$$\zeta_f(z,s)=\zeta_f(z\beta^{-s},0)\ . \tag{1.2}$$

(See Theorem 3.1.) Here β measures the expansion rate of the map, and $\log\beta$ is the *entropy* of f. The formula (1.1) now arises by analogy with (1.2) if one considers the Frobenius automorphism acting on $V(\bar{\mathbb{F}}_q)$ as behaving as if[1] it were a homogeneously expanding map with entropy $\log q$.

The two-variable version of the function field zeta function given above in (1.1) is produced by a formal analogy, and it does not provide any new information about function field arithmetic. However treating the two variables s and z as separate variables may have more than a casual significance. Most of the "eigenvalue" interpretations for zeta functions that I know of are formulated in terms of the z-variable. For function field zeta functions the eigenvalues of Frobenius are eigenvalues in the z-variable. In the case of dynamical zeta functions, the z-variable appears as an eigenvalue variable for a transfer operator. I am not aware of any corresponding "eigenvalue" interpretation of the s-variable. However the various "trace formulas" that have been developed, including the "explicit formulae" of prime number theory and the Selberg trace formula, are formulated in terms of the s-variable, cf. [**32**], [**92**], [**95**], [**100**].

At this point one may wonder whether, in the function field case, the definition of the two-variable zeta function in (1.1) above can be made more than formal. That is, does there exist an explicit construction which assigns to a function field over a finite field a statistical mechanics model (or a dynamical system model) whose two-variable zeta function will satisfy (1.1) and recover both zeta functions attached to the function field? In a similar vein, do there also exist such constructions that would apply to zeta functions attached to algebraic number fields? At present no positive answer to either of these questions is known. The main part of this paper describes the relations among these different kinds of zeta functions, in a form intended

[1]The Frobenius operator is an automorphism, so that it should correspond to the *natural extension* of an expanding map, rather than the expanding map itself, see Section 4.4.

to facilitate thinking about these questions. It emphasizes analogies, and describes what is known in specific examples.

The detailed contents of the paper are as follows. In §2 we describe one-variable number theory zeta functions in the number field case and in the function field case; for the function field case we present both the "arithmetic theory" and "geometric theory" interpretations.

In §3 we consider multivariable dynamical zeta functions, which were introduced by Ruelle [**76**] as a generalization of statistical mechanics zeta functions. We consider particularly the case of two-variable zeta functions for expanding maps of the interval. We show that "homogeneously expanding maps" have two-variable zeta functions that satisfy (1.2). We review known results for the two-variable zeta functions of the β-transformation $f_\beta(x) = \beta x \pmod 1$, and the Gauss continued fraction map $f_{CF}(x) = 1/x \pmod 1$. The results for the Gauss continued fraction map are mainly due to D. Mayer [**51**], [**55**], [**56**], [**57**], [**15**]. He proved results showing that the zeta function $\zeta_{CF}(z,s)$ at the "special values" $z = 1$ and $z = -1$ is related to Selberg's zeta function for the modular surface $\mathbb{H}/PSL(2,\mathbb{Z})$.

In §4 we consider multivariable statistical mechanics zeta functions for one-dimensional lattice gases. We describe the equilibrium statistical mechanics of lattice gases and the thermodynamic formalism of Ruelle. We define statistical mechanics zeta functions and discuss the relation of their variables and singularities to the thermodynamic formalism. As an explicit example, we determine the three-variable zeta function of the one-dimensional Ising model using transfer matrices. In Table 4.1 we give a dictionary of analogies between one-dimensional lattice gas models in statistical mechanics and discrete dynamical systems, which motivated Ruelle's definition of multivariable dynamical zeta functions. In preparing this dictionary I have benefited from Ruelle [**74**], [**78**] and Mayer [**52**].

In §5 we present two-variable number theory zeta functions $\zeta(z,s)$ constructed by analogy with two-variable dynamical zeta functions. We first consider two variable zeta functions for the function field case, and elaborate on the discussion given above. Then we address the number field case, and consider ways of adding a "geometric variable" z to define a two-variable extension of the Riemann zeta function. We describe two such functions, a "naive" zeta function $\zeta_{\mathbb{Z}}(z,s)$ and an Arakelov zeta function $\tilde{\zeta}_{\mathbb{Z}}(z,s)$, constructed in formal analogy to the function field case. These definitions are based on two different notions of the degree of a prime ideal. For these definitions we do not know of any associated dynamical system. The "naive" function $\zeta_{\mathbb{Z}}(z,s)$ was already studied in 1943 by A. Wintner [**101**], who determined its properties under analytic continuation. Its analytic continuation properties superficially resemble those of the two-variable zeta function $\zeta_{CF}(z,s)$ attached to the continued fraction map discussed in §3; for all values of the z-variable except $1, 0$, and -1 it does not analytically continue in the s-variable to the left of the line $\mathrm{Re}(s) = 0$. The function $\zeta_{\mathbb{Z}}(z,s)$ does encode the Riemann hypothesis for all values of z other than $z = 0$ in a "trivial" way. The Arakelov zeta function $\tilde{\zeta}_{\mathbb{Z}}(z,s)$ preserves the relation (1.2), and therefore also encodes the Riemann hypothesis for all values of z other than $z = 0$.

In §6 we give concluding remarks,including a brief summary, some directions for further work, and some speculations.

We conclude this introduction by briefly mentioning three other parallels between number theory zeta functions and dynamical zeta functions that are not discussed in this paper. These parallels mainly concern zeta functions expressed using the "arithmetic variable" s. First, there are analogies between Artin L-functions for number fields and zeta functions of coverings and vector bundles on manifolds, see Fried [**28**], [**29**], Adachi and Sunada [**1**] and Sunada [**89**]. Second, for certain dynamical zeta functions attached to flows there are analogues of the prime number theorem and the Chebotarev density theorem, and for these see Parry and Pollicott [**65**], [**66**], [**67**]. Third, certain "special values" of number theory zeta functions have parallels in "special values" of dynamical zeta functions, which encode topological invariants such as Reidemeister torsion; One aspect of this line of investigation was started by Milnor [**61**]. It also includes the Atiyah-Bott fixed point formula, see Atiyah and Bott [**7**], and the Atiyah-Singer index theorem, see Gilkey [**31**] and Hirzebruch and Zagier [**36**].

2. Number Theory Zeta Functions

This section reviews one-variable number theory zeta functions, in a form intended to aid in later comparison with dynamical zeta functions.

2.1. Number Fields. The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \ . \tag{2.1}$$

can be viewed as a sum taken over the integral ideals (n) of the ring $\mathbb{Z}$, in which each term in the Dirichlet series represents a *weight*

$$wt(n) := \#|\mathbb{Z}/(n)| = n \ ,$$

assigned to the ideal (n). The ring $\mathbb{Z}$ is a Dedekind domain, and the unique factorization of ideals into powers of prime ideals is reflected in the *Euler product*

$$\zeta(s) = \prod_{(p)} (1 - p^{-s})^{-1} \ , \tag{2.2}$$

taken over the set of prime ideals p in $\mathbb{Z}$. The Riemann zeta function satisfies a functional equation of the form

$$\hat{\zeta}(s) := \hat{\zeta}(1-s) \ , \tag{2.3}$$

in which

$$\hat{\zeta}(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \ . \tag{2.4}$$

The *Riemann hypothesis* asserts that all zeros of $\hat{\zeta}(s)$ lie on the line $\mathrm{Re}(s) = 1/2$.

These facts generalize to the Dedekind zeta function $\zeta_K(s)$ of an algebraic number field K. Here K is a finite extension of the rationals $\mathbb{Q}$, and the ring $\mathbb{Z}$ is replaced by the ring of algebraic integers O_K in K; it is a Dedekind domain. We define

$$\zeta_K(s) = \sum_{\mathbf{A}} (N\mathbf{A})^{-s} \tag{2.5}$$

where $\mathbf{A}$ runs over the integral ideals of O_K, and the weight of an ideal is its norm

$$wt(\mathbf{A}) = N\mathbf{A} := \#|O_K/\mathbf{A}| \ . \tag{2.6}$$

Unique factorization of ideals gives the Euler product

$$\zeta_K(s) = \prod_{\mathbf{P}} (1 - (N\mathbf{P})^{-s})^{-1} \ , \tag{2.7}$$

where the product is taken over all prime ideals $\mathbf{P}$ in O_K. It satisfies a functional equation of the form

$$\hat{\zeta}_K(s) = \hat{\zeta}_K(1-s) \ , \tag{2.8}$$

in which

$$\hat{\zeta}_K(s) := |d_K|^{s/2} \left(\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\right)^{n_1} \left(2^{-s}\Gamma\left(\frac{s+1}{2}\right)\right)^{n_2} \zeta_K(s) \ , \tag{2.9}$$

where d_K is the discriminant of K, n_1 is the number of real conjugate fields to K and n_2 the number of complex conjugate fields to K, and $n_1 + 2n_2 = [K : \mathbb{Q}]$. The function $\hat{\zeta}_K(s)$ has poles at $s = 0$ and $s = 1$. The *extended Riemann hypothesis* asserts that all zeros of $\hat{\zeta}_K(s)$ lie on the line $\mathrm{Re}(s) = 1/2$.

2.2. Function Fields — Arithmetic Variable. In 1924 E. Artin [**5**] studied zeta functions attached to function fields of one variable K over a finite field $\mathbb{F}_q$ where $q = p^k$. Here K is a finite extension of the function field $K_0 = \mathbb{F}_q(T)$ which has a distinguished subring $R_0 = \mathbb{F}_q[T]$ of "integers". The ring of "integers" R_K of K is the integral closure of R_0 in K; it is a Dedekind domain. He sets

$$\zeta_K(s) := \sum_{\mathbf{A}} (N\mathbf{A})^{-s} \tag{2.10}$$

where $\mathbf{A}$ runs over all integral ideals in R_K, and the *weight*

$$N\mathbf{A} := \#|R_K/\mathbf{A}| \ . \tag{2.11}$$

The unique factorization of ideals in a Dedekind domain gives the Euler product

$$\zeta_K(s) = \prod_{\mathbf{P}} (1 - (N\mathbf{P})^{-s})^{-1} \ , \tag{2.12}$$

where the product is taken over all prime ideals $\mathbf{P}$ in R. We call s an arithmetic variable because the Euler product (2.12) encodes information about the arithmetic of the ring R_K. E. Artin [**5**] derived a functional equation for $\zeta_K(s)$ in the special case of hyperelliptic function fields $K = K_0(T, y)$ over a prime field $\mathbb{F}_p$, where

$$y^2 = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0.$$

In 1931 F. K. Schmidt introduced the theory of divisors to study function field zeta functions defined over an arbitrary finite field $\mathbb{F}_q$ where $q = p^k$. He obtained a functional equation [**83**, Satz 22]), of the form

$$\hat{\zeta}_K(s) := \hat{\zeta}_K(1-s) \ , \tag{2.13}$$

in which

$$\hat{\zeta}_K(s) := q^{-(1-g)s} \zeta_K^{\infty}(s) \zeta_K(s) \ . \tag{2.14}$$

In this formula $g \geq 0$ is an integer called the *genus* of K, and $\zeta_K^{\infty}(s)$ is a finite Euler product coming from the "primes at infinity".

