Pinwheel Scheduling: Achievable Densities

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Abstract. A *pinwheel schedule* for a vector $v = (v_1, v_2, \ldots, v_n)$ of positive integers $2 \le v_1 \le v_2 \le \cdots \le v_n$ is an infinite symbol sequence $\{S_j : j \in \mathbb{Z}\}$ with each symbol drawn from $[n] = \{1, 2, \ldots, n\}$ such that each $i \in [n]$ occurs at least once in every v_i consecutive terms $(S_{j+1}, S_{j+2}, \ldots, S_{j+v_i})$. The *density* of v is $d(v) = \frac{1}{v_1} + \frac{1}{v_2} + \cdots + \frac{1}{v_n}$. If v has a pinwheel schedule, it is *schedulable*. It is known that $v(2, 3, m)$ with $m \ge 6$ and density $d(v) = \frac{5}{6} + \frac{1}{m}$ is unschedulable, and Chan and Chin [2] conjecture that every v with $d(v) \le \frac{5}{6}$ is schedulable. They prove also that every v with $d(v) \le \frac{7}{10}$ is schedulable.

We show that every v with $d(v) \le \frac{3}{4}$ is schedulable, and that every v with $v_1 = 2$ and $d(v) \le \frac{5}{6}$ is schedulable. The paper also considers the *m-pinwheel scheduling problem* for v, where each $i \in [n]$ is to occur at least m times in every mv_i consecutive terms $(S_{j+1}, \ldots, S_{j+mv_i})$, and shows that there are unschedulable vectors with $d(v) = 1 - \frac{1}{(m+1)(m+2)} + \epsilon$ for any $\epsilon > 0$.

Key Words: Pinwheel, Scheduling, Density Guarantee, Packing.

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1. Introduction. Pinwheel scheduling originated with the problem of scheduling a ground station to receive information from a number of satellites without data loss [5]. A pinwheel schedule provides a window of time for receipt of each satellite's information that depends on its orbital characteristics. Like many other scheduling problems that require more-or-less periodic processing of items by one or more service facilities [1, 4, 7, 8, 10, 12], contributions to pinwheel scheduling [2, 3, 6, 9, 11] have focused on algorithms which schedule significant numbers of cases that have feasible schedules. This paper provides guarantees for the existence of pinwheel schedules for an increased range of cases, measured in terms of their densities, and implicitly gives a method to construct such schedules.

The pinwheel scheduling problem is usually formulated for a multiset $\{v_1, \ldots, v_n\}$ of positive integers. (A multiset is a set in which elements occur with multiplicities.) We will represent such a multiset as a vector $v = (v_1, v_2, \ldots, v_n)$. The *density* $d(v)$ of vector v is

$$d(v) = \frac{1}{v_1} + \frac{1}{v_2} + \cdots + \frac{1}{v_n} .$$

A *pinwheel schedule* for v is a doubly-infinite symbol sequence $\{S_j : j \in \mathbb{Z}\}$ of symbols drawn from the alphabet $[n] = \{1, 2, \ldots, n\}$ with the property that each $i \in [n]$ occurs at least once in every block $(S_{j+1}, \ldots, S_{j+v_i})$ of v_i consecutive terms of the sequence. When a pinwheel schedule exists for v, we say that $v = (v_1, v_2, \ldots, v_n)$ is *schedulable.* Thus, $(v_1, v_2) = (2, 3)$ is schedulable with sequence $\cdots 1212 \cdots$. However, it turns out that every $(v_1, v_2, v_3) = (2, 3, v_3)$ is unschedulable.

By permuting symbols we can always reduce to the case in which v's components are ordered as $v_1 \leq v_2 \leq \cdots \leq v_n$. Moreover, a trivial necessary condition for schedulability is $d(v) \leq 1$. Thus, aside from $v = (1)$, we require $v_1 \geq 2$ and will assume henceforth that $2 \leq v_1 \leq v_2 \leq \cdots \leq v_n$.

Although constructing a pinwheel schedule is ostensibly an infinite problem, Theorem 2.1 in [5] notes that if a vector v is schedulable, then there exists a periodic schedule whose period is no greater than $v_1 v_2 \cdots v_n$. Consequently, one can decide if V is schedulable by an exponential-time algorithm that checks whether some sequence of $v_1 v_2 \cdots v_n$ terms from $[n]$ satisfies the pinwheel conditions. Such a decision algorithm is impractical, and considerable effort has been devoted to finding classes of problems for which simple schedules can be devised.

We note results of this effort in terms of subsets of the set

$$V = \{v = (v_1, v_2, \ldots, v_n) : n \geq 1,\ 2 \leq v_1 \leq \cdots \leq v_n,\ d(v) \leq 1\} .$$

We say that b is a *density guarantee* for a nonempty subset U of V if every $v \in U$ with $d(v) \leq b$ is schedulable. Holte et al. [6] proves that 1 is a density guarantee for $\{(v_1, \ldots, v_n) \in V : |\{v_i : i \in [n]\}| \leq 2\}$, the set of all v with no more than two distinct v_i values, and Lin and Lin [9] prove that $\frac{5}{6}$ is the maximum density guarantee for $\{(v_1, \ldots, v_n) \in V : |\{v_i : i \in [n]\}| \leq 3\}$. Of greater relevance for the present paper, Chan and Chin [2] proves that $\frac{7}{10}$ is a density guarantee for V. This paper and Chan and Chin [3] present an array of pinwheel scheduling algorithms with varying density guarantees, including 0.65 and $\frac{2}{3}$, but $\frac{7}{10}$ is the maximum guarantee for their algorithms. Table I in [2] also notes the following density guarantees for all $v \in V$ with smallest member v_1:

v_1	guarantee
2	0.75000
3	0.70000
4	0.70833
5	0.72196
6	0.73359

v_1	guarantee
7	0.73807
8	0.74470
9	0.75188
10	0.76134
11	0.76438

.

A few of these values, and the observation in [2] that every $v \in V$ with $v_1 \geq 9$ and $d(v) \leq \frac{3}{4}$ is schedulable, are relied on in the proof of our main result.

THEOREM **1.** *$\frac{3}{4}$ is a density guarantee for V.*

Because all $(2, 3, v_3) \in V$ are unschedulable, no $b > \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ is a density guarantee for V. Chan and Chin [2, 3] conjecture that $\frac{5}{6}$ is the maximum density guarantee for V. This is supported by all evidence to date, including the three-values theorem of Lin and Lin [9], and it leaves ample room to increase the $\frac{3}{4}$ bound of Theorem 1. The next theorem verifies the $\frac{5}{6}$ conjecture for one important special case.

THEOREM **2.** *$\frac{5}{6}$ is the maximum density guarantee for $\{v \in V : v_1 = 2\}$.*

Theorem 2 is proved in section 3 under the presumption that Theorem 1 is true. Theorem 1 is proved in section 4.

A natural step beyond Theorem 2 is to show that $\{v \in V : v_1 = 3\}$ has density guarantee $\frac{5}{6}$, which would be maximum in view of $(3, 4, 4)$. This seems hard to prove and we leave it open. A type of difficulty encountered is illustrated by $v = (3, 4, 7, 10, 140)$ with $d(v) = \frac{5}{6}$. It is easy to schedule $(3, 4, 7, 10)$, and even a greedy algorithm does this with schedule $\ldots 132142132142 \ldots$. But it is not at all obvious how one schedules v. One solution is

$$\ldots|1231421312413215|1231421312413215|\ldots ,$$

which in fact is a schedule for $(3,4,6,10,15)$. However, this is a custom-designed schedule, and many other cases of $v_1 = 3$ and $d(v) \leq \frac{5}{6}$ may require similar treatment.

We include another result motivated by the $\frac{5}{6}$ conjecture for a relaxed version of the pinwheel scheduling problem. Define $(v_1, v_2, \ldots, v_n)$ in V to be *m-schedulable* if there is a doubly-infinite sequence of members of $[n]$ in which every mv_i contiguous terms contain at least m i's, for $i = 1, 2, \ldots, n$. When every $v \in V$ with $d(v) \leq b$ is m-schedulable, we say that b is a *density m-guarantee* for V.

THEOREM **3.** *For every $m \geq 2$, no $b > 1 - 1/[(m+1)(m+2)]$ is a density m-guarantee for V.*

When $m = 1$, $1 - 1/[(m+1)(m+2)] = \frac{5}{6}$. We conjecture that $1 - 1/[(m+1)(m+2)]$ is the maximum density m-guarantee for V for every $m \geq 2$. Theorem 3 is proved in section 5.

2. Formulation and Lemmas. This section presents a slightly modified formulation for pinwheel scheduling in preparation for the proofs of Theorems 1 and 2. We assume throughout that v with density $d(v)$ is in V. We include some illustrative material not required for later proofs to facilitate understanding.

A *packing* of $v = (v_1, v_2, \ldots, v_n)$ is a map $f : \mathbb{Z} \to \{0\} \cup [n]$ such that, for all $i \in [n]$ and all $k \in \mathbb{Z}$,

$$i \in \{f(k), f(k+1), \ldots, f(k+v_i-1)\} \ .$$

When this holds, we say that f *packs* v and that v is *packable*. If v has no packing, it is *unpackable*.