F. K. Schmidt interpreted K as the field of regular functions attached to a nonsingular projective curve $\mathcal{C}$ defined over $\mathbb{F}_q$, and interpreted the functional equation (2.13) as equivalent to the Riemann-Roch theorem. The quantity $q^{-(1-g)s}$ appearing in (2.14) is a normalizing factor, in which

$$1 - g = \frac{1}{2}\chi(\mathcal{C})$$

where $\chi(\mathcal{C})$ is the Euler characteristic of the curve $\mathcal{C}$. He defined the *projective zeta function* of the curve $\zeta_{\mathcal{C}}(s)$ as the Euler product taken over all prime divisors of the projective curve $\mathcal{C}$ including "primes at infinity", i.e.

$$\zeta_{\mathcal{C}}(s) := \zeta_K^{\infty}(s) \zeta_K(s) \ . \tag{2.15}$$

F. K. Schmidt observed that the projective zeta function had the general form

$$\zeta_{\mathcal{C}}(s) = \frac{P(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})} \ , \tag{2.16}$$

in which

$$P(z) = \sum_{j=0}^{2g} b_j z^j$$

is a polynomial with integer coefficients, which has exact degree $2g$. The functional equation (2.13) is equivalent to the property that the polynomial $P(z)$ in (2.16) satisfies

$$z^{-g} P(\sqrt{q} z) = z^g P\left(\frac{1}{\sqrt{q} z}\right) \ . \tag{2.17}$$

As an example, in the simplest case of $K_0 = \mathbb{F}_q(T)$, the ideals of $R_0 := \mathbb{F}_q[T]$ are the principal ideals $(f(T))$, where $f(T)$ is a monic polynomial

$$f(T) := T^n + a_{n-1} T^{n-1} + \cdots + a_1 T + a_0 \tag{2.18}$$

and has norm $N(f(T)) = q^n$. There are q^n monic polynomials of degree n, hence

$$\zeta_{K_0}(s) = \sum_{n=1}^{\infty} q^n q^{-ns} = \frac{1}{1-q^{1-s}} \ . \tag{2.19}$$

Prime ideals of $\mathbb{F}_q[T]$ correspond to irreducible polynomials $(f(T))$ over $\mathbb{F}_q$, hence the Euler product factorization of $\zeta_{K_0}(s)$ is

$$\zeta_K(s) = \prod_{k=1}^{\infty} (1-q^{-ks})^{-\phi_k(q)} \ , \tag{2.20}$$

in which $\phi_k(q)$ counts the number of irreducible monic polynomials of degree k over $\mathbb{F}_q$. The Möbius inversion formula gives

$$\phi_k(q) = \frac{1}{k} \sum_{d|k} \mu(d) q^{k/d} \ . \tag{2.21}$$

To determine the functional equation, we note that there is a unique divisor at infinity $\frac{1}{T}$, which contributes

$$\zeta_{K_0}^{\infty}(s) = \frac{1}{1-q^{-s}} \ , \tag{2.22}$$

and the genus $g(K_0) = 0$. Thus $\hat{\zeta}_{K_0}(s) = \hat{\zeta}_{K_0}(1-s)$ for

$$\hat{\zeta}_{K_0}(s) := q^{-s} \frac{1}{(1-q^{-s})(1-q^{1-s})} = \frac{-1}{(1-q^{s})(1-q^{1-s})} \ . \tag{2.23}$$

The Riemann hypothesis for $\zeta_K(s)$ asserts that all zeros of $\hat{\zeta}_K(s)$ lie on the line $\mathrm{Re}(s) = 1/2$. E. Artin [**4**] formulated it and proved it in special cases. It is true for genus 0 curves by (2.16). In 1933 H. Hasse proved it for genus 1 curves ([**33**], [**34**]), and around 1939 A. Weil proved it for all nonsingular projective curves ([**97**]). The Riemann hypothesis for function fields of one variable is often stated in the alternate form: the roots θ of $P(z)$ in (2.17) all have $|\theta| = q^{-1/2}$.

2.3. Function Fields: Geometric Variable. In 1949 A. Weil [**91**] defined a zeta function attached to a nonsingular projective variety V of arbitrary dimension defined over $\mathbb{F}_q$ which counts points on V over extension fields $\mathbb{F}_{q^k}$ for $k \geq 1$. Let V be described by a system of homogeneous polynomial equations with coefficients in $\mathbb{F}_q$, say

$$F_i(x_1, \ldots, x_n) = 0, \quad 1 \leq i \leq k \ . \tag{2.24}$$

Let $\bar{\mathbb{F}}_q$ denote the algebraic closure of $\mathbb{F}_q$. If $(x_1, \ldots, x_n) \in (\bar{\mathbb{F}}_q)^n \setminus \{\mathbf{0}\}$ satisfies (2.24) then the point on $V(\bar{\mathbb{F}}_q)$ associated to $\mathbf{x}$ is the equivalence class

$$[\mathbf{x}] = \{(\lambda x_1, \ldots, \lambda x_n) : \lambda \in \bar{\mathbb{F}}_q \setminus \{0\}\} \ . \tag{2.25}$$

We say that $[\mathbf{x}]$ lies in $V(\mathbb{F}_{q^k})$ if there is some $\mathbf{x}' \in [\mathbf{x}]$ with $\mathbf{x}' \in (\mathbb{F}_{q^k})^n$. Weil defined the *zeta function* $Z_V(z)$ over $\mathbb{F}_q$ to be

$$Z_V(z) := \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} N_k(V;\mathbb{F}_q)\right) \tag{2.26}$$

in which $N_k(V;\mathbb{F}_q) := |V(\mathbb{F}_{q^k})|$, viewed as a projective variety. In this case

$$\frac{d}{dz}(\log Z_V(z)) = \frac{Z_V'(z)}{Z_V(z)} = \sum_{k=1}^{\infty} N_k(V;\mathbb{F}_q) z^{k-1} \ . \tag{2.27}$$

The Frobenius automorphism $Fr : \bar{\mathbb{F}}_q \to \bar{\mathbb{F}}_q$ is defined by

$$Fr(x) := x^q \ . \tag{2.28}$$

This map induces a well-defined map Fr_V on the points of $V(\bar{\mathbb{F}}_q)$ by

$$Fr_V([\mathbf{x}]) := [(Fr(x_1), Fr(x_2), \ldots, Fr(x_n))] \ . \tag{2.29}$$

The fixed points of Fr_V^k on V are exactly the points of $V(\mathbb{F}_{q^k})$. Thus

$$Z_V(z) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} |Fix(Fr_V^k)|\right) \ . \tag{2.30}$$

We call the variable z a *geometric variable* because it appears in counting points on the geometric object $V(\bar{\mathbb{F}}_q)$. Weil observed[2] that for curves $\mathcal{C}$ this zeta function has a simple relation to the projective zeta function introduced by F. K. Schmidt.

THEOREM 2.1 (Weil). *Let $\mathcal{C}$ be a complete nonsingular projective curve defined over $\mathbb{F}_q$. Then*

$$Z_{\mathcal{C}}(z) = \zeta_{\mathcal{C}}(q^{-s}) \ , \tag{2.31}$$

where $\zeta_{\mathcal{C}}(s)$ is the projective zeta function associated to the function field of $\mathcal{C}$ over $\mathbb{F}_q$.

PROOF. A fixed point of Fr_V^k is *primitive* if it is not a fixed point of any Fr_V^l for $l < k$. Then, using (2.30)

$$\begin{aligned} Z_{\mathcal{C}}(z) &= \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{\text{primitive} \\ \text{point}}} -\log(1-z^k)\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \sum_{\substack{\text{primitive} \\ \text{orbits}}} -\log(1-z^k)\right) \\ &= \prod_{k=1}^{\infty} (1-z^k)^{-\pi_k(Fr_{\mathcal{C}})} \ , \end{aligned} \tag{2.32}$$

[2] Weil was aware of this relation by 1942, see Weil [**98**, Section 8].

in which $\pi_k(Fr_{\mathcal{C}})$ counts the number of primitive orbits of period k of $Fr_{\mathcal{C}}$. Now

$$\zeta_{\mathcal{C}}(s) = \prod_{k=1}^{\infty} (1 - p^{-ks})^{-\phi_k(K)} , \tag{2.33}$$

in which $\phi_k(K)$ counts the number of prime divisors of degree k of the function field $K = K(\mathcal{C})$. The primitive orbits of period k of $Fr_{\mathcal{C}}$ are in one-to-one correspondence with prime divisors of the function field $K(\mathcal{C})$, so $\pi_k(Fr_{\mathcal{C}}) = \phi_k(K(\mathcal{C}))$, which gives (2.31). □

As an example, the affine curve $x_1 + x_2 = -1$ gives, after homogenization, the projective line $\mathbf{P}^1$:

$$x_1 + x_2 + x_3 = 0 . \tag{2.34}$$

The equation $x_1 + x_2 = -1$ has q^k solutions over $GF(q^k)$ yielding the points $(x_1, 1-x_1, -1)$, while the plane at infinity $x_3 = 0$ contain one point $(1, -1, 0)$. Thus $N_k(V; \mathbb{F}_q) = q^k + 1$, hence

$$\begin{aligned} Z_{\mathbf{P}^1}(z) &= \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} (q^k + 1) \right) \\ &= \exp(-\log(1 - zq) - \log(1 - z)) \\ &= \frac{1}{(1 - z)(1 - qz)} . \end{aligned} \tag{2.35}$$

The function field $K(\mathbf{P}^1) = \mathbb{F}_q(T)$, since $\mathbb{F}_q[x,t]/(x + t - 1) = \mathbb{F}_q[t]$. A periodic point $\{(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) : 1 \le i \le k\}$ in $\mathbb{F}_{q^k}$ with $x_3^{(i)} = 0$ corresponds to a prime divisor $(f(T))$ in $\mathbb{F}_q[T]$ which is the principal ideal given by

$$f(T) = \prod_{j=1}^{k} (T - \theta_j) \in \mathbb{F}_q[T] \tag{2.36}$$

in which $\theta_j = Fr^j(x_1^{(1)})$, and $f(T)$ is irreducible over $\mathbb{F}_q$.

Weil [**99**] conjectured that the zeta function $Z_V(z)$ of an n-dimensional complete nonsingular variety V over $\mathbb{F}_q$ satisfies a functional equation

$$\hat{Z}_V\left(\frac{1}{q^n z}\right) = \pm \hat{Z}_V(z) , \tag{2.37}$$

in which

$$\hat{Z}_V(z) = z^{\frac{n\chi(V)}{2}} Z_V(z) \tag{2.38}$$

where $\chi(V)$ is the Euler characteristic of V. He also conjectured that

$$Z_V(z) = \frac{Z_1(z) Z_3(z) \cdots Z_{2n-1}(z)}{Z_0(z) Z_2(z) \cdots Z_{2n}(z)} \tag{2.39}$$

in which $Z_0(z) = 1 - z$, $Z_{2n}(z) = 1 - q^n z$

$$Z_h(z) = \prod_{i=1}^{B_h(V)} (1 - \alpha_{hi} z) \tag{2.40}$$

where $B_h(V)$ are the Betti numbers of V. Finally, he conjectured the *Riemann hypothesis* for the variety V, which asserts that

$$|\alpha_{hi}| = q^{h/2}, \quad 1 \le i \le B_h(V) \tag{2.41}$$

for $1 \le h \le 2n - 1$. The first conjecture was proved by Dwork and the Riemann hypothesis was proved by Deligne in 1974.

The method of solving these problems involved constructing suitable cohomology theories on V, such that $Z_h(z)$ is expressed as the characteristic polynomial of a linear operator Fr_* associated to the Frobenius operator Fr_V acting on cohomology, e.g.

$$Z_h(z) := \det(I - zFr_* | H^i_{et}(V, \mathbb{Q}_l)) \tag{2.42}$$

for l-adic etale cohomology. This formula gives a "spectral" interpretation of the polynomials $Z_h(z)$.

3. Dynamical Zeta Functions

The notion of dynamical zeta function was introduced by M. Artin and B. Mazur [**6**] in 1965. Let $f : M \to M$ be a diffeomorphism of a compact manifold, such that its iterates f^k all have isolated fixed points. They set

$$\zeta_f(z) := \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} \#Fix(f^k)\right) , \tag{3.1}$$

in analogy with the geometric zeta function attached function field. They showed that for a dense set of $f \in Diff(M)$, the power series converged in a neighborhood of $z = 0$, i.e.

$$\#Fix(f^k) << c^k \quad \text{as} \quad k \to \infty .$$

Smale [**88**] conjectured that for Axiom A diffeomorphisms $\zeta_f(z)$ is a rational function, a result later proved by Manning [**50**], and he also proposed a definition of a zeta function for a flow. About the same time Ruelle [**75**], [**76**], [**77**], motivated by problems in statistical mechanics, introduced dynamical zeta functions with weights. Ruelle's general zeta function for a map $f : X \to X$ on a compact space X, takes the form

$$\zeta_f(z, \Phi) := \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} \sum_{x \in Fix(f^M)} \exp\left(\sum_{k=1}^{m} \Phi(f^k(x))\right)\right) \tag{3.2}$$

in which $\Phi : X \to \mathbb{C}$ is a *weight function.* Ruelle related one-parameter families of weight functions Φ_β which depend on a (thermodynamic) parameter β to the thermodynamic formalism, see Ruelle [**78**, Sect. 7.23]. The two-variable dynamical zeta function (3.4) below is a special case.

For recent surveys on dynamical zeta functions see Baladi [**8**], Hurt [**37**], Parry and Pollicott [**67**], and Ruelle [**82**].

3.1. Zeta Functions for Expanding Maps of the Interval. Dynamical zeta functions are especially well suited for studying the dynamics of expanding maps (Ruelle [**80**], [**82**], Parry and Pollicott [**67**]). The simplest case are piecewise C^1-maps of the interval $f : [0,1] \to [0,]$ which are *expanding* in the sense that

$$|f'(x)| > 1 \quad \text{a.e.} \quad x \in [0,1] \ . \tag{3.3}$$

To such maps we associate a two-variable dynamical zeta function

$$\zeta_f(z,s) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} \sum_{\substack{x \in Fix(f^k) \\ x_i = f^i(x)}} \left| \prod_{i=1}^{k} f'(x_i) \right|^{-s} \right) \ . \tag{3.4}$$

The periodic points are each weighted by a factor based on the rate of expansion of the map near the periodic orbit (which is unstable). The weight depends on the parameter s, and the special case $s = 0$ gives the Artin-Mazur zeta function. This particular weight $\exp(-\log|f'(x_i)|)$ appearing in (3.4) is important in studying absolutely continuous invariant measures for $f(x)$ (when $s = 1$). It traces back to ideas of Bowen [**12**] and Sinai [**87**], see Mayer [**54**], p. 312.

We say that a piecewise C^1-map $f : [0,1] \to [0,1]$ is a *homogeneously expanding map* if

$$|f'(x)| = \beta \quad \text{for almost all } x \in [0,1] \ , \tag{3.5}$$

where the exceptional set is countable. Such maps are piecewise linear. A simple example is the *beta transformation*

$$f(x) = \beta x \pmod{1} \ , \tag{3.6}$$

whose ergodic theory properties were first studied in detail by Parry [**64**].

THEOREM 3.1. *If a piecewise C^1-map $f : [0,1] \to [0,1]$ is a homogeneously expanding map, then*

$$\zeta_f(z,s) = \zeta_f(z\beta^{-s}, 0) \ . \tag{3.7}$$

PROOF. The homogeneously expanding property gives

$$\begin{aligned} \zeta_f(z,s) &= \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} \sum_{x \in Fix(f^k)} (\beta^k)^{-s} \right) \\ &= \exp\left(\sum_{K=1}^{\infty} \frac{(z\beta^{-s})^k}{k} \#Fix(f^k) \right) \\ &= \zeta_f(z\beta^{-s}, 0), \end{aligned}$$

as asserted. □

Theorem 3.1 implies that homogeneously expanding maps have two-variable zeta functions that are singly-periodic in the s-variable with pure imaginary period $\frac{2\pi i}{\log \beta}$. In another direction, Margulis constructed for transitive Y-flows a maximal entropy measure with respect to which the Y-flow has a homogeneously expanding part, see Sinai [**87**, p. 42].

For homogeneously expanding maps the Artin-Mazur zeta function can be explicitly computed using symbolic dynamics together with the kneading theory of Milnor and Thurston [**62**]. This approach was taken by Takahashi [**90**], and has recently been extended by Baladi and Ruelle [**9**]. When it applies, the symbolic dynamics approach can be used to obtain substantial information on the singularities of the Artin-Mazur zeta function in the unit disk $\{z : |z| < 1\}$, and hence, by Theorem 3.1 of the two-variable zeta function. This was carried out for the beta transformation in Flatto et al [**27**], see §3.2 below.

There is a second method for computing dynamical zeta functions, which is more general but usually gives less explicit information. It applies to general piecewise-C^1 expanding maps $f : [0,1] \to [0,1]$ and determines the two-variable zeta function. It uses the family of Ruelle-Araki *transfer operators* $\mathcal{L}_s$ depending on the parameter s, where each $\mathcal{L}_s$ is a linear operator defined on a suitable Banach space $\mathcal{B}$ of functions $h : [0,1] \to \mathbb{C}$ by

$$\mathcal{L}_s h(x) := \sum_{\substack{y \\ f(y)=x}} |f'(y)|^s h(y), \quad \text{for} \quad h \in \mathcal{B} \ . \tag{3.8}$$

Under suitable hypotheses

$$\zeta_f(z,s) = \det(I - z\mathcal{L}_s) \ , \tag{3.9}$$

where the right side is a Fredholm determinant. In the case of the β-transformation the locations z' of poles of $\zeta_\beta(z)$ in $\{|z| < 1\}$ have $z' = 1/\lambda'$ where λ' is an eigenvalue of $\mathcal{L}_1$. For more information see Ruelle [**82**].

In the rest of this section we describe in detail what is known about the two-variable zeta functions associated to two expanding maps of particular interest in number theory, the beta transformation and the Gauss continued fraction map.

3.2. Zeta Function of the Beta Transformation. The beta transformation is the map $f_\beta : [0,1] \to [0,1]$ given by

$$f_\beta(x) = \beta x \pmod{1} \ ,$$

for a given $\beta > 1$, see Figure 3.1. It is a homogeneously expanding map, hence Theorem 3.1 applies. Thus to determine the analytic properties of the two-variable zeta function $\zeta_{f_\beta}(z,s)$ it suffices to consider its Artin-Mazur zeta function $\zeta_\beta(z) := \zeta_{f_\beta}(z,0)$. Flatto et al. [**27**] establish the following results.

THEOREM 3.2 (Flatto, Lagarias and Poonen). *(i) The zeta function $\zeta_\beta(z)$ of the β transformation is given by*

$$\zeta_\beta(z) = \frac{1 - z^N}{(1-\beta z)\left(\sum_{n=0}^{\infty} f_\beta^n(1) z^n\right)} \tag{3.10}$$

where N is the minimal value such $f_\beta^N(1) = 0$, and if no such iterate exists, then $z^N \equiv 0$.

(ii) $\zeta_\beta(z)$ is a meromorphic function in the open unit disk $\{|z| < 1\}$. It is holomorphic in the open disk $\{|z| < 1/\beta\}$ and has a simple pole at $z = 1/\beta$. It has no other singularities on the circle $|z| = 1/\beta$.

(iii) If the sequence of values $\{f_\beta^n(1) : n = 1, 2, \ldots\}$ is eventually periodic, then $\zeta_\beta(z)$ is a rational function and continues meromorphically to $\mathbb{C}$. Otherwise $\zeta_\beta(z)$ has the unit circle $\{|z| = 1\}$ as a natural boundary to analytic continuation.

PROOF. (i) is derived in [**27**] from a result of Takahashi [**90**]. (ii) and (iii) appear as Theorem 2.2 and Theorem 2.4 in [**27**] respectively. □

More generally, one can analyze the zeta function for the linear mod one transformation

$$f_{\beta,\alpha}(x) = \beta x + \alpha \pmod{1} .$$

In this case more complicated behaviors occur, including a renormalization phenomenon for certain (β, α) with $1 < \beta < 2$, see Flatto and Lagarias ([**23**]–[**25**]). For a number-theoretic application of the dynamics of linear mod one transformations, see Flatto, Lagarias and Pollington [**26**].

3.3. Zeta Function for the Gauss Continued Fraction Map.

The Gauss continued fraction map is

$$f_{CF}(x) := 1/x \pmod{1} , \tag{3.11}$$

and its graph is pictured in Figure 3.2. Its zeta function has been extensively studied by D. Mayer and collaborators.

The continued fraction expansion for a real number $x \in [0,1]$ is expressed in terms of a symbolic dynamics for the map (3.11). We partition

$$[0,1] = \bigcup_{n \geq 1} X_n,$$

in which $X_n = (\frac{1}{n+1}, \frac{1}{n}]$, and we assign the symbol $S(x) = n$ to all $x \in X_n$, for $n \geq 1$. The continued fraction expansion of $x \in [0,1)$ is

$$x := [0, a_1, a_2, \ldots, a_n, \ldots] = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}, \tag{3.12}$$

in which the *partial quotients* a_n are given by

$$a_n = S(f_{CF}^{n-1}(x)), \quad \text{for} \quad n \geq 1 . \tag{3.13}$$

The n^{th} *convergent* $\frac{p_n}{q_n}$ of the continued fraction expansion is

$$\frac{p_n}{q_n} := [0, a_1, \ldots, a_n] = \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_n}}} . \tag{3.14}$$

We now define a zeta function for the Gauss continued fraction map. This map is clearly not homogeneously expanding. It has infinitely many periodic points of each fixed period, hence the Artin-Mazur zeta function is not well-defined. However the two-variable dynamical zeta function (3.4) is well-defined for some range of z and s. It was introduced and extensively studied by D. Mayer [**51**], [**54**], [**56**], [**57**]. The two-variable zeta function is

$$\zeta_{CF}(z, s) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} Z_k(s)\right) , \tag{3.15}$$

in which

$$Z_k(s) := \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} (wt(x[n_1, n_2, \ldots, n_k]))^{-s} , \tag{3.16}$$

where $x[n_1, \ldots, n_k]$ denotes the point in $[0, 1]$ which has the periodic ordinary continued fraction expansion

$$x[n_1, \ldots, n_k] := [0, \overline{n_1, n_2, \ldots, n_k}], \tag{3.17}$$

and which is assigned the weight

$$wt(x[n_1, \ldots, n_k]) := \prod_{i=1}^{k} (x[n_i, n_{i+1}, \ldots, n_k, n_1, \ldots, n_{i-1}])^{-2} . \tag{3.18}$$

The numbers $x[n_1, \ldots, n_k]$ are real quadratic irrationals, and $wt(x[n_1, \ldots, n_k])$ is a unit in the ring of integers of the real quadratic field generated by $x[n_1, \ldots, n_k]$. For example,

$$Z_1(s) = \sum_{n=1}^{\infty} \epsilon_n^{-2s} \tag{3.19}$$

in which $\epsilon_n := 1/2(n + \sqrt{n^2 + 4})$. The function[3] $Z_1(s)$ meromorphically continues to $\mathbb{C}$ and its only singularities are simple poles at $s = 1/2$ and $-1/2$ with residue 1 (Mayer [**55**, Proposition 2]). Mayer determined the analytic properties of $\zeta_{CF}(z, s)$.

THEOREM 3.3 (Mayer). *(i) The two-variable zeta function* $\zeta_{CF}(z, s)$ *of the Gauss continued fraction map* $1/x \pmod 1$ *is meromorphic in the z-plane for each fixed s with* $\mathrm{Re}(s) > 1/2$.

(ii) For $z = -1$, *0 and 1, the function* $\zeta_{CF}(z, s)$ *extends to a meromorphic function of s in the entire s-plane.*

[3] It seems likely that each $Z_k(s)$ meromorphically continues to $\mathbb{C}$ and has simple poles at $s = 1/2$ and $-1/2$.

PROOF. (i) Mayer [**54**] [**56**] considers the transfer operator

$$\mathcal{L}_{\beta} f(x) := \sum_{n=1}^{\infty} \left(\frac{1}{x+n} \right)^{2s} f\left(\frac{1}{x+n} \right) , \tag{3.20}$$

where $f \in A_{\infty}(D)$, a Banach space of functions holomorphic inside the closed disk $D = \{z : |z-1| \le 3/2\}$ and continuous on its boundary. He expresses $\zeta_{CF}(z,s)$ as a quotient of two Fredholm determinants

$$\zeta_{CF}(z,s) = \frac{\det(I + z\mathcal{L}_{s+1})}{\det(I - z\mathcal{L}_s)} , \tag{3.21}$$

and obtains the result using Grothendieck's Fredholm theory of nuclear operators on suitable Banach spaces.

(ii) For $z = 0$ $\zeta_{CF}(z,s) = 1$. For $z = 1$ it follows from Corollary 7 of Mayer [**71**]. The proof for $z = -1$ is similar to Corollary 7, as Mayer [**56**, Theorem 1] indicates. □

Some further analytic continuation of $\zeta_{CF}(z,s)$ might be possible in the s variable, in analogy with Theorem 5.1 below. Apparently $z = 1$ and -1 are "special values", as indicated by Theorem 3.4 below.