The definition of a packing modifies our prior definition of a pinwheel schedule by allowing zeros in the sequence. We refer to $k \in \mathbb{Z}$ for which $f(k) = 0$ as a *hole* in f. A packing with no holes is *tight*. All packings of $v = (2,3)$, such as $\ldots 1212 \ldots$ and $\ldots 1211212112 \ldots$, are necessarily tight, but the packing $\ldots 12101210 \ldots$ of $v = (2,4)$ has a hole in every fourth position. Every nontight packing can be made tight by deleting 0's and closing up. A schedule is a tight packing, so v is schedulable if and only if it is packable.

An *extension* of $v = (v_1, \ldots, v_k)$ is any $(v_1, \ldots, v_k, \ldots, v_n)$ in V with $n > k$. A packable v is *extendable* if it has a packable extension, and *nonextendable* otherwise. Nonextendable packable v's include $(2,3)$, $(2,5,7)$ and $(3,3,5)$.

LEMMA **1.** *A packable v is extendable if and only if it has a packing with denumerably many holes.*

PROOF. If f is a packing of the extension $(v_1, \ldots, v_n, v_{n+1})$ of $v = (v_1, \ldots, v_n)$, set $f(k) = 0$ for all k that have $f(k) = n + 1$ to get a packing of v with denumerably many holes.

Suppose f is a packing of $v = (v_1, \ldots, v_n)$ that has a denumerable number of holes. Assume without loss of generality that denumerably many $k \geq 0$ are holes. Then some v_n-term sequence in $\{0, 1, \ldots, n\}^{v_n}$, say S, must occur infinitely often as a contiguous subsequence of $f(0)f(1)f(2)\ldots$. A denumerable number of occurrences of S will be mutually nonoverlapping, and when any two of these with intermediate subsequence T is used to form the sequence $\ldots STSTST\ldots$, this sequence gives another packing of v. If S itself has no hole, we can always choose T with a hole, so we get a packing of v that has a hole in every j^{th} position for some $j \geq v_n$. When the 0 in each such position is changed to j, we obtain a packing of $(v_1, \ldots, v_n, j)$. ∎

The following example illustrates preceding ideas in ways that are used repeatedly in subsequent analyses.

EXAMPLE 1. To show that $v = (3, 3, 5)$ is nonextendable, we suppose otherwise and obtain a contradiction by trying to construct a packing outward from a hole:

$\ldots 101 \ldots$	2 won't pack
$\ldots 21021 \ldots$	3 won't pack
$\ldots 103 \ldots$	2 won't pack
$\ldots 303 \ldots$	1 and 2 won't pack.

The same procedure often verifies extendability. The pattern

$$\ldots |12314210|12314210| \ldots$$

shows that $(3, 4, 8, 8)$, with packing $\ldots S0S0S0\ldots$, where $S = 1231421$, can be extended to $v = (3, 4, 8, 8, 8)$ with $d(v) = 0.95833$. The two-hole pattern

$$\ldots |314256130241653213451260|314256 \ldots$$

shows that $(5, 6, 7, 8, 9, 10)$ can be extended to $(5, 6, 7, 8, 9, 10, 15)$ with density 0.9123015. ∎

Let

$$P = \{v : \ v \text{ is packable}\}$$

and for every $v \in P$ let

$$P(v) = \{v' \in P : \ v' \text{ is an extension of } v\}.$$

Every member of $P(v)$ is a *packable extension* of v. If every packing of v is tight then $P(v)$ is empty and v has no packable extensions.

As in the introduction, we refer to $b \in (0,1]$ as a *density guarantee* for V when $\{v : d(v) \leq b\} \subseteq P$. A similar definition applies to subsets of V. Given $v \in P$ and $d(v) \leq b \leq 1$, we also say that b is a *guarantee for* v if every extension v' of v for which $d(v') \leq b$ is in $P(v)$. We noted in the introduction that $\frac{3}{4}$ is a guarantee for $v = (2)$, 0.70 is a guarantee for $v = (3), \ldots,$ and 0.76438 is a guarantee for $v = (11)$. A guarantee $b < 1$ for v is *maximum* if for every $b' > b$ there is an unpackable extension v' of v with $d(v') \leq b'$. In these terms, Theorem 2 says that $\frac{5}{6}$ is the maximum guarantee for $v = (2)$.

We now describe a method for increasing guarantees and establishing new guarantees. A packing f of v will be said to be *porous* if every c consecutive positions of f have a hole for some positive integer c. When f is porous, let

$$p(f) = 1 + (\text{maximum number of positions between adjacent holes in } f)\,.$$

The two-hole pattern at the end of Example 1 has $p(f) = 1 + 14 = 15$.

For notational clarity, we often identify a v under consideration for extensions by $v^* = (v_1^*, \ldots, v_k^*)$. When f is a porous packing of v^* with $p(f)$ as just defined, every one-term extension $(v_1^*, \ldots, v_k^*, v_{k+1})$ of v^* for which $v_{k+1} \geq \max\{v_k^*, p(f)\}$ is packable, i.e., in $P(v^*)$. The following lemma generalizes this observation. Given f for v^* and an extension $(v_1^*, \ldots, v_k^*, \ldots, v_n)$ of v^*, for each $i \in \{k+1, \ldots, n\}$ let

$$h_i = \text{minimum number of holes in } v_i \text{ consecutive positions of } f\,.$$

LEMMA **2.** *Suppose f is a porous packing of $v^* = (v_1^*, \ldots, v_k^*)$, $n > k$, and $\max\{v_k^*, p(f)\} \leq v_{k+1} \leq \cdots \leq v_n$. Let b be either a density guarantee for V, or a guarantee for some $(h_{k+1}, \ldots, h_m)$ with $k+1 \leq m \leq n$. Then $(v_1^*, \ldots, v_k^*, v_{k+1}, \ldots, v_n) \in P(v^*)$ if*

$$\sum_{i=k+1}^{n} \frac{1}{h_i} \leq b\,.$$

PROOF. The lemma's hypotheses and inequality for b imply that $(h_{k+1}, \ldots, h_n)$ in V is packable, say with packing $g : \mathbb{Z} \to \{0\} \cup \{k+1, \ldots, n\}$ with an obvious notational extension. Let β be an order-preserving map from the set of holes of f onto $\mathbb{Z}$. Let $f' = f$ except for the holes of f where $f'(k) = g(\beta(k))$. Then f' is a packing of $(v_1^*, \ldots, v_k^*, v_{k+1}, \ldots, v_n)$. ■

Our next lemma considers a number t as a guarantee for v^*. It may be helpful to think of t as a "target" for a proposed guarantee.

LEMMA **3.** *Suppose the hypotheses in the first two sentences of Lemma 2 hold, and t satisfies $d(v^*) < t \le 1$ along with*

$$d(v^*) + \sum_{i=k+1}^{n} \frac{1}{v_i} \le t \ .$$

Then $(v_1^*, \ldots, v_k^*, v_{k+1}, \ldots, v_n) \in P(v^*)$ if

$$v_i \le h_i \left[\frac{b}{t - d(v^*)} \right] \quad \textit{for} \quad i = k+1, \ldots, n \ .$$

PROOF. The final inequalities give

$$\sum_{i=k+1}^{n} \frac{1}{h_i} \le \left(\sum_{i=k+1}^{n} \frac{1}{v_i} \right) \frac{b}{t - d(v^*)} \le b \ ,$$

so, when they hold, Lemma 2 implies that $(v_1^*, \ldots, v_k^*, v_{k+1}, \ldots, v_n)$ is packable. ∎

Applications of Lemmas 2 and 3 put a premium on porous packings of v^* that have high densities of holes that are approximately evenly spaced. This occurs automatically with f packing $\ldots 010101 \ldots$ for $v^* = (2)$, but some other cases require effort to uncover porously efficacious packings.

The utility of Lemma 3 is demonstrated in the next section by the proof that $t = \frac{5}{6}$ is a guarantee for $v = (2)$, given that $b = \frac{3}{4}$ is a density guarantee for V. Lemma 3 can also be used for a straightforward proof of Theorem 1, but we will give a shorter proof of our main theorem by using an extended version of the lemma that effectively replaces b in Lemma 3 by t itself.

LEMMA **4.** *Fix t with $0.7 < t \le \frac{5}{6}$. Suppose t is a density guarantee for a proper subset U of V. Let $V(U,t) = \{v \in V \setminus U : d(v) \le t\}$, let V^* be a finite set of v^*'s in $V(U,t)$ such that every $v \in V(U,t) \setminus V^*$ is an extension of some $v^* \in V^*$, and let F be a finite set of periodic packings of the members of V^* such that every $v^* \in V^*$ is packed by at least one $f \in F$. Suppose also that there is a finite set T of triples $(v^*, u, f) \in V^* \times \mathbb{Z} \times F$ such that*

(i) $u \ge \max\{v_k^*, p(f)\}$ *when* $v^* = (v_1^*, \ldots, v_k^*)$,

(ii) $(v^*, u) \in V(U,t)$,

(iii) f *packs* v^*,

(iv) with $h(v_i)$ the minimum number of holes in v_i consecutive positions of f,

$$v_i < h(v_i) \left[\frac{b^*}{t - d(v^*)}\right] \quad \text{for all} \quad v_i \geq u \ ,$$

where b^* is the larger of t and a known guarantee for $(h(u))$,

and for every extension $v = (v_1^*, \ldots, v_k^*, v_{k+1}, \ldots, v_n)$ in $V(U,t) \setminus V^*$ of a $v^* = (v_1^*, \ldots, v_k^*)$ in V^* there is a triple $(v^*, u, f) \in T$ for which $v_{k+1} \geq u$. Then t is a density guarantee for V.