Mayer [**56**] observed that the two-variable zeta function is related to the Selberg zeta function $Z(s)$ of the modular surface $\mathcal{M} = \mathbb{H}/PSL(2,\mathbb{Z})$. The Selberg zeta function for a surface $\mathbb{H}/\Gamma$ has the general form

$$Z(s) = \prod_{\gamma} \prod_{k=0}^{\infty} (1 - e^{-(s+k)l(\gamma)}) \tag{3.22}$$

where the product is over the lengths of all isolated periodic geodesics on $\mathcal{S}$ with prime period $l(\gamma)$. In the special case of the modular surface $\mathcal{M}$ it is known that the values $e^{l(\gamma)}$ are algebraic integers, which are units in real quadratic fields. Vigneras [**93**] showed that the Selberg zeta function for the modular surface satisfies the functional equation

$$\hat{Z}(s) = \hat{Z}(1-s) , \tag{3.23}$$

where

$$\hat{Z}(s) := Z_{par}(s) Z_{\infty}(s) Z_{e_2}(s) Z_{e_3}(s) Z(s) \tag{3.24}$$

is meromorphic on $\mathbb{C}$, and

$$Z_{par}(s) = \zeta(2s-1) \tag{3.25}$$

$$Z_{e_2}(s) = \left[1 + \tan\left(\frac{\pi}{2} - \frac{\pi}{4} \right) \right]^{1/2} \tag{3.26}$$

$$Z_{e_3}(s) = \left[1 + \sqrt{3} \tan\left(\frac{\pi s}{3} - \frac{\pi}{s} \right) \right]^{2/3} \tag{3.27}$$

$$Z_{\infty}(s) = [\Gamma_2(s)^2 \Gamma(s)^{-1} (2\pi)^{-s}]^{1/6} , \tag{3.28}$$

where $\Gamma_2(s)$ is Barne's double gamma function [4], and the principal branch is taken for $Z_2(s)$, $Z_3(\infty)$ and $Z_\infty(s)$ on the positive real axis. (The multi-valuedness of $Z_{e_2}(s)$, $Z_{e_3}(s)$, $Z_\infty(s)$ cancels out in $\hat{Z}(s)$.)

THEOREM 3.4 (Mayer). *The Selberg zeta function $Z(s)$ for $\mathbb{H}/PSL(2,\mathbb{Z})$ is given by*

$$Z(s) = \prod_{k=0}^{\infty} \frac{1}{\zeta_{SR}(s+k)} \tag{3.29}$$

in which

$$\zeta_{SR}(s) = \zeta_{CF}(1,s)\ \zeta_{CF}(-1,s)\ . \tag{3.30}$$

PROOF. The formula (3.29) appears in Mayer [**56**]. By Mayer's definition

$$\zeta_{CF}(1,s) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} Z_n(T_G, A_S)\right) = \frac{\det(I+\mathcal{L}_{s+1})}{\det(I-\mathcal{L}_s)} \tag{3.31}$$

$$\zeta_{CF}(-1,s) = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} Z_n(T_G, A_S)\right) = \frac{\det(I-\mathcal{L}_{s+1})}{\det(I+\mathcal{L}_s).} \tag{3.32}$$

Thus we obtain

$$\zeta_{CF}(1,s)\zeta_{CF}(-1,s) = \exp\left(\sum_{n=1}^{\infty} \frac{2}{2n} Z_2(T_G, A_S)\right) := \zeta_2(1,s)\ ,$$

and Mayer proves that $\zeta_{SR}(s) = \zeta_2(1,s)$. □

The derivation of the formula (3.29) by Mayer used a symbolic encoding of geodesics on the modular surface developed by Adler and Flatto [**2**] and Series [**86**]. Mayer [**56**] observed that the function $\zeta_{SR}(s)$ can be defined in terms of $Z(s)$ by

$$\zeta_{SR}(s) = \frac{Z(s+1)}{Z(s)}, \tag{3.33}$$

and noted that it can be viewed as the dynamical zeta function of a suspension map over the Gauss map; such maps are discussed in Parry and Pollicott [**65**].

Theorem 3.4 implies that the poles of the functions $\zeta_{CF}(\pm 1, s)$ have an interpretation connected with the discrete spectrum of the Laplacian on the modular surface. In fact $\det(I+\mathcal{L}_s)$ is zero at the "odd" eigenvalues of the Laplacian while $\det(I-\mathcal{L}_s)$ is zero at the "even" eigenvalues of the Laplacian *and* at the complex zeros of $\zeta(2s)$, see Efrat [**21**] and Eisele and Mayer [**22**]. (Here the odd eigenvalues are eigenvalues on the space L^2_{odd} of L^2-functions

[4] Here $\frac{1}{\Gamma_2(s+1)} = (2\pi)^{s/2} e^{-1/2s(s+1)-1/2\gamma s^2} \prod_{n\geq 1}(1+s/n)^n e^{-s+s^2/2n}$ where γ is Euler's constant, and satisfies the functional equation $\Gamma_2(s+1) = \frac{\Gamma_2(s)}{\Gamma(s)}$ and $\Gamma_2(1)=1$, see also Voros [**95**].

$f(x+iy)$ on $\mathbb{H} \setminus PSL(2,\mathbb{Z})$ with $f(x+iy) = -f(-x+iy)$. Here L^2_{even} consists of L^2-functions with $f(x+iy) = -f(-x+iy)$.) The formula (3.29) is equivalent to

$$Z(s) = \det(I + \mathcal{L}_s)\det(I - \mathcal{L}_s) \ , \tag{3.34}$$

The functions $\det(I + \mathcal{L}_s)$ have poles at $s = \frac{1}{2}(1-j)$, $j \geq 1$ hence $Z(s)$ is homomorphic except at these points. The zeros of $\det(I + \mathcal{L}_s)$ are all on the line $s = \frac{1}{2}$, as are those of $\det(I + \mathcal{L}_s)$ coming from the "even" zeros. The zeros of $\zeta^*(2s)$ lie in the strip $0 < \mathrm{Re}(s) < \frac{1}{2}$. In particular the poles of $\zeta_{CF}(1,s)$ for $\mathrm{Re}(s) > 0$ are at the complex zeros of $\zeta(2s)$ and at the "even" eigenvalues of the Laplacian acting on $L^2(\mathbb{H}/PSL(2,\mathbb{Z}))$. A detailed analysis of the zeros and poles of $\det(I \pm \mathcal{L}_s)$ at the special points $s = \frac{1}{2}(1-j)$, $j \geq 0$, was recently given by Chang and Mayer [**15**], who relate them to the space of cusp forms of weight j and Maass wave forms of $\mathbb{H}/PSL(2,\mathbb{Z})$, and to work of Lewis [**48**] and Zagier [**103**].

We next consider the two-variable zeta function $\zeta_{CF}(z,s)$ at $s = 1$. By Theorem 3.3 $\zeta_{CF}(z,1)$ is meromorphic on $\mathbb{C}$. This function contains information on the ergodic behavior of the continued fraction map relative to its unique absolutely continuum invariant measure on $[0,1]$, given by the *Gauss measure* μ_G

$$d\mu_G := \left(\frac{1}{\log 2}\right)\frac{dx}{1+x} \tag{3.35}$$

on $[0,1]$. Gauss stated in a letter to Laplace that the event $\{x : f^n_{CF}(x) \leq a\}$ had probability $\log_2(1+a)$, which is now taken to mean

$$\lim_{n\to\infty} \ meas\{x : T^n x \leq a\} = \frac{1}{\log 2}\int_0^a \frac{dx}{1+x} = \log_2(1+a) \ . \tag{3.36}$$

The Lebesgue measure of $\{x : f^n_{CF}(x) \leq a\}$ depends on n, and the "relaxation dynamics" concerns the rate at which it converges to the limit (3.36) as $n \to \infty$. This problem has a long history, including work of Levy, Kuzmin, Wirsing and Babenko. It was studied at length by Mayer and Roepstorff [**58**], [**59**]. One has

$$meas\{x : f^n_{CF}(x) \leq a\} = \log_2(1+a) + \sum_{i=2}^{\infty} \lambda_i^n c_i(a) \tag{3.37}$$

where $\lambda_1 = 1 > |\lambda_2| \geq |\lambda_3| \geq \cdots$ are real numbers with $|\lambda_i| < \infty$, and $c_i(a)$ are certain continuous functions. One has $\lambda_2 = .30366\ldots$. The quantities $\{\frac{1}{\lambda_i} : 1 \leq i \leq \infty\}$ are the locations of the poles of $\zeta_{CF}(z,1)$, counted with multiplicity. Mayer and Roepstorff [**59**, p. 343] conjecture that all the poles of $\zeta_{CF}(z,1)$ are simple, and that the eigenvalues alternate in sign

$$(-1)^{i+1}\lambda_i > 0 \ .$$

Khinchin showed that under the Gauss measure $d\mu_G$ the partial quotients a_n of the continued fraction expansion are identically distributed with distribution

$$p_m := \mu_G(\{x : a_n(x) = m]) = \log_2 \left(\frac{1}{1 + m(m+2)} \right) . \tag{3.38}$$

However the partial quotients $\{a_n\}$ are not independent random variables. The two-point correlation between $\{a_n\}$ and $\{a_{n+k}\}$ drops off exponentially as $k \to \infty$. The location of the pole of $\zeta_{CF}(z, 1)$ closest to the origin encodes information about the rate of decay of these correlations.

There is ergodic theory information about the dynamical system $\mathcal{S} = (f_{CF}, d\mu_G)$ which can be extracted from $\zeta_{CF}(z, s)$ using both variables. The *Kolmogorov-Sinai entropy*[5] $h_{K-S}(\mathcal{S})$ is given by Pesin's formula

$$h_{K-S}(\mathcal{S}) = \int_0^1 \log |f'_{CF}(x)| d\mu_G(x) = \frac{\pi^2}{6 \log 2} . \tag{3.39}$$

Mayer [**71**, Appendix] observes that it can be recovered from $\zeta_{CF}(z, s)$, as follows. The *pressure* $P(\beta)$ for real $\beta > 1/2$ is defined by

$$\exp(P(\beta)) := \sup\{r : \zeta(z, \beta) \text{ is holomorphic in } |z| < r\} ,$$

i.e. log $P(\beta)$ is the location of the pole of $\zeta(z, \beta)$ nearest the origin (which is necessarily on the positive real axis). Then

$$h_{K-S}(\mathcal{S}) = -\frac{d}{d\beta} P(\beta) \big|_{\beta=1} . \tag{3.40}$$

This is a special case of the thermodynamic formalism discussed in Section 4.

4. Statistical Mechanics of One Dimensional Lattice Gases

In this section we describe classical equilibrium statistical mechanics for one-dimensional lattice gases and the associated thermodynamic formalism of Ruelle [**78**]. The zeta function of a statistical mechanical system $\mathcal{S}$ is a generating function that encodes information about passing to the thermodynamic limit through a sequence of finite models of larger and larger size. We define

$$\zeta_{\mathcal{S}}(z, \beta) = \exp \left(\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(\beta) \right) \tag{4.1}$$

in which $Z_n(\beta)$ is the partition function for a finite model of size n, and the systems are presumed to have a thermodynamic limit as $n \to \infty$. We compute the zeta function for a simple "exactly solvable" model, the one dimensional Ising model.

[5]The denominator of the n^{th} partial quotient $\frac{p_n}{q_n}$ of the continued fraction expansion of a real number drawn randomly from $(0, 1)$ with distribution $d\mu_G$ has $\lim_{n\to\infty} (q_n)^{1/n} = \frac{\pi^2}{12 \log 2}$, with probability 1.

The statistical mechanics viewpoint provides an interpretation of the variables appearing in multivariable dynamical zeta functions. The geometric variable z is a "scaling variable" for taking the thermodynamic limit. All other variables s, h, etc. represent macroscopically measurable parameters of the physical system such as temperature or magnetization. The singularities of the statistical mechanics zeta function encode properties of the associated physical system.

Ruelle's definition of a dynamical zeta function with general weight φ in [**75**] was motivated by analogy with statistical mechanics. A connection between invariant measures of dynamical systems and Gibbs measures of one-dimensional statistical mechanics models was first observed by Sinai [**87**]. This analogy permits the "thermodynamic formalism" to be carried over to dynamical zeta functions of expanding maps. In the last subsection we detail the analogy between expanding maps and one-dimensional lattice gas models.

4.1. Equilibrium Statistical Mechanics and Thermodynamics. Following Ruelle [**74**], we recall that statistical mechanics originated from the desire to obtain a mathematical understanding of a class of physical systems of the following nature.

(a) The system is an assembly of identical subsystems.
(b) The number of subsystems is large.
(c) The interactions between the subsystems are such as to produce a thermodynamic behavior of the system.

The notion of thermodynamic behavior refers to a macroscopic description of the system, in which the subsystems are regarded as small and not individually observed. Typically, thermodynamic behavior may be described as follows.

(a′) The state of an isolated system tends to an equilibrium state as time tends to $+\infty$ ("approach to equilibrium").
(b′) An equilibrium state of a system consists of one or more macroscopically homogeneous regions (called "phases").
(c′) Equilibrium states can be parametrized by a finite number of thermodynamic parameters which determine all thermodynamic functions. The thermodynamic functions depend piecewise analytically (or smoothly) on the parameters, and the singularities of the functions correspond to changes in the phase structure of the system (phase transitions).

A mathematical justification of (a′)–(c′) involves considering the limiting case of an infinite number of subsystems, the *thermodynamic limit.*

We consider here equilibrium statistical mechanics, which is concerned with the structure and nature of the set of equilibrium states, as a function of the thermodynamic parameters. Furthermore we concentrate on the special case of one-dimensional lattice gases. The prototype of such models is the one-dimensional Ising model, solved in 1925 by Ising [**38**]. One dimensional lattice gases have been exhaustively studied, see Simon [**86**]. The translation operator or *shift operator* on a one-dimensional lattice yields a dynamical structure on the lattice gas configuration space, which leads to a relation with dynamical systems.

A one-dimensional lattice gas is described by an infinite collection Ω of configurations on a one-dimensional lattice $\mathbb{Z}$ with each configuration being assigned an energy (viewed as an energy-density per site on the lattice). The infinite system consists of a one-dimensional lattice of *sites* indexed by $\mathbb{Z}$, and an infinite *configuration* is a set of values

$$(\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots) \tag{4.2}$$

in the values are drawn from a finite or countable *alphabet*[6] $\mathcal{A}$. We let

$$\Omega \subseteq \mathcal{A}^{\mathbb{Z}} \tag{4.3}$$

denote the space of allowed configurations. The set of configurations is required to be invariant under translations and an *equilibrium state* will be a measure on Ω.

These infinite systems are viewed as arising as limits as $n \to \infty$ of finite systems on a finite "lattice" consisting M lattice sites in a line. That is, the *lattice gas model* consists of a collection of finite models $\{(\Omega_n, E_n) : n = 1, 2, 3, \ldots, \}$. The model on n sites has a configuration space

$$\Omega_n \subseteq \mathcal{A}^n \ , \tag{4.4}$$

which is required to be *translation-invariant* under cyclic shift of states

$$S((\sigma_1, \sigma_2 \ldots, \sigma_n)) = (\sigma_2, \sigma_3, \ldots, \sigma_n, \sigma_1) \ . \tag{4.5}$$

We view the points of Ω_n as corresponding to periodic configurations in the limiting model. The finite model is completely specified by its *energy function* $E_n : \Omega_n \to \mathbb{R}$ which assigns to each configuration $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ an energy $E_n(\sigma)$; We extend the energy function E_n to $\mathcal{A}^n$ by assigning the energy $+\infty$ to all configurations in $\mathcal{A}^n \setminus \Omega_n$ ("hard core constraint"). The energy function defines a probability distribution called the *Gibbs distribution*, which makes the probability of a configuration $\boldsymbol{\sigma}$ proportional to $\exp(-\beta E(\boldsymbol{\sigma}))$, in which β is a positive constant. In gas models the parameter β traditionally denotes

$$\beta := \frac{1}{\kappa T}, \tag{4.6}$$

where κ is Boltzmann's constant and T is the temperature. The Gibbs distribution is then

$$\mu_n(\sigma) := \frac{1}{Z_n(\beta)} \exp(-\beta E_n(\sigma)) \ , \tag{4.7}$$

[6]More general lattice gas models allow a continuous space $\mathcal{A}$ as alphabet, e.g., $\mathcal{A} = SO(3, \mathbb{R})$ is used to describe continuous spins.

in which the *partition function* $Z_n(\beta)$ is the normalizing factor necessary to get a probability distribution. We have

$$Z_n(\beta) := \sum_{\sigma} \exp(-\beta E_n(\sigma)) \ , \tag{4.8}$$

in which the sum as taken over all legal configurations Ω_n, or over all $\mathcal{A}^n$, using the convention above on when $E(\sigma) = +\infty$. We view the energy function $E_n(\boldsymbol{\sigma})$ as being a sum of local interactions over various distances, so that

$$E_n(\sigma) := \sum_{i=1}^{n} \phi_1(\sigma_i) + \sum_{i=1}^{n} \phi_2(\sigma_i, \sigma_{i+1}) + \sum_{j=1}^{n} \phi_3(\sigma_i, \sigma_{i+1}, \sigma_{i+2}) + \cdots \ , \tag{4.9}$$

We require that the energy functions $E_n(\sigma)$ for different n be compatible in the sense that the local interaction functions $\phi_n(\cdot)$ be the same for each n. This is no loss of generality because once $\phi_1, \ldots, \phi_{n-1}$ are fixed, there are still enough degrees of freedom in the function ϕ_n to describe any energy function E_n. However in physical applications one is interested in such energy functions in which the interactions become small at large distances, i.e. $\phi_k(\sigma) \to 0$ as $k \to \infty$. In particular, an interaction is *finite-range* if $\phi_k \equiv 0$ for all large enough k.

All thermodynamic quantities associated to a finite system are encoded in its Gibbs distribution. These include the *average energy per site*[7]

$$\left\langle \frac{1}{n} E_n \right\rangle := \frac{1}{Z_n} \sum_{\sigma \in \mathcal{A}^n} E_n(\sigma) \exp(-\beta E_n(\sigma)) \ , \tag{4.10}$$

and quantities like the *nearest-neighbor two-point correlation*

$$\langle \sigma_x \sigma_{x+1} \rangle := \frac{1}{Z_n} \sum_{\sigma \in \mathcal{A}^n} \sigma_1 \sigma_2 \exp(-\beta E_n(\sigma)) \ . \tag{4.11}$$

The *thermodynamic limit* concerns the limiting behavior of the systems (Ω_n, E_n) as $n \to \infty$. Under some restrictions on the local interaction functions $\{E_k : k \geq 1\}$ a thermodynamic limit will exist. The thermodynamic limit is often understood in the "classical" sense to be a suitably scaled limiting behavior of the partition functions $Z_n(\beta)$ as $n \to \infty$. In thermodynamics one considers a *family* of energy functions $E_n(\beta, h, \ldots)$ depending on a finite number of thermodynamic parameters $\beta, h, \ldots$ representing "macroscopic" quantities such as temperature, magnetization per site, etc. A fundamental quantity obtained in taking the thermodynamic limit is the *pressure function*

$$p(\beta, h) := \lim_{n \to \infty} \frac{1}{n} \log Z_n(\beta, h) \ . \tag{4.12}$$

The subject of classical thermodynamics deals with the behavior of the pressure function $p(\beta, h)$ for specific systems. The pressure function $p(\beta)$ for the thermodynamic limit of Gibbs distributions (4.8) contains information

[7] We set $E \log(-\beta E) = 0$ if $E = +\infty$. Note that E_n represents an energy across n sites, so $\frac{1}{n} E_n$ measures energy per site.

about the limiting energy distribution per site. One can recover from it all the moments $\langle \tilde{E}^k \rangle$ of this limiting distribution, which are given by

$$\langle \tilde{E}^k \rangle := \lim_{n\to\infty} \left\langle \frac{1}{n}(E_n)^k \right\rangle . \tag{4.13}$$

These quantities can be expressed in terms of β-derivatives of $p(\beta)$, via

$$\langle \tilde{E} \rangle = \lim_{n\to\infty} \left\langle \frac{1}{n} \frac{Z_n'(\beta)}{Z_n(\beta)} \right\rangle = p'(\beta) \tag{4.14}$$

$$\langle \tilde{E}^2 \rangle = p''(\beta) + p'(\beta)^2 \tag{4.15}$$

$$\langle \tilde{E}^3 \rangle = p'''(\beta) + 3p'(\beta)p''(\beta) + p'(\beta)^3 , \tag{4.16}$$

and so on. We are assuming here that the interchange of the order of differentiation and taking limits can be justified. There is a large mathematical literature concerning circumstances when this can be done, cf. Ruelle [**74**], [**78**], and Simon [**86**].

In Gibbs' development of thermodynamics the fundamental macroscopic quantity is the *free energy per site*, which is

$$f(\beta, h) = -\frac{1}{\beta} p(\beta, h) . \tag{4.17}$$

Mathematically it is equivalent to knowing the pressure, and from it one can derive other thermodynamic quantities. In gas models the *internal energy per site* is

$$u(\beta) = \frac{\partial}{\partial \beta}(\beta f(\beta)) . \tag{4.18}$$

The *specific heat per site* is

$$c(\beta) = -\beta^2 \frac{\partial}{\partial \beta} u(\beta) . \tag{4.19}$$

In the Ising model the *magnetization* (or *average magnetic moment per site*) is

$$M(\beta, h) = -\frac{\partial}{\partial h} f(\beta, h) . \tag{4.20}$$

These are macroscopically measurable quantities.

More generally, the thermodynamic limit concerns the existence of well-behaved invariant measures on infinite configurations Ω that describes a limit of the Gibbs distributions on finite site models Ω_n. An *equilibrium state* is a translation invariant Borel measure on Ω such that when conditioned on a finite set of consecutive sites $\{1, 2, \ldots, n\}$ yields the Gibbs measure $d\mu_n$ for the n-site model (Ω_N, E_n). We say that a thermodynamic limit exists (for given parameter values (β, h)) if at least one equilibrium state exists. For given thermodynamic parameters a statistical mechanics model may have one or many equilibrium states; these correspond to different phases possible at those parameter values. Associated to an equilibrium state $d\mu$ are its *n-point correlation functions*

$$\langle \sigma_x \sigma_{x+1} \cdots \sigma_{x+n-1} \rangle := \int_{\Omega} \sigma_x \sigma_{x+1} \cdots \sigma_{x+n-1} d\mu \ , \tag{4.21}$$

which measure how states on the lattice are related at fixed distances. The correlation functions are independent of the choice of equilibrium state at the given thermodynamic parameter values. That is, all equilibrium states produce the same thermodynamic quantities. In general the multi-point correlation functions contain information that cannot be obtained from the pressure function.

Equilibrium states have been proved to exist for a wide variety of energy functions, for one-dimensional lattice gases, see Sinai [**87**]. He considers a symbolic dynamical system $\Sigma \subseteq \mathcal{A}^{\mathbb{Z}}$ given with an invariant measure $\overline{\mu}$ of maximum entropy equal to the topological entropy. For a large class of energy functions f he proves there is a unique Gibbs measure $\overline{\mu}(f)$, see Sinai [**87**, Theorem1].

Finally we consider *phase transitions*, which are abrupt changes in physical state as a thermodynamic parameter is varied, e. g. gas to liquid. In statistical mechanics models this is reflected in abrupt changes in the nature of the set of equilibrium states. For definiteness we consider the inverse temperature β as the parameter to be varied. For many systems the pressure $p(\beta)$ is an analytic function of the parameter β in a neighborhood of an interval on the real axis $\sigma_1 < \beta < \sigma_2$. In 1952 Lee and Yang [**47**], [**102**] proposed that phase transitions in some statistical mechanical systems would manifest themselves as singularities of the pressure function viewed as an analytic function of the inverse temperature parameter β. Phase transitions might similarly be detected in failures of analyticity of correlation functions as a function of the parameter β. For further elaboration of this mathematical mechanism for phase transitions, see Kac [**40**].