PROOF. Fix t with $0.7 < t \leq \frac{5}{6}$, and assume that *all* other hypotheses of Lemma 4 hold. Because t is a density guarantee for U and some $f \in F$ packs v^* for each $v^* \in V^*$, the only members of V that require further consideration to assure t as a density guarantee for V are the extensions in $V(U,t) \setminus V^*$ of members of V^*. By the penultimate sentence of the lemma, every such extension is related to a triple (v^*, u, f) in $V^* \times \mathbb{Z} \times F$ that satisfies (i) through (iv).

Consider one such $(v^*, u, f) \in T$ which satisfies (i)–(iv) and necessarily has $h(u) > 0$. Let $B = b^*/[t - d(v^*)]$ in (iv), so that (iv) is $v_i < h(v_i)B$ for all $v_i \geq u$. Also let p be the number of terms in f's period, and for $1 \leq j \leq p$ let α_j be the minimum number of holes in j consecutive terms of f. Thus every p consecutive terms of f has exactly α_p holes. Moreover, $p\alpha_j \leq \alpha_p j$ for all $1 \leq j < p$, for if $p\alpha_j > \alpha_p j$ then we conclude that some block of jp consecutive terms has more than $j\alpha_p$ holes. It follows that if $v_i = Kp + j$ with $0 \leq j < p$, then $h(v_i) = K\alpha_p + \alpha_j$, and with $h(v_i) > 0$ that

$$\frac{v_i + p}{h(v_i + p)} = \frac{(K+1)p + j}{(K+1)\alpha_p + \alpha_j} \leq \frac{Kp + j}{K\alpha_p + \alpha_j} = \frac{v_i}{h(v_i)} \ ,$$

where the inequality is tantamount to $p\alpha_j \leq \alpha_p j$. As a consequence, we see that the maximum value of $v_i/h(v_i)$ must occur in the first p $v_i \geq u$. Because the inequality of (iv) is strict, i.e., $v_i/h(v_i) < B$, it follows that we can decrease B slightly by subtracting some $\epsilon > 0$ from its numerator without violating (iv). To be specific, let $\epsilon > 0$ be such that

$$v_i < h(v_i) \left[\frac{b^* - \epsilon}{t - d(v^*)}\right] \quad \text{for all} \quad v_i \geq u \ .$$

It now follows from Lemmas 2 and 3 that if $t - \epsilon$ is a density guarantee for V and if v is an extension of v^* as described just after (iv) in Lemma 4, then v is a packable extension of v^*.

A similar analysis for every $(v^*, u, f) \in T$ yields a finite collection E of positive ϵ's that give the same packability conclusion for each triple in T. Therefore, if $0 < \epsilon_0 \leq \min\{\epsilon : \epsilon \in E\}$, uniform replacement of b^* by $b^* - \epsilon_0$ in (iv) implies that if $t - \epsilon_0$ is a density guarantee for V then t also is a density guarantee for V.

Assume without loss of generality that $M\epsilon_0 = t - 0.7$ for a positive integer M. The proof of Lemma 4 is completed by observing that, beginning with 0.7 as a known density guarantee for V, the hypotheses of the lemma imply inductively that $0.7 + \epsilon_0$ is a density guarantee for V, $0.7 + 2\epsilon_0$ is a density guarantee for $V, \ldots$, and $0.7 + M\epsilon_0 = t$ is a density guarantee for V. The final step, from $0.7 + (M-1)\epsilon_0 = t - \epsilon_0$ to $0.7 + M\epsilon_0 = t$, is validated in the preceding paragraphs. The earlier steps are covered by the preceding analysis because we can assume that their U's include the U of the final step, their V^*'s are included in the V^* of the final step, and so long as $t - K\epsilon_0 - d(v^*) > 0$, the ratio

$$\frac{b^* - (K+1)\epsilon_0}{t - K\epsilon_0 - d(v^*)}$$

increases as K increases, thus assuring that (iv) holds for earlier steps when it holds for the final step. ■

It seems obvious that as t increases toward $\frac{5}{6}$, verification of t as a density guarantee for V increases in difficulty. This is reflected by the statement and proof of Lemma 4. Although our proof in section 4 for $t = \frac{3}{4}$ that is based on Lemma 4 is rather long, it is certainly possible to use Lemma 4 to verify larger density guarantees for V. However, proof length and complexity strongly encourage the development of other ideas to determine the validity of Chan and Chin's conjecture [2, 3] that $\frac{5}{6}$ is the maximum density guarantee for V.

3. Proof of Theorem 2, Assuming Theorem 1. We prove that $\frac{5}{6}$ is a guarantee for $v = (2)$, hence a maximum guarantee for (2) in view of $v = (2, 3)$, given that $\frac{3}{4}$ is a density guarantee for V.

Assume with no loss of generality that $v = (2, v_2, \ldots, v_n) \in V$ with $n \geq 2$, $v_2 \geq 4$ and

$$\frac{1}{2} + \sum_{i=2}^{n} \frac{1}{v_i} \leq \frac{5}{6} \; .$$

If v is packable then any packing with adjacent 1's can be converted to a packing with no adjacent 1's by removing some 1's and closing up. We therefore assume that all 1's in a potential packing of v are separated by other terms. Then, because every even $v_i \geq 4$ can not be packed more closely than odd $v_i + 1$, and $1/(v_i + 1) < 1/v_i$, we assume also that all v_i for $i \geq 2$ are odd. We apply Lemma 3 with $t = \frac{5}{6}$.

Suppose $v^* = (2)$ with $d(v^*) = \frac{1}{2}$. Then, by the definition of h_i prior to Lemma 2, $h_i = \lfloor v_i/2 \rfloor$ for all $i \geq 2$. In addition, $t - d(v^*) = \frac{1}{3}$ and, with $b = \frac{3}{4}$ as a density guarantee for

V, the final inequalities of Lemma 3 are

$$v_i \le (2.25)\left\lfloor \frac{v_i}{2} \right\rfloor \quad \text{for} \quad i = 2, \ldots, n \; .$$

This holds for all $v_i \ge 9$. Hence, by Lemma 3, v is packable if $v_2 \ge 9$. In other words, every extension of v^* with $v_2 \ge 9$ and $\frac{1}{2} + \sum_2^n 1/v_i \le \frac{5}{6}$ is packable, so $\frac{5}{6}$ is a guarantee for $(2,9)$, $(2,11), \ldots$. Only $v_2 = 5$ and $v_2 = 7$ require further consideration.

Suppose $v_2 = 5$. Take $v^* = (2,5)$ with $t - d(v^*) = 5/6 - (1/2 + 1/5) = 2/15$. With $v = (2,5,v_3,\ldots,v_n)$, $\sum_1^n 1/v_i \ge 5/6$ requires $v_3 \ge 15/2$, hence $v_3 \ge 9$. For the packing

$$\ldots 121012101210 \ldots \quad \text{of} \quad v^* \; ,$$

we have $h_i = 2$ for $v_i \in \{9, 11\}$, $h_i = 3$ for $v_i \in \{13, 15\}$, and so forth. Again using $b = \frac{3}{4}$ in Lemma 3, its final inequalities are

$$v_i \le \left(\frac{3/4}{2/15}\right) h_i = (5.625) h_i \quad \text{for} \quad i = 3, \ldots, n \; .$$

This holds for all $v_i \ge 9$ and it follows that $\frac{5}{6}$ is a guarantee for $(2,5)$.

Suppose $v_2 = 7$. Take $v^* = (2,7)$ with $t - d(v^*) = 4/21 = 0.1904761$. The v^* packing

$$\ldots 121010121010121010 \ldots$$

gives the following values of h_i and $h_i b/[t - d(v^*)] = (63/16)h_i$:

v_i	h_i	$(63/16)h_i$
$7, 9$	2	7.88
11	3	11.82
$13, 15$	4	15.75
17	5	19.69
$19, 21$	6	23.63

and so forth. Hence the final inequality of Lemma 3 holds whenever $v_i \ge 11$, so every $(2,7,v_3)$ with $v_3 \ge 11$ has guarantee $\frac{5}{6}$. It remains only to consider $(2,7)$ with $v_3 \in \{7, 9\}$.