In the special case of one-dimensional lattice gases, it is known that for finite-range interactions phase transitions cannot occur, i.e. the pressure function $p(\beta)$ is analytic on the entire range $0 < \beta < \infty$. The term "gas" for such systems reflects the fact that such systems are ergodic and have exponential decay of correlations as a function of distance; they have a unique equilibrium state. (That is, they never "freeze.") However phase transitions can occur for infinite range interactions. Typically the pressure function $p(\beta)$ will be analytic in some vertical strip $\sigma_1 < \mathrm{Re}(\beta) < \sigma_2$, with $0 \le \sigma_1 < \sigma_2 \le \infty$, and one studies the system in this range. The failure of analyticity of the pressure function may be reflected in the behavior of the singularities of the zeta function $\zeta(z, \beta)$ in the z-variable, as the β-variable is varied. This motivates the study of the analytic properties of $\zeta(z, \beta)$ in both variables.

4.2. Statistical Mechanics Zeta Functions. We formally define statistical mechanics zeta functions of a statistical mechanics model $\mathcal{S}$ by

$$\zeta_{\mathcal{S}}(z,\beta) := \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(\beta)\right) . \tag{4.22}$$

We can apply this definition to any classical statistical mechanics model which comes with a family of partition functions$\{Z_n(\beta) : n = 1, 2, 3, \ldots\}$ for finite size systems, e.g. the two-dimensional Ising model. We call the variable z the *scaling variable* because it is the variable incorporating the thermodynamic limit. This zeta function also is a function of all the macroscopic thermodynamic variables that are present in the partition functions.

If the model $\mathcal{S}$ is a one-dimensional lattice gas, then the definition (4.22) is a special case of the Ruelle dynamical zeta function defined in §3. Here we take the dynamical system (Ω, s, φ) where $S : \Omega \to \Omega$ where $\Omega \subseteq \mathcal{A}^{\mathbb{Z}}$ is the limiting configuration space and S is the shift operator, and $\varphi = E(\sigma)$ is the energy function on periodic orbits σ. Ruelle's definition of the dynamical zeta function in §3 was motivated by this analogy.

A converse assertion seems useful as a heuristic principle: it appears that mostt dynamical systems (X, f, φ) can be "reverse engineered" to give an energy function (Hamiltonian) for a one-dimensional lattice gas whose statistical mechanics zeta function (4.22) is identical to the Ruelle zeta function $\zeta_f(z,s)$ in (3.4), with $s = \beta$. The resulting Hamiltonian may be quite complicated and physically unnatural as a lattice gas model, however. For example, consider the map $f(x) \equiv \beta x \pmod 1$ with $\beta > 1$. Let $\Omega_\beta \subseteq \mathcal{A}^{\mathbb{N}}$ be the shift describing the allowed symbolic dynamics for f, where $\mathcal{A} = \{0, 1, 2, \ldots, \lfloor \beta \rfloor\}$. Let $\Omega_\beta^* \subseteq \mathcal{A}^{\mathbb{Z}}$ the closed two-sided subshift that is the natural extension of Ω_β, which consists of all infinite two-sided strings such that each suffix $(\sigma_j, \sigma_{j+1}, \sigma_{j+2}, \ldots) \in \Omega_\beta$. The set Ω_β^* is closed, and is specified by its set $\mathcal{F}_\beta^*$ of forbidden blocks for $j \in \mathbb{Z}$. We produce a one-dimensional lattice gas (Ω_β^*, S, E) that produces the same zeta function using the energy function

$$E(\sigma_1, \ldots, \sigma_n) = \begin{cases} \log \beta & \text{if} \quad n = 1 , \\ +\infty & \text{if} \quad (\sigma_1, \ldots, \sigma_n) \text{is a minimal forbidden block for } \Omega_\beta^* , \\ 0 & \text{otherwise} . \end{cases} \tag{4.23}$$

This Hamiltonian consists of a constant self-interaction term and "hard-core conditions" which specify the symbolic dynamics Ω_β^*. It gives a finite range interaction if and only if Ω_β^* has a finite set of minimal forbidden blocks, i.e. it is a shift of finite type (see Lind and Marcus [**49**], Chapter 2.) When this occurs $\zeta_f(z)$ is rational function of z. For most values of β the zeta function $\zeta_f(z)$ has the unit circle$\{z : |z| = 1\}$ as a natural boundary; however there are values of β such that Ω_β^* is not a shift of finite type but $\zeta_f(z)$ is still a rational function.

The two-variable zeta function $\zeta_{\mathcal{S}}(z,s)$ encodes information about the thermodynamic limit in its singularities. Under suitably strong hypotheses one can prove results like:

(1) The function $\zeta_{\mathcal{S}}(z,s)$ is meromorphic on the domain $\{z : |z| < 1\} \times \{s : \mathrm{Re}(s) > 0\}$.
(2) For $s = 0$ the closest singularity to the origin in the z-variable is at $z = e^{-h(\mathcal{S})}$, where $h(\mathcal{S}) > 1$ is the topological entropy of the space Ω of limiting configurations of $\mathcal{S}$. This singularity is a simple pole, and there are no other singularities of $\zeta(z,0)$ in the closed disk $\{|z| \le e^{-h(\mathcal{S})}\}$.
(3) Let $r(\beta)$ denote the radius of the closest singularity in the z-variable to the origin of $\zeta_{\mathcal{S}}(z,s)$ for $s = \beta$ a positive real, $0 \le \beta \le \infty$. Then the pressure is given by

$$p(\beta) = \frac{\partial}{\partial \beta}\left(\frac{1}{r(\beta)}\right), \quad 0 < \beta < 1 . \tag{4.24}$$

Such results hold for the one-dimensional Ising model considered below as well as for the homogeneously expanding maps considered in Section 3.

4.3. Zeta Function for the One-Dimensional Ising Model. We compute the three-variable zeta function of the one-dimensional Ising model and describe its thermodynamic properties. The one-dimensional model was solved exactly by Ising [**38**] in 1925. Ising observed that this model has no phase transition, and (erroneously) inferred from this that the n-dimensional model for each $n \ge 2$ should also have no phase transitions.

We first describe the Ising model on a finite part of the one-dimensional lattice consisting of n consecutive sites. with periodic boundary conditions. Each site is assigned a spin $\sigma_i \in \{+1, -1\}$ so there are 2^N configurations $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. The *energy* $E(\boldsymbol{\sigma})$ of a configuration is given by

$$E(\boldsymbol{\sigma}) := -J \sum_{i=1}^{n} \sigma_i \sigma_{i+1} - H \sum_{i=1}^{n} \sigma_i \ , \tag{4.25}$$

in which $\sigma_{n+1} = \sigma_1$ (periodic boundary condition) and J, h are parameters with h representing magnetization. We set $\beta := \frac{J}{kT}$ and $h := \frac{H}{kT}$, and the partition function becomes

$$Z_n(\beta, h) := \sum_{\boldsymbol{\sigma} \in \{1,-1\}^n} \exp\left(\beta \sum_{i=1}^{n} \sigma_i \sigma_{i+1} + h \sum_{i=1}^{n} \sigma_i\right) . \tag{4.26}$$

Ising [**38**] found that the partition function $Z_n(\beta, h)$ has a closed form which permitted him to compute the thermodynamic limit and obtain the pressure (or equivalently, the free energy per site). We follow Baxter [**10**, p. 33]. The partition function (4.26) factors into a product of nearest-neighbor interactions as

$$Z_n(\beta, h) = \sum_{\boldsymbol{\sigma} \in \{-1,1\}^n} \prod_{i=1}^{n} V(\sigma_i, \sigma_{i+1}) \ , \tag{4.27}$$

in which

$$V(\sigma, \sigma') = \exp(\beta\sigma\sigma' + \frac{1}{2}h(\sigma + \sigma')),$$

using the convention that $\sigma_{n+1} = \sigma_1$. We introduce the *transfer matrix*

$$V = \begin{bmatrix} V(1,1) & V(1,-1) \\ V(-1,1) & V(-1,-1) \end{bmatrix} = \begin{bmatrix} e^{\beta+h} & e^{-h} \\ e^{-h} & e^{\beta-h} \end{bmatrix} . \tag{4.28}$$

The nearest-neighbor factorization (4.27) corresponds to

$$Z_n(\beta, h) = \text{Trace } [V^n] \ . \tag{4.29}$$

Thus

$$Z_n(\beta, h) = \lambda_+^n + \lambda_-^n \ , \tag{4.30}$$

where $\lambda_+ = \lambda_+(\beta, h)$ and $\lambda_- = \lambda_-(\beta, h)$ are the eigenvalues of V, given by

$$\lambda_\pm = e^\beta \left(\frac{e^h + e^{-h}}{2} \right) \pm \left(e^{2\beta} \left(\frac{e^h - e^{-h}}{2} \right)^2 + \frac{1}{4} e^{-2\beta} \right)^{1/2} . \tag{4.31}$$

For $\beta > 0$, h real, the matrix V is a real symmetric matrix so both eigenvalues are real, with $\lambda_+ > \lambda_-$, and the pressure is

$$p(\beta, h) = \log \lambda_+ \ . \tag{4.32}$$

The three-variable zeta function associated to the Ising model is

$$\zeta_{Ising}(z, \beta, h) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(\beta, h) \right) . \tag{4.33}$$

The formulae above yield:

THEOREM 4.1. *The one-dimensional Ising model $\mathcal{I}$ has three-variable zeta function*

$$\zeta_{\text{Ising}}(z, \beta, h) = \frac{1}{1 - (e^{\beta+h} + e^{\beta-h})z + (e^{2\beta} + e^{-2h})z^2} . \tag{4.34}$$

When there is no magnetic field ($h = 0$), the two-variable zeta function is

$$\zeta_{\text{Ising}}(z, \beta) = \frac{1}{1 - 2e^\beta z + (e^{2\beta} + 1)z^2} . \tag{4.35}$$

PROOF. Using (4.30) we have

$$\begin{aligned}\zeta_{Ising}(z,\beta,h) &= \exp(\sum_{n=1}^{\infty}\frac{z^n}{n}(\lambda_+^n+\lambda_-^n)) \\ &= \exp(-\log(1-\lambda_+ z)-\log(1-\lambda_- z)) \\ &= (1-\lambda_+ z)^{-1}(1-\lambda_- z)^{-1} \\ &= (\det(I-zV))^{-1} \ .\end{aligned}$$

This gives (4.34). □

By inspection we see that $\zeta_{\mathcal{I}}(z,\beta,h)$ meromorphically continues to $\mathbb{C}^3$. For fixed β and h it has two poles in the z-variable, except that it can have a double pole for certain special values of the parameters (β,h). For $\beta>0$ and real h the pole closest to the origin in the z-plane is at the positive real number

$$z_+ = \frac{1}{\lambda_+} = \exp(-\log\lambda_+) = \exp(-p(\beta,h)) \ . \tag{4.36}$$

Thus the location of the pole of $\zeta_{\mathcal{I}}(z,\beta,h)$ in the scale variable z closest to the origin determines the pressure function, from which the free energy per site and other thermodynamic quantities (4.17)–(4.20) can be derived.

Recall that associated to an equilibrium state there are *n-point correlation functions*

$$\langle\sigma_x\sigma_{x+i_1}\dots\sigma_{x+i_{n-1}}\rangle := \int_{\Omega}\sigma_x\sigma_{x+i_1}\dots\sigma_{x+i_n}d\mu(x) \ . \tag{4.37}$$

Some combinations of these functions are partial derivatives of the pressure. For the Ising model the one-point correlation $\langle\sigma_x\rangle$ is the magnetization per site

$$\langle\sigma_x\rangle = M(\beta,h) = -\frac{\partial}{\partial h}p(\beta,h) \ .$$

The two-point correlation at distance 1 is given by

$$\langle\sigma_x\sigma_{x+1}\rangle = \frac{\partial}{\partial\beta}p(\beta,h) \ .$$

The two-point correlation functions $\langle\sigma_x\sigma_{x+m}\rangle$ for $m\geq 2$ cannot in general be deduced from the pressure. For the one-dimensional Ising model one can use a transfer matrix calculation to determine all two-point point correlations, to obtain, for $\beta>0$ and h real, that

$$\langle\sigma_x\sigma_{x+m}\rangle - \langle\sigma_x\rangle\langle\sigma_{x+m}\rangle = (\sin 2\phi)^2\left(\frac{\lambda_-}{\lambda_+}\right)^{|m|} \ , \quad \text{for } m\in\mathbb{Z} \ . \tag{4.38}$$

where $(\sin\phi,\cos\phi)$ is a left-eigenvector of the transfer matrix (4.28) that has eigenvalue λ_+, cf. Baxter [**10**, p. 36]. These correlation functions exhibit exponential decay with distance $|m|$. This manifests the "gas" nature of the interaction, and demonstrates that there is no phase transition in the one-dimensional Ising model at finite temperature.