For $v^* = (2,7,7)$ and $v^* = (2,7,9)$, with

$$\frac{b}{t - d(v^*)} = \frac{3/4}{5/6 - (1/2 + 1/7 + 1/9)} = 9.45 \quad \text{for} \quad v^* = (2,7,9) \; ,$$

the packing $\ldots 121310121310 \ldots$ of $(2,7,7)$ and also of $(2,7,9)$ gives

v_i	h_i	$9.45h_i$
$13 - 17$	2	18.90
$19 - 23$	3	28.35
$25 - 29$	4	37.80

.

Because $v^* = (2,7,9)$ requires $v_4 \geq 13$ to satisfy $\sum_1^n 1/v_i \leq 5/6$, we conclude that all extensions of $(2,7,9)$ with $\sum 1/v_i \leq 5/6$ are packable. Because $v^* = (2,7,7)$ has a larger coefficient than 9.45 in the preceding table (and requires $v_4 \geq 21$), all extensions of $(2,7,7)$ with $\sum 1/v_i \leq 5/6$ are packable. It follows that $(2,7)$ has guarantee $\frac{5}{6}$, and the proof is complete.

4. Proof of Theorem 1. We use Lemma 4 with $t = \frac{3}{4}$ to prove Theorem 1. Because all $v \in V$ with $d(v) \leq \frac{3}{4}$ and $v_1 \in \{2, 9, 10, 11, \ldots\}$ are packable, U is the set of all v with $v_1 = 2$ or $v_1 \geq 9$. Then

$$V(U,t) = \left\{ v \in V : 3 \leq v_1 \leq 8 \quad \text{and} \quad d(v) \leq \frac{3}{4} \right\} .$$

The finite sets $V^* \subseteq V(U,t)$, F and T will emerge during the course of the proof. Other notations and assumptions follow.

1. The strict inequality in (iv) of Lemma 4 is the *key inequality*. When it holds for a particular $v^* = (v_1^*, \ldots, v_k^*)$ and u, we say that v^* *packs to* $3/4$ *when* $v_{k+1} \geq u$. A collection of such statement which imply that all extensions of v^* in $V(U, \frac{3}{4})$ pack to $3/4$ is abbreviated by v^* *packs to* $3/4$.

2. Exactly five values of b^* are used in the key inequality:

(a) $b^* = \frac{3}{4}$ unless noted otherwise;

(b) when $h(u) = 2$, we sometimes take $b^* = 0.78$. This is justified by Lemma 3 because its final inequality is $v_i \leq (0.7/[0.78 - \frac{1}{2}])\lfloor v_i/2 \rfloor = 2.5\lfloor v_i/2 \rfloor$ when $v^* = (2)$, $t = 0.78$ and $b = 0.7$, and $v_i \leq 2.5\lfloor v_i/2 \rfloor$ holds for all $v_i \geq 4$;

(c) when $h(u)$ is 9, 10, or 11, we sometimes use $b^* = 0.75188$, 0.76134, or 0.76438, respectively [2].

3. $B = b^*/[3/4 - d(v^*)]$ for the key inequality. Tabular arrays similar to those for $(v_i, h_i, h_i b/[t - d(v^*)])$ in the proof of Theorem 2 will be headed (u, h, Bh) with $h(v_i) = h(u)$ when $v_i = u$. They are truncated when it is evident that the key inequality holds for larger values of u or v_i.

4. The arrow notation, such as $(5,6,6) \to (3,5)$, signifies that $(5,6,6)$ can be packed like $(3,5)$ and, because $(3,5)$ has already been shown in the proof to pack to $3/4$, $(5,6,6)$ does likewise and requires no further consideration.

5. Each packing $f \in F$ is identified by its periodic repeating pattern. Thus 120130 stands for $\ldots 120130120130 \ldots$.

The following lemma completes a first step in the application of Lemma 4 by noting the one-term extensions of (3) through (8) which require further processing to show that they pack to $3/4$. We then branch on each (v_1) for $3 \le v_1 \le 8$ to complete its proof.

LEMMA **5.** *All* (v_1, v_2) *pack to* $3/4$ *with the possible exception of* $(3,4)$, $(4,5)$ *to* $(4,9)$, $(5,5)$ *to* $(5,11)$, $(6,6)$ *to* $(6,11)$, $(7,7)$ *to* $(7,11)$, *and* $(8,8)$ *to* $(8,11)$.

PROOF. We omit $(4,4)$ from the exceptions list because $(4,4) \to (2)$. We retain $(6,6)$ and $(8,8)$ because (3) and (4) have not yet been shown to pack to $3/4$.

We use the f packings 100, 1000, ..., 10000000 for v^* equal to (3), (4), ..., (8), respectively. When $v^* = (x)$, $h(v_i) = \lfloor v_i(x-1)/x \rfloor$. The computations for the key inequality for $v^* = (3)$, with $B = (3/4)/(3/4 - 1/3) = 9/5$ are

u	h	$(9/5)h$
$3,4$	2	3.6
5	3	5.4
$6,7$	4	7.2
8	5	9.0
$9,10$	6	10.8

If $v_i = 4$ is not used in an extension of (3), then the extension packs to $3/4$, and because $d(3,3,4) > \frac{3}{4}$, we conclude that $(3,3)$ packs to $3/4$. (This is a slight extension of Lemma 4 that follows from the analysis of Lemma 3.) It follows that the only one-term extension of (3) that requires further consideration is $(3,4)$.

The proof of Lemma 5 for $v_1 \in \{4, \ldots, 8\}$ amounts to observing that

$$v_i < \left\lfloor \frac{v_i(v_1 - 1)}{v_1} \right\rfloor \frac{b^*(v_1)}{3/4 - 1/v_1} \quad \text{for all} \quad v_i \ge c(v_1) \ ,$$

where $(b(4), c(4)) = (0.75, 10)$, $(b(5), c(5)) = (0.75188, 12)$, $(b(6), c(6)) = (b(7), c(7)) = (b(8), c(8)) = (0.76134, 12)$. In each case, $c(v_1)$ is the smallest value of u for which the key inequality holds for all $v_i \ge u$, and $b^*(v_1)$ is either $\frac{3}{4}$ or the corresponding guarantee for $h(c(v_1))$, whichever is larger. For example, when $v_1 = 6$, the smallest u for which the key inequality holds for all $v_i \ge u$ is $u = 12$. The value of $h(12)$ for $v_1 = 6$ is $\lfloor 12(5/6) \rfloor = 10$, and the guarantee from [2] for (10) is 0.76134. It is routine to check that the displayed inequality holds for the cited values, so all of $(4,10)$, $(4,11), \ldots, (5,12)$, $(5,13), \ldots, (8,12), (8,13), \ldots$ pack to $3/4$. ■

The proof of Theorem 1 will be completed by showing that each excepted pair of Lemma 5 packs to $3/4$. The repeating patterns of all porous packings used for this are shown in Table

1. Each row gives the packing number, the v^*'s to which it applies, the packing pattern and its period length. We process each excepted pair in turn and complete its branching analysis before going on to the next pair.

Table 1 about here

(3,4). $t - d(3,4) = 3/4 - (1/3 + 1/4) = 1/6$, so $B = (3/4)/(1/6) = 4.5$. Packing (3.1) of Table 1 has $h(v_i) = \lfloor v_i/3 \rfloor$, so the key inequality is $v_i < (4.5)\lfloor v_i/3 \rfloor$. This holds for all $v_i \geq 6$. Because $d(3,4,v_3,\ldots) \leq \frac{3}{4} \Rightarrow v_3 \geq 6$, it follows that (3,4) packs to 3/4.

This completes the proof that $v^* = (3)$ packs to 3/4.

(4,5). $t - d(4,5) = 0.3$, so $B = (3/4)/(0.3) = 2.5$. Packing (4.1) has $h(v_i) = \lfloor v_i/2 \rfloor$, so the key inequality is $v_i < (2.5)\lfloor v_i/2 \rfloor$. This holds for all $v_i \geq 5$, except that equality holds when $v_i = 5$. However, when $v_3 = 5$, $h(v_3) = 2$, so we can use $b^* = 0.78$ to get strict inequality for this case. It follows that (4,5) packs to 3/4.

(4,6). The key inequality for $v^* = (4,6)$ and $b^* = \frac{3}{4}$ is $v_i < (2.25)h(v_i)$. This holds for all $v_i \geq 10$ with packing (4.2), where $h(10) = 5$, $h(11) = 6$, $h(12) = h(13) = h(14) = 7$, and so forth. Hence (4,6) packs to 3/4 if $v_3 \geq 10$. When $v_3 = 6$, $(4,6,6) \to (3,4)$. When $v^* = (4,6,7)$, packing (4.3) has $h(v_i) = \lfloor v_i/3 \rfloor$. Then, when $v_4 \leq 8$ with $h(v_4) = 2$, we use $b^* = 0.78$ for the key inequality $v_i < (4.095)\lfloor v_i/3 \rfloor$, which holds for all $v_i \geq 7$. And when $v_4 \geq 9$, the key inequality with $b^* = \frac{3}{4}$ is $v_i < (3.9375)\lfloor v_i/3 \rfloor$, which holds for all $v_i \geq 9$. Hence $(4,6,7)$ packs to 3/4. Finally, suppose $v_3 \in \{8,9\}$ with v^* either $(4,6,8)$ or $(4,6,9)$. With $B = (3/4)/[3/4 - d(4,6,9)]$, the key inequality is $v_i < (3.375)h(v_i)$. This holds for all $v_i \geq 8$ for packing (4.4), which has $h(v_i) = 3$ for $v_i \in \{8,9\}$, $h(v_i) = 4$ for $v_i \in \{10,11\},\ldots$. It follows that (4,6,9) packs to 3/4, and likewise for $(4,6,8)$ because B for the key inequality in this case exceeds 3.375.