The one-dimensional Ising model is sometimes considered to have a phase transition in the Lee-Yang sense at "absolute zero" $\beta = +\infty$, when there is no magnetic field ($h = 0$). The two poles of $\zeta_{Ising}(z, \beta)$ in the z-variable then approach each other as $\beta \to \infty$ through real values.

4.4. Lattice Gases and Dynamical Systems. The thermodynamic formalism was developed by Ruelle [**77**] for lattice gases, and then carried over to expanding linear maps. In Table 4.1 we give a dictionary of parallel concepts relating the one-dimensional lattice gas framework to the discrete dynamical system framework. Below we comment on specific entries in this table.

(1) We assume that $f : X \to X$ has a symbolic dynamics with a finite or countably infinite alphabet $\mathcal{A}$ corresponding to a partition $X = \bigcup_{\alpha \in \mathcal{A}} X_\alpha$, with the property that each $x \in X$ is uniquely determined by its symbolic itinerary $i(x) := (S_0(x), S_1(x), S_2(x), \ldots)$ where $S_n(x) := S(f^n(x))$ is the symbol α such that $f^n(x) \in X_\alpha$.

(2) An automorphism $f : X \to X$ has a two-sided infinite symbolic dynamics obtained by following the orbit of x forwards and backwards. An endomorphism $f : X \to X$ only has a one-sided infinite symbolic dynamics on a half-infinite lattice $\mathbb{N}$.

(3) An *extension* of f is any pair $(\tilde{f}, \tilde{X}, \pi)$ such that $\tilde{f} : \tilde{X} \to \tilde{X}$ is an automorphism together with an onto projection $\pi : \tilde{X} \to X$ such that the following diagram commutes

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\ \downarrow \pi & & \downarrow \pi \\ X & \xrightarrow{f} & X \end{array}$$

A minimal $(\tilde{f}, \tilde{X})$ with this property is called the *natural extension* of f. It is defined by the universal property that any other extension factors through it. When it exists it is unique up to isomorphism. It always exists when f is onto. The natural extensions of maps of the interval can sometimes be expicitly represented as maps of a two-dimensional set to itself, e.g. maps of the square. This can be done for the additive continued fraction map, see Lagarias and Pollington [**46**].

In the dynamical system case the space $\tilde{X}$ of the natural extension corresponds to the set of infinite configurations in the statistical mechanics model. Thus the thermodynamic limit $n \to \infty$ has already been taken in the dynamical system model; its domain represents a space of infinite configurations.

Baladi [**8**, Sect. 1.3] discusses the problem of going in the reverse direction from two-sided infinite dynamics to one-sided infinite dynamics in special cases; there can be many ways to do this and they are classified by a suitable cohomology group.

	One-Dimensional Lattice Gas	Discrete Dynamical System f: $X \to X$, X compact
(1)	configuration symbols $\mathcal{A}$	symbolic dynamics alphabet $\mathcal{A}$ (Markov partition)
(2)	$S(\boldsymbol{\sigma}) = \boldsymbol{\sigma}'$ with $\sigma'_n = \sigma_{n+1}$ left-translation operator (left-shift) on $\left\{ \begin{array}{l} \text{one-sided infinite lattice } \mathbb{N} \\ \text{two-sided infinite lattice } \mathbb{Z} \end{array} \right.$	$f(x) = x'$ iteration of $\left\{ \begin{array}{l} f : X \to X \text{ endomorphism} \\ f : X \to X \text{ automorphism} \end{array} \right.$
(3)	configuration space $\Omega \subseteq \mathcal{A}^{\mathbb{Z}}$	domain $\tilde{X}$ of natural extension of f
(4)	periodic n-site configurations $N_n(\Omega) = \#\{\text{periodic } n\text{-site configurations}\}$	periodic points $N_n(f) := \#\{\text{periodic points of period } n\}$
(5)	interaction energy $E : \Omega \to \mathbb{R}$	weight $\phi : X \to \mathbb{C}$
(6)	Partition function (n-sites) $Z_n(\beta) = \sum_{\substack{\text{period} \\ \text{period } n \\ \sigma}} e^{-\beta E(\boldsymbol{\sigma})}$	Gibbs weight function (periodic points) $Z_n(\beta) = \sum_{\substack{(x) \\ \text{periodic} \\ \text{period } n}} \exp\left(-\beta \sum_{i=0}^{n-1} \phi(f^i(x))\right)$
(7)	Configurational entropy $h_{config}(f) = \limsup_{n\to\infty} \frac{1}{n} \log(N_n(\Omega))$	Topological entropy $h_{top}(f) = \limsup_{n\to\infty} \frac{1}{n} \log(N_n(f))$
(8)	Transfer matrix Ruelle-Araki transfer operator	Transfer operator Perron-Frobenius operator
(9)	Gibbs state $S^{-1}(d\mu) = d\mu$;	invariant Borel measure $f^{-1}(d\mu) = d\mu$
(10)	exponential decay rate of two-point correlation	mixing rate for f with respect to $d\mu$
(11)	metric entropy $h(d\mu)$	metric entropy $h(d\mu)$

TABLE 1. Analogies between statistical mechanics of a lattice gas and discrete dynamical systems

(5) One can allow complex-valued interaction functions in statistical mechanics models. These can occur in quantum statistical mechanics models.

(6) In the case of piecewise differentiable functions $f : [0,1] \to [0,1]$ an important case is where the weight function on periodic points of period n given by

$$\phi(x) := \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(x_i)|$$

where $x_i = f^i(x)$. When extended to all points this $\phi(x)$ measures the "average expansion rate" on an orbit.

(8) There is a precise relation between transfer matrices and transfer operators, which is described in Mayer [**52**, Chapter II]. A transfer matrix is the matrix of a linear operator $\mathcal{L}_n$ on the finite-dimensional space $\mathcal{C}(\Lambda_n)$ of complex-valued functions on the set Λ_n of n-point configurations. For the Ising model $\mathcal{C}(\Lambda_n)$ is a vector space of dimension 2^n. In order to obtain a limiting operator on infinite configurations we have to consider its n-th root $(\mathcal{L}_n)^{1/n}$, whose eigenvalues measure per-site quantities. A suitable infinite limit of $(\mathcal{L}_n)^{1/n}$ will define a transfer operator on a suitable Banach space $\mathcal{B}$ of functions on the space Ω of infinite configurations, the *Ruelle-Araki transfer operator*. Formally such a transfer operator has the general form

$$(\mathcal{L}_f h)(x) := \sum_{\substack{y \\ f(y)=x}} w(y)h(y) \ ,$$

in which $w(\cdot)$ is a suitable weight function. When the weights are nonnegative real numbers such operators are sometimes called *Perron-Frobenius operators*. The functions $h(x) \in \mathcal{B}$ are interpreted as *observables* of the system, and the transfer operator can be interpreted as taking an (unnormalized) *conditional expectation* of h with respect to the action of the map $f : \Omega \to \Omega$, see Mayer [**52**, p. 53].

(9) The Gibbs probability distribution p for a finite system with a given energy function has two different characterizations. One is the formula (4.7) expressing it in terms of a partition function. The other in the *Variational Principle*, which is that this distribution maximizes the function

$$U(p) := S(p) - \sum_{\boldsymbol{\sigma}} p(\boldsymbol{\sigma})E(\boldsymbol{\sigma}) \tag{4.39}$$

over all probability distributions p on configurations, where

$$S(p) = -\sum_{\boldsymbol{\sigma}} p(\boldsymbol{\sigma}) \log p(\boldsymbol{\sigma})$$

is the entropy of p.

These two characterizations of Gibbs distributions lead to two different generalizations to infinite systems, when the thermodynamic limit is taken. *Gibbs states* are Borel probability measures $d\mu$ on the infinite configuration space Ω that generalize the property (4.7). *Equilibrium states* are Borel probability measures on the configuration space Ω that maximize the variational function, generalizing (4.39). Under some circumstances these sets of measures coincide, see Ruelle [**77**, Chapter 5]. For more information on Gibbs states and equilibrium states, see Bowen [**12**] and Sinai [**87**].

(10) The exponential decay rate of correlations arises only for ergodic systems which are very strongly mixing. In statistical mechanics this corresponds to a "gas" state, in which there is a unique equilibrium state which is a Gibbs state.

(11) For an expanding piecewise C^1-map on $[0,1]$ the metric entropy can be computed using *Pesin's formula*

$$h(d\mu) = \int |f'(x)| d\mu(x)$$

.

5. Two-Variable Zeta Functions in Number Theory

We propose and study some two-variable zeta functions attached to algebraic function fields and algebraic number fields that are formally analogous to the two-variable dynamical zeta function in §3.1.

5.1. Function Fields. In §2 we described two independent developments of the theory of zeta functions attached to an algebraic function field K over a finite field $\mathbb{F}_q$, one in term of an arithmetic variable s and the other in terms of a geometric variable z. By analogy with Theorem 3.1, we define a two-variable zeta function that keeps the arithmetic variable s and geometric variable z separate, from which one can recover both the arithmetic and geometric version of the zeta function. For a complete n-dimensional nonsingular projective variety V defined over $\mathbb{F}_q$ we define the *two-variable zeta function* $\zeta_V(z,s)$ by

$$\zeta_V(z,s) := \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} \sum_{Fix(Fr^k)} (q^k)^{-s} \right) \tag{5.1}$$

To each algebraic divisor A defined over $V(\mathbb{F}_q)$ consisting of k points this definition associates to each of these points individually the weight

$$wt(A) := N(A) = q^k \ . \tag{5.2}$$

The weights (5.2) treat the Frobenius automorphism Fr_q formally as if it were the natural extension of a uniformly expanding map with expansion factor q. With this definition we obtain

$$\zeta_V(z,s) = \zeta_V(zq^{-s}, 0) = \zeta_{Fr_q}(zq^{-s}) \ . \tag{5.3}$$

There is a two-variable version of the functional equation for the two-variable zeta function $\zeta_V(z,s)$. Let V be a complete nonsingular projective variety of dimension d defined over $\mathbb{F}_q$. If we set

$$\hat{\zeta}_V(z,s) := (zq^{-s})^{\frac{1}{2}\chi(V)} \zeta_V(z,s) \tag{5.4}$$

where $\chi(V)$ is the Euler characteristic of V, then we have the functional equation

$$\hat{\zeta}_V(z,s) = \pm \hat{\zeta}_V\left(\frac{1}{q^n z}, 1-s \right) \ . \tag{5.5}$$

We note that the corresponding two-variable extension for the arithmetic zeta function of a Dedekind ring R has the Euler product

$$\zeta_R(z,s) := \prod_P (1 - z^{\deg(P)}(NP)^{-s})^{-1} \ . \tag{5.6}$$

This differs from (5.1) in not including the Euler factors for "primes at infinity".

It would be nice if the formal analogy between two-variable function field zeta functions and dynamical zeta functions could be made exact. It is an open problem to construct a statistical mechanics model or dynamical system that would produce (5.1) as its associated two-variable zeta function. In attempting to construct such a dynamical system model, we note that the set of points on an algebraic variety V defined over $\bar{\mathbb{F}}_q$ comprise the set of periodic points of the Frobenius automorphism acting on V. Can one add "transcendental points" to the variety together with a suitable topology in a natural way such that the Frobenius map extends to act on these points to give a symbolic dynamics that is a full shift on q letters? Perhaps one can use a ring of Laurent series in an auxiliary indeterminate, with a suitable multiplication, to do this. Can one also construct in a canonical fashion a non-invertible function with q inverse images of each point, which has a natural extension which is the Frobenius automorphism acting on this object? This is not known, but see the discussion in Baladi [**8**, Sec. 1.3].