(4,7). $t - d(4,7) = 0.35714$ and $B = 2.1$, so the key inequality is $v_i < (2.1)h(v_i)$. This holds for all $v_i \geq 10$ for packing (4.5):

u	h	$(2.1)h$
10	5	10.5
11	6	12.6
12, 13	7	14.7
14, 15	8	16.8

Thus (4,7) packs to 3/4 if $v_3 \geq 10$. When packing (4.6) is used for $v^* \in \{(4,7,7),(4,7,8)\}$ and (4.7) is used for $v^* = (4,7,9)$, their key inequalities using $d(4,7,8)$ for the first case are

$v_i < (3.2304)h(v_i)$ and $v_i < (3.0482)h(v_i)$, respectively:

packing (4.6)

u	h	$(3.2304)h$
$7-9$	3	9.69
$10-12$	4	12.92
13	5	16.15
$14-16$	6	19.38

packing (4.7)

u	h	$(3.0482)h$
9	3	9.14
$10-12$	4	12.19
$13-14$	5	15.24

We conclude that (4,7) packs to 3/4.

(4,8). $t-d(4,8)=3/8$, so the key inequality is $v_i < 2h(v_i)$. It is easily seen that this holds for all $v_i \geq 8$ with packing (4.8), so (4,8) packs to 3/4.

(4,9). Packing (4.9) with $t-d(4,9)=0.3888$ and $B=1.9286$ gives

u	h	$(1.9286)h$
$9,10$	5	9.64
11	6	11.57
$12,13$	7	13.50
14	8	15.42
$15,16$	9	17.35

so (4,9) packs to 3/4 when $v_3 \geq 11$. When v^* is (4,9,9) or (4,9,10), packing (4.10) with $h(v_i) = \lfloor v_i/2 \rfloor$, $t-d(4,9,10)=0.2888$ and $B=2.596$ yields the key inequality $v_i < (2.596)\lfloor v_i/2 \rfloor$, which holds for all $v_i \geq 9$. Hence (4,9) packs to 3/4.

This completes the proof that (4) packs to 3/4.

(5,5). The key inequality for $v^*=(5,5)$ is $v_i < (2.143)h(v_i)$. This holds for all $v_i \geq 5$ with packing (5.1).

(5,6). Packing (5.1) for $v^*=(5,6)$ with key inequality $v_i < (1.956)h(v_i)$ shows that (5,6) packs to 3/4 if $v_3 \geq 9$. When $v_3=6$, $(5,6,6) \to (3,5)$. When $v_3=7$, packing (5.2) for $v^*=(5,6,7)$ with $h(v_i)=\lfloor 2v_i/5 \rfloor$ has key inequality $v_i < (3.1188)\lfloor 2v_i/5 \rfloor$, which holds for $v_i \geq 8$, so $(5,6,7)$ packs to 3/4 when $v_4 \geq 8$. And when $v^*=(5,6,7,7)$, $t-d(v^*)=0.097619$, so $\sum 1/v_i \leq \frac{3}{4}$ requires $v_5 \geq 11$ with key inequality $v_i < (7.6829)\lfloor v_i/5 \rfloor$ for packing (5.3), so (5,6,7,7) and also (5,6,7) packs to 3/4. Finally, when $v_3=8$, packing (5.4) for $v^*=(5,6,8)$ and key inequality $v_i < (2.9032)h(v_i)$ shows that v^* packs to 3/4:

u	h	$(2.9032)h$
8	3	8.71
$9-11$	4	11.61
$12,13$	5	14.52
14	6	17.42

Hence (5,6) packs to 3/4.

(5,7). With $v^* = (5,7)$, $t - d(v^*) = 0.40714$ and key inequality $v_i < (1.842)h(v_i)$, packing (5.5) shows that v^* packs to 3/4 if $v_3 \geq 9$:

u	h	$(1.892)h$
9	5	9.21
10, 11	6	11.05
12	7	12.89
13, 14	8	14.74
15, 16	9	16.57 .

When $v^* = (5,7,7)$ with $t-d(v^*) = 0.26428$, packing (5.6) with key inequality $v_i < (2.8379)h(v_i)$ shows that v^* packs to 3/4:

u	h	$(2.8379)h$
7.8	3	8.51
9	4	11.35
$10-12$	5	14.19
13, 14	6	17.03 .

The same packing for $v^* = (5,7,8)$ shows that v^* packs to 3/4 unless $v_4 = 8$, but because $(5,7,8,8) \to (4,5,7)$, $(5,7,8)$ and therefore (5,7) packs to 3/4.

(5,8). With $v^* = (5,8)$ and $t - d(v^*) = 0.425$, packing (5.7) with key inequality $v_i < (1.7646)h(v_i)$ gives

u	h	$(1.7646)h$
12	7	12.35
13	8	14.12
14	9	15.88
$15-17$	10	17.65
18	11	19.41
19	12	21.18 .

Hence (5,8) packs to 3/4 unless $8 \leq v_3 \leq 11$. If $v_3 = 8$ then $(5,8,8) \to (4,5)$. When v^* is $(5,8,9)$ or $(5,8,10)$, the key inequality for $t - d(5,8,10) = 0.325$ is $v_i < (2.3076)\lfloor v_i/2 \rfloor$ with packing (5.8), and this holds for all $v_i \geq 9$. When $v^* = (5,8,11)$, the key inequality for the same packing is $v_i < (2.2449)\lfloor v_i/2 \rfloor$, which holds for $v_i \geq 11$. It follows that $(5,8)$ packs to 3/4.

(5,9). Packing (5.9) for $v^* = (5,9)$, $t - d(v^*) = 0.43888$ and $B = 1.7088$ shows that v^* packs to 3/4 unless $v_3 \leq 12$:

u	h	$(1.7088)h$
13	8	13.67
14	9	15.38
15, 16	10	17.08
17	11	18.80 .

When $9 \le v_3 \le 12$ with $t - d(5,9,12) = 0.3555$ and $B = 2.1088$, we have the following for packing (5.10):

u	h	$(2.1088)h$
$9,10$	5	10.54
$11,12$	6	12.65
$13,14$	7	14.76
$15,16$	8	16.87
17	9	18.97

Hence $(5,9,9)$ through $(5,9,12)$ pack to 3/4, so $(5,9)$ packs to 3/4.

(5,10) and (5,11). With v^* equal to $(5,10)$ or $(5,11)$, the key inequality for $t - d(5,11) = 0.4591$ is $v_i < (1.6336)h(v_i)$. Packing (5.11) gives

u	h	$(1.6336)h$
$10,11$	7	11.43
$12,13$	8	13.07
14	9	14.70
$15,16$	10	16.34
17	11	17.97

so $(5,10)$ and $(5,11)$ pack to 3/4.

This completes the proof that (5) packs to 3/4.

(6,6). $(6,6) \to (3)$.

(6,7). With $v^* = (6,7)$ and $t - d(v^*) = 0.44047$, packing (6.1) gives key inequality $v_i < (1.7027)\lfloor 2v_i/3 \rfloor$. This holds for $v_i \ge 8$, so (6,7) packs to 3/4 unless $v_3 = 7$. Packing (6.2) for $v^* = (6,7,7)$ gives key inequality $v_i < (2.52)\lfloor v_i/2 \rfloor$, and this holds for $v_i \ge 7$. Hence $(6,7)$ packs to 3/4.

(6,8). With $v^* = (6,8)$ and $t - d(v^*) = 0.45833$, packing (6.1) with key inequality $v_i < (1.636)\lfloor 2v_i/3 \rfloor$ shows that (6,8) packs to 3/4 unless $8 \le v_3 \le 10$. When $v_3 = 8$, $(6,8,8) \to (4,6)$. When $v_3 \in \{9,10\}$, we have the following data for the key inequalities under packing (6.3):

		$v^* = (6,8,9)$	$v^* = (6,8,10)$
u	h	$(2.16)h$	$(2.093)h$
9	4	8.64	
10	5	10.80	10.46
11	6	12.96	12.56
$12-14$	7	15.12	14.65
15	8	17.28	16.74

Hence $(6,8,9)$ and $(6,8,10)$ pack to 3/4 except when $v_4 = 9$ for $(6,8,9)$. When $v^* = (6,8,9,9)$, with $t - d(v^*) = 0.23611$ and $B = 3.176$, packing (6.4) with $h(9) = 3$, $h(10) = h(11) = 4$,

$h(12) = h(13) = h(14) = 5$, $h(15) = 6, \ldots$, shows that v^* packs to 3/4. Hence (6,8) packs to 3/4.

(6.9). With $v^* = (6,9)$ and $t - d(v^*) = 0.47222$, packing (6.5) with $b^* = 0.75188$ [for $h(u) = 9$] shows that (6,9) packs to 3/4 if $v_3 \geq 14$: for $u = 14$, $h = 9$ and $Bh = 14.3; \ldots;$ for $u = 20$, $h = 13$ and $Bh = 20.7$. When $9 \leq v_3 \leq 13$, packing (6.6) shows that $(6,9,9)$ through (6,9,13) pack to 3/4:

u	h	$v_3 \in \{9,10\}$ $(2.015)h$	$v_3 = 11$ $(1.967)h$	$v_3 = 12$ $(1.928)h$	$v_3 = 13$ $(1.897)h$
$9, 10$	5	10.07			
11	6	12.09	11.80		
$12, 13$	7	14.11	13.77	13.50	13.28
$14, 15$	8	16.12	15.73	15.42	15.18
$\vdots$					
20	11	22.16	21.63	21.21	20.87

We conclude that $(6,9)$ packs to 3/4.

(6,10). With $v^* = (6,10)$ and $t - d(v^*) = 0.48333$, packing (6.7) with $h(14) = 9$, $h(15) = 10$, $h(16) = 11, \ldots$ and $b^* = 0.75188$ satisfies the key inequality $v_i < (1.5556)h_i$ for $v_i \geq 14$, so (6,10) packs to 3/4 unless $10 \leq v_3 \leq 13$. For $v_3 = 10$, $(6,10,10) \to (5,6)$. With $v^* = (6,10,11)$ and $t - d(v^*) = 0.39242$, packing (6.8) with $B = 1.9112$ shows that v^* packs to 3/4:

u	h	$v_3 = 11$ $(1.9112)h$	$v_3 \in \{12,13\}$ $(1.8454)h$
11	6	11.46	
$12, 13$	7	13.37	12.92
14	8	15.29	14.76
$15, 16$	9	17.20	16.61
17	10	19.11	18.45
$18, 19$	11	21.02	20.30
20	12	22.93	22.15

When v^* is (6,10,12) or (6,10,13) with $t - d(6,10,13) = 0.40641$, the same packing with $B = 1.8454$ shows that these v^* pack to 3/4 unless $v_i = 13$ for some $i \geq 4$. Then packing (6.9) with $t - d(6,10,13,13) = 0.32949$, $B = 2.276$, and $h(12) = 6$, $h(13) = h(14) = 7$, $h(15) = 8, \ldots$ shows that the key inequality $v_i < (2.276)h(v_i)$ holds for $v_i \geq 12$. Hence (6,10,12,12) (6,10,12,13) and (6,10,13,13) pack to 3/4, so (6,10) packs to 3/4.

(6,11). With $v^* = (6,11)$ and $t - d(v^*) = 0.49242$, packing (6.10) shows that (6,11) packs to 3/4 if $v_3 \geq 15$. The key inequality here with $b^* = 0.76134$ for $h = 10$ is $v_i < (1.546)h(v_i)$ with $h(15) = 10$, $h(16) = 11$, $h(17) = 12$, $h(18) = h(19) = 13, \ldots$. When $11 \leq v_3 \leq 14$

with $t - d(6,11,14) = 0.420996$ and $B = 1.7814$, packing (6.11) shows that (6,11,11) through (6,11,14) pack to 3/4:

u	h	$(1.7814)h$
$11, 12$	7	12.47
$13, 14$	8	14.25
15	9	16.03
$16, 17$	10	17.81
18	11	19.60
$19, 20$	12	21.37

Thus (6,11) packs to 3/4.

This completes the proof that (6) packs to 3/4.

(7,7). With $v^* = (7,7)$ and $t - d(v^*) = 0.46428$, packing (7.1) shows that (7,7) packs to 3/4:

u	h	$v^* = (7,7)$ $(1.6154)h$	$v^* = (7,8)$ or $(7,9)$ $(1.512)h$
$7, 8$	5	8.07	
9	6	9.69	
$10, 11$	7	11.31	
12	8	12.93	12.10
13	9	14.53	13.61
$14, 15$	10	16.15	15.12
16	11	17.77	16.62
$17, 18$	12	19.38	18.14

The right column, where $t - d(7,9) = 0.49603$, shows that (7,8) and (7,9) pack to 3/4 when $v_3 \geq 12$. Hence only $8 \leq v_3 \leq 11$ needs further consideration for these cases.

(7,8). Consider $8 \leq v_3 \leq 11$. For $v_3 = 8$, $(7,8,8) \to (4,7)$. When $v^* = (7,8,9)$ with $t - d(v^*) = 0.37103$ and $B = 2.0214$, packing (7.2) shows that v^* packs to 3/4:

u	h	$(2.0214)h$
$9, 10$	5	10.11
$11, 12$	6	12.13
13	7	14.15
$14, 15$	8	16.17
$16, 17$	9	18.19

When $v^* = (7,8,10)$ with $t - d(v^*) = 0.3824$, we have (7,8,10,10)$\to$(5,7,8), and packing (7.3) with $B = (3/4)/(0.38214) = 1.9626$ shows that (7,8,10) packs to 3/4 when $v_4 \geq 11$:

u	h	$v_3 = 10$ $(1.9626)h$	$v_3 = 11$ $(1.917)h$
11	6	11.76	11.50
12, 13	7	13.74	13.42
14, 15	8	15.70	15.34
16, 17	9	17.66	17.25
18	10	19.63	19.17
19	11	21.59	21.09
20 − 22	12	23.55	23.00

When $v^* = (7, 8, 11)$ with $t - d(v^*) = 0.39123$ and $B = 1.917$, the right column shows that v^* packs to 3/4.

(7,9). Consider $9 \leq v_3 \leq 11$. Because $(7, 9, 9, 9) \rightarrow (3, 7)$, we omit a row for $v = 9$ in the following key inequality data for $v^* \in \{(7, 9, 9), (7, 9, 10), (7, 9, 11)\}$ with packing (7.4):

u	h	$v^* = (7, 9, 9)$ $(1.948)h$	$v^* = (7, 9, 10)$ $(1.8938)h$	$v^* = (7, 9, 11)$ $(1.856)h$
10	5	9.74		
11	6	11.69	11.36	
12	7	13.64	13.26	
13 − 15	8	15.59	15.15	
16	9	17.53	17.04	16.70
17	10	19.48	18.93	18.56
18, 19	11	21.43	20.83	20.42

The column for $v^* = (7, 9, 9)$ shows that v^* packs to 3/4 unless $v_4 = 10$ and, when $v^* = (7, 9, 9, 10)$ with $t - d(v^*) = 0.28492$, the following data for packing (7.5) show that $(7, 9, 9, 10)$ packs to 3/4:

u	h	$v^* = (7, 9, 9, 10)$ $(2.6323)h$	$v^* = (7, 9, 11, 11)$ $(2.3869)h$
10	4	10.53	
11	5	13.16	11.93
12 − 14	6	15.79	14.32
15, 16	7	18.42	16.71
17	8	21.05	19.09
18 − 20	9	27.69	21.48

Therefore (7,9,9) packs to 3/4. For $v^* = (7, 9, 10)$, $(7, 9, 10, 10) \rightarrow (5, 7, 9)$, so assume that $v_4 \geq 11$. The next to last display shows that $(7, 9, 10)$ packs to 3/4. The right column of the next to last display with $b^* = 0.75188$ for $h = 9$ shows that (7,9,11) packs to 3/4 unless $11 \leq v_4 \leq 15$, and the right column in the preceding array shows that (7,9,11,11) packs to 3/4.

The unresolved v^*'s for (7,9) at this point are (7,9,11,12), (7,9,11,13), (7,9,11,14) and (7,9,11,15). With $v^* = (7, 9, 11, 12)$ and $t - d(v^*) = 0.32179$, packing (7.6) with $B = 2.3307$

shows that v^* packs to 3/4 if $v_5 \geq 15$:

u	h	$(2.3307)h$
$15, 16$	7	16.06
17	8	18.35
$18, 19$	9	20.64

.

Because $(7, 9, 11, 12, 12) \to (6, 7, 9, 11)$, this leaves $v_5 \in \{13, 14\}$ for further consideration with (7,9,11,12). With v^* as (7,9,11,12,13) or (7,9,11,12,14) and $t - d(7, 9, 11, 12, 14) = 0.25036$, packing (7.7) with $B = 2.9957$ shows that these v^*'s pack to 3/4:

u	h	$(2.9957)h$
$13, 14$	5	14.98
$15 - 17$	6	17.97
18	7	20.97
$19 - 21$	8	29.96

.

The following data for $v^* = (7, 9, 11, 13)$ with $t - d(v^*) = 0.3282$, and $v^* = (7, 9, 11, 15)$ with $t - d(v^*) = 0.33846$, pertain to packing (7.8):

		$v_4 = 13$	$v_4 = 15$
u	h	$(2.2851)h$	$(2.2159)h$
13	6	13.71	
$14, 15$	7	15.99	15.51
$16, 17$	8	18.28	17.72
$18, 19$	9	20.57	19.95

.