Finally we remark that the two-variable zeta function (5.1) is distinct from the two-variable zeta function for a curve recently introduced by Pellikaan [**68**].

5.2. Number Fields. The Riemann zeta function $\zeta(s)$ is not known to have a naturally associated "geometric variable" z, as in the function field case. However we can formally define a two-variable zeta function for the Dedekind ring $\mathbb{Z}$ according to the prescription (5.6) above, provided that we have a definition of the *degree* of a prime ideal.

The naive definition of degree is to view each finite prime ideal in $\mathbb{Z}$ as having degree one. In this case we obtain the two variable zeta-function

$$\zeta_{\mathbb{Z}}(z,s) := \prod_p (1 - zp^{-s})^{-1} \ . \tag{5.7}$$

We have

$$\zeta_{\mathbb{Z}}(z,s) = \sum_{n=1}^{\infty} z^{\Omega(n)} n^{-s} \ , \tag{5.8}$$

where $\Omega(n)$ denotes the number of prime factors of n, counted with multiplicity.

What is the behavior of $\zeta_{\mathbb{Z}}(z,s)$ as a function of two complex variables? The infinite product (5.2) defines it as a holomorphic function in the domain

$$\{z : |z| < 1\} \times \{s : \mathrm{Re}(s) > 1\} \ .$$

Its behavior under analytic continuation was determined in 1943 by Wintner [**101**]. His results are most elegantly stated by setting

$$\zeta'_{\mathbb{Z}}(z,s) := \frac{\partial}{\partial s}\zeta_{\mathbb{Z}}(z,s)\ . \tag{5.9}$$

He studied $\zeta_{\mathbb{Z}}(z,s)$ for fixed z and variable s. We reformulate his result as follows:

THEOREM 5.1 (Wintner). *(i) For each* $z \in \mathbb{C}$ *the function* $F_z(s) := \frac{\zeta'_{\mathbb{Z}}(z,s)}{\zeta_{\mathbb{Z}}(z,s)}$ *extends to a meromorphic function on the half-plane* $\mathrm{Re}(s) > 0$*, whose singularities are simple poles.*

(ii) For all values of $z \in \mathbb{C} \setminus \{-1,0,1\}$ *the function* $F_z(s)$ *has* $\mathrm{Re}(s) = 0$ *as a natural boundary to analytic continuation. If* $z \in \{-1,0,1\}$ *then* $F_z(s)$ *continues to a meromorphic function of* s *on* $\mathbb{C}$*.*

(iii) For each $z \in \mathbb{C}$ *the singularities of* $F_z(s)$ *in the region* $\mathrm{Re}(s) \geq 1/2$ *consist of a simple pole with residue* $-z$ *at* $s = 1$ *and simple poles with residue* z *at the nontrivial zeros of* $\zeta(s)$ *which have* $\mathrm{Re}(s) \geq 1/2$*.*

PROOF SKETCH. Wintner [**101**] actually studies $(s-1)^{-z}\zeta(z,s)$; here we have reformulated his results in terms of $F_z(s)$. He observes that for any $z \in \mathbb{C} \setminus \{0\}$ there is a unique formal expansion $\{\beta_n(z) : n = 1,2,3,\ldots\}$ with $\beta_n(z) \in \mathbb{C}$ such that

$$1 - zT = \prod_{n=1}^{\infty}(1-T^n)^{\beta_n(z)} \tag{5.10}$$

agree as formal power series, with the two properties:

(a) Infinitely many $\beta_n(z) \neq 0$ unless $z = 1$ or -1.

(b) The infinite product converges absolutely in the disk $\{T : |T| < \frac{1}{|z|}\}$.

Applying this identity to individual terms in the Euler product for $\zeta_{\mathbb{Q}}(z,s)$ yields

$$\zeta_{\mathbb{Q}}(z,s) = \prod_{n=1}^{\infty}\zeta(ns)^{\beta_n(z)}\ . \tag{5.11}$$

Thus,

$$F_z(s) = \frac{\zeta'_{\mathbb{Z}}(z,s)}{\zeta_{\mathbb{Z}}(z,s)} = \sum_{n=1}^{\infty} n\beta_n(z)\frac{\zeta'(ns)}{\zeta(ns)}\ . \tag{5.12}$$

To prove (i), for any $\epsilon > 0$, only finitely many terms in (5.11) have poles with $\mathrm{Re}(s) > \epsilon$, and the rest converges to an analytic function on $\mathrm{Re}(s) > \epsilon$.

To prove (ii), the function $F_z(s)$ has poles contained in $s = 1$ together with the set of zeros of $\zeta(ns)$ whenever $\beta_n(z) \neq 0$. If this set is infinite, then these points approach all points on the line $\mathrm{Re}(s) > 0$. Some care is needed to show that sufficiently many pole locations are actual poles to give the general result. Wintner actually shows it only for $z \in \mathbb{Z} \setminus \{-1,0,1\}$. For the three "special values" $z = -1, 0, 1$ we have $\zeta_{\mathbb{Z}}(-1, s) = \frac{\zeta(2s)}{\zeta(s)}$, $\zeta_{\mathbb{Z}}(0, s) = 0$ and $\zeta_{\mathbb{Z}}(1, s) = \zeta(s)$.

Finally, for (iii) we observe that all terms on the right side of (5.12) are analytic for $\text{Re}(s) \geq 1/2$ except the first term, which is $z\frac{\zeta'(s)}{\zeta(s)}$ since $\beta_1(z) = z$. □

The result (ii) above seems to be in parallel with the analytic continuation properties of the continued fraction zeta function $\zeta_{CF}(z, s)$ in Theorem 3.3.

Result (iii) implies that the Riemann hypothesis holds for $\zeta(s)$ if and only if for some fixed $z \neq 0$ the function

$$G_z(s) := (1 - z)^s \zeta(z, s)$$

is holomorphic for $\text{Re}(s) > 1/2$. Thus the two-variable zeta function $\zeta_{\mathbb{Z}}(z, s)$ encodes information on the Riemann hypothesis for variable z, but in an unenlightening way.

One can define another two-variable zeta function using a different notion of degree, suggested by the the arithmetic intersection theory of Arakelov [**3**], see also van der Geer and Schoof[**30**]. Here the prime ideal (p) in $\mathbb{Z}$ is assigned the degree $\log p$. The resulting two-variable zeta function is

$$\tilde{\zeta}_{\mathbb{Z}}(z, s) := \prod_p (1 - z^{\log p} p^{-s})^{-1} \ , \tag{5.13}$$

where we take

$$z^{\log p} := exp(\log p \log z). \tag{5.14}$$

In this case we formally obtain the identity

$$\tilde{\zeta}_{\mathbb{Z}}(z, s) = \tilde{\zeta}_{\mathbb{Z}}(ze^{-s}, 0), \tag{5.15}$$

which is analogous to the function field case.

Finally, in connection with statistical mechanics models, we note that Knauf ([**42**]-[**45**]) has recently constructed a collection of finite lattice gas models whose partition functions $Z_n(s)$ approach the function $\frac{\zeta(s-1)}{\zeta(s)}$ in the thermodynamic limit (for $Re(s) > 2$). (See also Contucci and Knauf [**17**] [**18**].) One can formally construct from Knauf's models a two-variable zeta function according to the prescription (4.22); its properties have not been investigated.

6. Concluding Remarks

In this paper we considered two-variable zeta functions. We observed that function field zeta functions can be developed in terms of either an arithmetic variable s or of a geometric variable z, and that both of these variables can be made simultaneously present in a two-variable zeta function. This two-variable zeta function is analogous to Ruelle's two-variable dynamical zeta function of an expanding map. There is a functional identity relating the two variables so that each can be expressed in terms of the other: $z = q^{-s}$. In this way, questions about the "Riemann hypothesis" in the s-variable can be transferred to questions of the z-variable. The z-variable has an interpretation in terms of "spectral geometry"; the behavior of the zeta function in the z-variable can be related to cohomology of the underlying variety. No such interpretation is currently known for the s-variable. It remains an open problem to construct a statistical mechanics model associated to a function field K which would give the two-variable zeta function $\zeta_K(z, s)$ as its associated zeta function.

For the number field case, in §5 we presented two different notions of a two-variable zeta function that extend the Riemann zeta function in a way analogous to the function field case. The naive version preserves the "Riemann hypothesis" but offers no substantial new insight into the truth of the Riemann hypothesis. The version using Arakelov's notion of degree preserves the function field relation (1.2). A dynamical system interpretation is lacking for either of these two-variable zeta functions.

Deninger [**19**] [**20**] has recently outlined an approach which attempts in the number field case to associate a (conjectural) cohomology theory to the s-variable in number field zeta functions, with an associated dynamical system. If this can be done, it would seem to assign a geometric meaning directly to the s-variable. One may also consider the suggestive evidence for a spectral interpretation of zeta zeros in Katz and Sarnak [**41**]. Thus it may be that in the number field case it is not appropriate to introduce an auxiliary "scaling variable" z.

We conclude with various speculations. One may view the integers $\mathbb{Z}$ as a dynamical system with an $\mathbb{N}^\infty$-action is given by a semigroup of commuting endomorphisms $\{\mathcal{M}_a : a \in \mathbb{Z}\}$, which are defined by

$$\mathcal{M}_a(n) = an, \quad n \in \mathbb{Z} . \tag{6.1}$$

This semigroup is generated by $\{\mathcal{M}_p\}$ for p prime, together with the torsion element $\mathcal{M}_{-1}$. This suggests that one must add a "geometric variable" z_p separately for each prime, and consider the zeta function

$$\zeta(s, \mathbf{z}) = \prod_p (1 - z_p p^{-s})^{-1} , \tag{6.2}$$

which has an infinite number of variables. Some interesting information can be gained from this viewpoint, see Hedenmalm, Lindqvist and Seip [**35**].

We also note that each multiplication operator $\mathcal{M}_p$ acts as an expanding map on $\mathbb{Z}$ with entropy $\log p$. These different expansion rates suggest that any underlying dynamical system associated to $\mathbb{Z}$ to which a hypothetical two-variable zeta function is attached will not be homogeneously expanding. This analogy seems an obstacle to a direct "spectral geometry" interpretation of the Riemann hypothesis. On the other hand, if we extend $\mathbb{Z}$ to obtain a minimal space $\mathbb{A}$ on which the natural extensions $\mathcal{M}_a$ of all the maps $\{\mathcal{M}_a : a \in \mathbb{Z}\}$ act as automorphisms, then $\mathbb{A}$ is a space of adeles. In this connection there is a recent interpretation of the Riemann hypothesis by Connes [**16**] via a trace formula on adeles. See also Bost and Connes [**11**] and Goldfeld [**32**].

Finally we note that the functional equation for the projective zeta function of an algebraic curve requires a global "normalizing factor" $q^{-(1-g)s}$ in (2.14) to obtain a symmetric functional equation (2.13). For the Riemann zeta function the term $\pi^{-s/2}$ seems to play a similar role in the functional equation. Weil [**96**, p. 238] suggests a definition of "genus" for a number field K; for $K = \mathbb{Q}$ his formula assigns genus $g_{\mathbb{Q}} = 0$. With this interpretation of g, the factor $\pi^{1/2}$ plays a role analogous to q. Since log q is the entropy assigned to the Frobenius operator in §5, this would suggest the possibility of viewing $\frac{1}{2}\log\pi$ as the "global entropy" of an hypothetical dynamical system attached to $\zeta(s)$. On the other hand van der Geer and Schoof [**30**, Proposition 1] recently formulated a Riemann-Roch formula for number fields using Arakelov divisors; in this case *log* $|d_K|$ plays the role of $2g_K - 2$, so that $\mathbb{Q}$ cab be assigned a genus $g_{\mathbb{Q}} = 1$. (This notion of genus is however not the same as the one given in [**30**].) With this interpretation the ‘"global entropy" of an hypothetical dynamical system associated to $\zeta(s)$ must be infinite, and the value $\frac{1}{2}\log\pi$ might then represent a "renormalized" entropy for a quotient dynamical system.

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