It follows that (7,9,11,13) through (7,9,11,15) pack to 3/4. This completes the proof for (7,9).

(7,10). $(7, 10, 10) \to (5, 7)$, so we assume $v_3 \geq 11$. With $v^* = (7, 10)$ and $t - d(v^*) = 0.50714$, packing (7.9) with $b^* = 0.76134$ for $h = 10$ shows that $(7, 10)$ packs to 3/4 if $v_3 \geq 14$:

u	h	$(1.5012)h$
$14, 15$	10	15.01
16	11	16.51
17	12	18.01
18	13	19.51

.

With $v_3 \in \{11, 12, 13\}$, we begin with packing (7.10) with $B = (3/4)/(0.41625) = 1.802$ for $v^* = (7, 10, 11)$ and $B = (3/4)/(0.42381) = 1.77$ for $v^* = (7, 10, 12)$:

		$v^* = (7, 10, 11)$	$v^* = (7, 10, 12)$
u	h	$(1.802)h$	$(1.77)h$
12	7	12.61	
$13, 14$	8	14.41	14.16
15	9	16.22	15.93
$16, 17$	10	18.02	17.70
18	11	19.82	19.47

.

Hence (7,10,11) packs to 3/4 when $v_4 \geq 12$ and, when $v_4 = 11$ with $v^* = (7,10,11,11)$ and $t - d(v^*) = 0.32532$, packing (7.11) shows that v^* packs to 3/4:

u	h	$(2.3054)h$
11	5	11.53
12, 13	6	13.83
14, 15	7	16.14
16, 17	8	18.44

.

The final column of the next to last array along with $(7,10,12,12) \rightarrow (6,7,10)$ shows that $(7,10,12)$ packs to 3/4. The same column shows that (7,10,13) packs to 3/4 if $v_4 \geq 15$ because B there is $0.75188/(0.43022) = 1.7477$ with $9h = 15.73, \ldots$. Then the only remaining v^*'s for (7,10) are (7,10,13,13) and (7,10,13,14). With $t - d(7,10,13,14) = 0.35879$, packing (7.12) with $B = 2.090$ yields

u	h	$(2.090)h$
13, 14	7	14.63
15	8	16.72
16, 17	9	18.81
18	10	20.90

so (7,10,13,13) and (7,10,13,14) pack to 3/4.

(7,11). With $v^* = (7,11)$ and $t - d(v^*) = 0.51623$, packing (7.13) with $b^* = 0.76438$ for $h = 11$ shows that v^* packs to 3/4 if $v_3 \geq 16$: $(1.4807)h(v_i)$ equals 16.29, 17.77, 19.25 and 20.73 for $v_i = 16$, 17, 18, 19, respectively. Packing (7.14) gives the following data for v^* from (7,11,11) through (7,11,15):

					$v_3 = 14$	$v_3 = 15$
		$v_3 = 11$	$v_3 = 12$	$v_3 = 13$	$b^* = 0.76134$	$b^* = 0.76438$
u	h	$(1.7634)h$	$(1.7325)h$	$(1.7072)h$	$(1.7116)h$	$(1.7002)h$
11, 12	7	12.34	12.12			
13	8	14.11	13.86	13.65		
14, 15	9	15.87	15.59	15.36		
16, 17	10	17.63	17.32	17.07	17.12	
18	11	19.40	19.06	18.78	18.83	18.70
19	12	21.16	20.79	20.49	20.54	20.40

.

Because $(7,11,12,12) \rightarrow (6,7,11)$ and $(7,11,14,14) \rightarrow (7,7,11)$, all extensions of $(7,11)$ pack to 3/4 except perhaps (7,11,14,15), (7,11,15,15), (7,11,15,16), (7,11,15,17). The worst-case $t - d(v^*)$ for these is $t - d(7,11,15,17) = 0.39074$, and with $B = 1.9194$ packing (7.15) shows that all pack to 3/4.

u	h	$(1.9194)h$
15	8	15.36
16	9	17.27
17, 18	10	19.19
19	11	21.11

.

This completes the proof that (7) packs to 3/4.

(8,8). $(8,8) \to (4)$.

(8,9). With $v^* = (8,9)$, packing (8.1) with $h(v_i) = \lfloor 3v_i/4 \rfloor$ is easily seen to satisfy the key inequality $v_i < (1.459)\lfloor 3v_i/4 \rfloor$ when $v_i \geq 10$, so (8,9) packs to 3/4 when $v_3 \geq 10$. With $v^* = (8,9,9)$ and $9t - d(v^*) = 0.40278$, packing (8.2) gives

u	h	$(1.862)h$
9	5	9.31
10, 11	6	11.17
12, 13	7	13.03
14	8	14.90
15	9	16.76
16	10	18.62

so (8,9,9) packs to 3/4.

(8,10). Because $(8,10,10) \to (5,8)$, assume that $v_3 \geq 11$. With $v^* = (8,10)$ and $t - d(v^*) = 0.525$, packing (8.3) with $b = 0.76134$ for $h = 10$ shows that v^* packs to 3/4 unless $v_3 \leq 13$:

u	h	$(1.4502)h$
14	10	14.50
15	11	15.95
16, 17	12	17.40
18	13	18.85
19	14	20.30

Assume $11 \leq v_3 \leq 13$ henceforth. With $(v^*, t - d(v^*))$ as $((8,10,11), 0.43409)$, $((8,10,12), 0.44167)$, or $((8,10,13), 0.44807)$, packing (8.4) gives the following:

u	h	$v^* = (8,10,11)$ $(1.7277)h$	$v^* = (8,10,12)$ $(1.6981)h$	$v^* = (8,10,13)$ $(1.674)h$
11	6	10.37		
12	7	12.09	11.89	
13	8	13.82	13.58	13.39
14, 15	9	15.55	15.28	15.06
16	10	17.28	16.98	16.78
17, 18	11	19.01	18.68	18.41
19	12	20.73	20.38	20.09
20, 21	13	22.96	22.08	21.76

Thus (8,10,11) packs to 3/4 unless $v_4 = 11$, (8,10, 12) packs to 3/4 because $(8,10,12,12) \to (6,8,10)$, and (8,10,13) packs to 3/4. With $t - d(8,10,11,11) = 0.34318$, packing (8.5) with

$B = 2.1854$ shows that (8,10,11,11) packs to 3/4 with the exception of (8,10,11,11,11):

u	h	$(2.1854)h$
11	5	10.93
12	6	13.11
13, 14	7	15.30
15, 16	8	17.35
17	9	19.53

Finally, packing (8.6) with $v^* = (8, 10, 11, 11, 11)$, and $t - d(v^*) = 0.25227$ and $B = 2.973$, shows that v^* packs to 3/4:

u	h	$(2.973)h$
11	4	11.89
12, 13	5	14.87
14	6	17.84
15 − 18	7	20.81

(8,11). With $v^* = (8, 11)$ and $t - d(v^*) = 0.53409$, packing (8.7) with $b = 0.76134$ for $h = 10$ shows that v^* packs to 3/4 if $v_3 \geq 14$:

u	h	$(1.4255)h$
14	10	14.25
15	11	15.68
16	12	17.11

This leaves (8,11,11), (8,11,12) and (8,11,13). With $t-d(8, 11, 12) = 0.44318$ and $t-d(8, 11, 13) = 0.45717$, packing (8.8) gives the following:

u	h	$v^* = (8, 11, 11), (8, 11, 12)$ $(1.6923)h$	$v^* = (8, 11, 13)$ $(1.6405)h$
13	8	13.53	13.12
14	9	15.23	14.76
15	10	16.92	16.41
16 − 18	11	18.62	18.05
19	12	20.31	19.69
20	13	22.00	21.32

Hence, with $(8, 11, 12, 12) \to (6, 8, 11)$, $(8, 11, 12)$ packs to 3/4, $(8, 11, 11)$ packs to 3/4 unless $v_4 \in \{11, 12\}$, and (8,11,13) packs to 3/4. With $t - d(8, 11, 11, 12) = 0.35985$, packing (8.9) with $B = 2.0842$ shows that (8,11,11,11) and (8,11,11,12) pack to 3/4:

u	h	$(2.0842)h$
11, 12	6	12.51
13, 14	7	14.59
15	8	16.67
16 − 18	9	18.76
19	10	20.84

This completes the proof that (8) packs to 3/4, so the proof of Theorem 1 is complete.

5. Proof of Theorem 3. The theorem says that, for every $m \geq 2$, if b is a density m-guarantee for V then $b \leq 1 - 1/[(m+1)(m+2)]$. We prove this by showing that for every $N \geq (m+1)(m+2)$ the $(m+2)$-term vector

$$v^{(m)} = (v_1, \ldots, v_{m+2}) = (m+1, m+1, \ldots, m+1, m+2, N) \ ,$$

with

$$d(v^{(m)}) = \frac{m}{m+1} + \frac{1}{m+2} + \frac{1}{N} = 1 - \frac{1}{(m+1)(m+2)} + \frac{1}{N}$$

is not m-schedulable. We show this for $m = 2$ and then consider $m \geq 3$. Throughout, a contiguous subsequence of K terms in a sequence is referred to as a K*-block*.

EXAMPLE **2.** Let $m = 2$, so $v^{(2)} = (3, 3, 4, N)$ with $N \geq 12$ for $d(v^{(2)}) \leq 1$. Contrary to Theorem 3, suppose that $v^{(2)}$ is 2-schedulable as verified by a sequence S from $\mathbb{Z}$ into $\{1, 2, 3, 4\}$. Define the sequence T over $\{\alpha, \beta, \gamma\}$ from S by replacing all instances of 1,2,3 and 4 in S by α, α, β and γ, respectively. Then, by 2-schedulability,

every 6-block of T has at least 4 α's,
every 8-block of T has at least 2 β's ,
every $2N$-block of T has at least 2 γ's

Let $x_1 x_2 \cdots x_{11}$ be an 11-block of T with $x_6 = \gamma$. Then each of $x_1 \cdots x_5$ and $x_7 \cdots x_{11}$ has at least 4 α's, hence at most one β. Because each of $x_1 \cdots x_8$ and $x_4 \cdots x_{11}$ has at least 2 β's, we have $\beta \in \{x_7, x_8\}$ and $\beta \in \{x_4, x_5\}$. But then the 6-block $x_4 \cdots x_9$ has at most 3 α's, a contradiction. We conclude that $v^{(2)}$ is not 2-schedulable and, because N can be arbitrarily large, that no density 2-guarantee for V exceeds $1 - 1/12$. ∎

Suppose $m \geq 3$. Contrary to Theorem 3, suppose sequence S from $\mathbb{Z}$ into $\{1, 2, \ldots, m+2\}$ shows that $v^{(m)}$ is m-schedulable. Define T from S by replacing all instances of $1, 2, \ldots, m, m+1, m+2$ in S by $\alpha, \alpha, \ldots, \alpha, \beta, \gamma$, respectively. Then

every $m(m+1)$-block of T has at least m^2 α's,
every $m(m+2)$-block of T has at least m β's

and some terms of T are γ's. Let γ_0 denote a γ in a fixed position. Let A be any $m(m+1)$-block that includes γ_0, let C be the m-block next to A on its right, and let $B = AC$ with $m(m+2)$ terms. Because A has γ_0 and at least m^2 α's, it has at most $m-1$ β's. The $m(m+2)$-block B has at least m β's, so C has at least one β.

Let C_j be the m-block covering positions $(j-1)m+1$ to jm to the right of γ_0, so C_1 abuts γ_0, C_2 abuts C_1, and so forth. By the result just proved, every one of $C_1, C_2, \ldots, C_m$ has at least one β. Define $C'_1, C'_2, \ldots, C'_m$ similarly to the left of γ_0, so C'_1 is the m-block next to γ_0 on its left, C'_2 abuts C'_1 on its left, and so forth. By symmetry, every one of $C'_1, C'_2, \ldots, C'_m$ has at least one β.

Suppose m is odd. Let D be the $[m(m+1)+1]$-block that includes γ_0, the first $(m+1)/2$ C_j to its right and the first $(m+1)/2$ C'_j to its left. Then D has at least $(m+1)+1 = m+2$ β's and γ's and therefore has no more than $m^2 - 1$ α's, a contradiction.

Suppose m is even. Let D be the $m(m+1)$-block that includes γ_0, $m/2$ C_j to its right, $m/2$ C'_j to its left, and $m-1$ other terms to the right of $C_{m/2}$. Then D has at least $m+1$ β's and γ's and therefore has no more than $m^2 - 1$ α's, a contradiction.

It follows that $v^{(m)}$ is not m-schedulable and, because N can be arbitrarily large, that no density m-guarantee for V exceeds $1 - 1/[(m+1)(m+2)]$.

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Table 1

Packing Patterns

(3.1). (3, 4) : 120 [3]

(4.1). (4, 5) : 1020 [4]

(4.2). (4, 6) : 120010 021000 [12]

(4.3). (4, 6, 7) : 120130 [6]

(4.4). (4, 6, 8), (4, 6, 9) : 3210 0012 3010 0210 3012 0010 [24]

(4.5). (4, 7) : 12001 00210 00102 01000 [20]

(4.6). (4, 7, 7), (4, 7, 8) : 1300120 [7]

(4.7). (4, 7, 9) : 23100 01201 30012 00130 12001 01320 10010 [35]

(4.8). (4, 8) : 1020 1000 [8]

(4.9). (4, 9) : 210001001 200100010 201000100 [27]

(4.10). (4, 9, 9), (4, 9, 10) : 1020 1030 [8]

(5.1). (5, 5), (5, 6) : 10200 [5]

(5.2). (5, 6, 7) : 10230 [5]

(5.3). (5, 6, 7, 7) : 12340 [5]

(5.4). (5, 6, 8) : 1320 0100 2310 0021 0301 2000 [24]

(5.5). (5, 7) : 21000 01200 01002 01000 [20]

(5.6). (5, 7, 7), (5, 7, 8) : 31002 01300 21003 01200 [20]

(5.7). (5, 8) : 21000 01020 01000 [15]

(5.8). (5, 8, 9) to (5, 8, 11) : 1020 1030 [8]

(5.9). (5, 9) : 12000 10002 10000 10020 10000 10200 10000 [35]

(5.10). (5, 9, 9) to (5, 9, 12) : 300 102 010 [9]

(5.11). (5, 10), (5, 11) : 10020 10000 [10]

(6.1). (6, 7), (6, 8) : 100200 [6]

(6.2). (6, 7, 7) : 102030 [6]

(6.3). (6, 8, 9), (6, 8, 10) : 210030 012000 310020 013000 [24]

(6.4). (6, 8, 9, 9) : 312040 013020 410030 214000 [24]

(6.5). (6, 9) : 210000 010200 010000 [18]

(6.6). (6, 9, 9) to (6, 9, 13) : 210003 010200 013000 [18]

(6.7). (6, 10) : 100200 100000 120000 100002 100000 [30]

(6.8). (6, 10, 11) to (6, 10, 13) : 100200 103000 120003 100002 100300 [30]

(6.9). (6, 10, 12, 12), (6, 10, 12, 13), (6, 10, 13, 13) : 210030 010240 010030 210040 010230 010040 [36]

(6.10). (6, 11) : 120_4 $10_4 2$ 10_5 $10_3 20$ 10_5 100200 10_5 1020_3 10_5 [54]

(6.11). (6, 11, 11) to (6, 11, 14) : 10200100300 [11]

(7.1). (7, 7) to (7, 9) : 1002000 [7]

(7.2). (7, 8, 9) : 1002030 [7]

(7.3). (7, 8, 10), (7, 8, 11) : 21003 00120 00301 02000 [20]

(7.4). (7, 9, 9) to (7, 9, 11) : 120003010 020013000 021003000 [27]

(7.5). (7, 9, 9, 10), (7, 9, 11, 11) : 210300 410200 310400 [18]

(7.6). (7, 9, 11, 12) : 103020 100403 120000 103420 100003 120400 [36]

(7.7). (7, 9, 11, 12, 13), (7, 9, 11, 12, 14) : 3210500 0412305 0010024 0310052 0014030 0210500 3412000 0510324 0010002 5314000 0210035 0412000 0310524 0010032 0510400 [105]

Table 1 continued

(7.8). $(7,9,11,13)$ to $(7,9,11,15)$: 3400210030012 0403010020010 [26]

(7.9). $(7,10)$: $120_5\ 10_3200\ 10_52\ 10_6\ 1020_4\ 10_420\ 10_6$ [49]

(7.10). $(7,10,11)$ to $(7,10,13)$: 12000 30100 02001 03000 [20]

(7.11). $(7,10,11,11)$: 12040 30100 02041 03000 [20]

(7.12). $(7,10,13,13),(7,10,13,14)$: 4020100300102 4000100302100 4000102300100 [39]

(7.13). $(7,11)$: 2100000 0100200 0100000 [21]

(7.14). $(7,11,11)$ to $(7,11,15)$: 1020030 1000020 1030000 [21]

(7.15). $(7,11,14,15),(7,11,15,15)$ to $(7,11,15,17)$: 1023000 1004002 1003000 [21]

(8.1). $(8,9)$: 1000 2000 [8]

(8.2). $(8,9,9)$: 1030 2000 [8]

(8.3). $(8,10)$: $210_71\ 20_6100\ 20_410_4\ 20010_6$ [40]

(8.4). $(8,10,11)$ to $(8,10,13)$: 2100300001 2000300100 2000310000 2001300000 [40]

(8.5). $(8,10,11,11)$: 1200300410 0200301400 0201300400 [30]

(8.6). $(8,10,11,11,11)$: 21300 40501 20300 41500 20301 40500 [30]

(8.7). $(8,11)$: 21000000 01020000 01000020 01000000 [32]

(8.8). $(8,11,11)$ to $(8,11,13)$: 21000030 01020000 31000020 01030000 [32]

(8.9). $(8,11,11,11),(8,11,11,12)$: 21000040 31020000 41030020 01040030 [32]