Threading Homotopies and DC Operating Points of Nonlinear Circuits

Ross Geoghegan
Dept. of Mathematical Sciences
SUNY at Binghamton
Binghamton, NY 13902

Jeffrey C. Lagarias
AT&T Labs – Research
Florham Park, NJ 07932

Robert C. Melville
Bell Laboratories
Lucent Technologies
Murray Hill, NJ 07974

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Abstract

This paper studies continuation methods for finding isolated zeros of nonlinear functions. Given a nonlinear function $F : \mathbb{R}^n \to \mathbb{R}^n$, a *threading homotopy* is a function $H(\mathbf{x}, \lambda) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ with $H(\mathbf{x}, 0) \equiv F(\mathbf{x})$, such that the zero set of H is a single connected curve containing all zeros of $F(\mathbf{x})$. For a C^1 function F, a necessary condition for the existence of a nondegenerate C^1 threading homotopy is that the topological degree of $F(\mathbf{x})$ be 1, 0 or -1. For C^2 mappings in all dimensions except possibly $n = 2$ this condition is also a sufficient condition for existence of a C^2 threading homotopy which is weakly proper over 0. A homotopy H is *weakly proper over 0* if for every interval $[a, b]$ the set $H^{-1}(\mathbf{0}) \cap (\mathbb{R}^n \times [a, b])$ is compact. This condition rules out any part of the zero set escaping to infinity at a finite value of the homotopy parameter.

Threading homotopies are potentially applicable in continuation methods for finding all dc operating points of nonlinear circuits. We show that most transistor circuits have dc operating point equations $F(\mathbf{x}) = \mathbf{0}$ with $\deg(F) = \pm 1$, so that threading homotopies exist in principle for such operating point equations. It remains an open problem to explicitly construct such threading homotopies.

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1. Introduction

This paper studies continuation methods to find all zeros of a nonlinear function $F : \mathbb{R}^n \to \mathbb{R}^n$ which has a finite number of isolated zeros. The continuation approach to finding zeros is to find a function $H(\mathbf{x}, \lambda) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ such that $H(\mathbf{x}, 0) \equiv F(\mathbf{x})$ while $H(\mathbf{x}, 1) \equiv G(\mathbf{x})$ is a function with known zeros, the zero set

$$\Gamma(H) = \{(\mathbf{x}, \lambda) : H(\mathbf{x}, \lambda) = \mathbf{0}\} \tag{1.1}$$

is a union of curves (1-dimensional components), and these curves can be individually traced from the known zero set

$$\Gamma_1 = \{\mathbf{x} : H(\mathbf{x}, 1) = \mathbf{0}\}$$

to find all solutions Γ_0 of $F(\mathbf{x})$. The function $H(\mathbf{x}, \lambda)$ is called a *homotopy,* and a *homotopy path* is a path $(\mathbf{x}(t),\ \lambda(t))$ for $t \in [0, 1]$ on which $H(\mathbf{x}, \lambda) = 0$. One method to find all the zeros is to choose a homotopy $H(\mathbf{x}, \lambda)$ such that each zero of $F(\mathbf{x})$ is on a separate connected component of the zero set of $H(\mathbf{x}, \lambda)$, and separate homotopy paths are followed to find each zero of $F(\mathbf{x})$, see for example Allgower and Georg [1], §6, [2], Chow et al. [8], Drexler [17], and

Garcia and Zangwill [20], [21]. This has the advantage of permitting parallel computation to find different zeros. This approach has been proposed in particular to find complex zeros of univariate polynomials $F(z)$, see Kojima et al. [33].

In this paper we study the opposite extreme, which are homotopies $H(\mathbf{x}, \lambda) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ with $H(\mathbf{x}, \mathbf{0}) \equiv F(\mathbf{x})$, such that the zero set of $H(\mathbf{x}, \lambda)$ is a single connected curve. We call a homotopy with this property a *threading homotopy* for F, because the zeros of $F(\mathbf{x})$ are threaded along a single curve in the zero set of $H(\mathbf{x}, \lambda)$, which passes back and forth through the hyperplane $\lambda = 0$. More generally we consider *semi-threading homotopies*, which are homotopies H in which all zeros of $F(\mathbf{x})$ are on a single connected component of the zero set $\Gamma(H)$ of H, but $\Gamma(H)$ may contain other connected components. Using a semi-threading homotopy all zeros of $F(\mathbf{x})$ can be located by tracing a single curve.

This study of threading homotopies is motivated by the problem of numerically computing all dc-operating points of nonlinear resistive circuits, e.g. circuits with transistors. A *dc operating point*[1] for a nonlinear resistive circuit is any solution of a given system of network equations $F(\mathbf{x}) = \mathbf{0}$ for the circuit. The detection of multiple operating points is of considerable practical concern in circuit simulation, because some solutions of the network equations may represent unintended pathological modes of operation, so that the circuit may fail in the field. To avoid this, one would like to detect all possible operating points during the circuit-design phase, or at least alert the designer to the presence of more than one operating point, before a decision is made to fabricate an integrated circuit. Existing circuit simulators do not guarantee to find all operating points, and there is now considerable interest in developing methods that will find all operating points, cf. Mathis and Wettlaufer [34], Trajković, Melville and Fang [43], Melville, Trajković, Fang and Watson [36]. The use of continuation methods to find individual operating points has a long history, see Chao and Saeks [6]. However the problem of developing continuation methods guaranteed to find all operating points has received relatively little study. The idea of finding several zeros of $F(\mathbf{x})$ along one curve was made in the early 1970's in Branin [5] and Chua and Ushida [12]. In some of their examples, there are zeros of $F(\mathbf{x})$ on several connected components of $\Gamma(H)$. It is natural in pursuing this approach to try to get all zeros

[1] Some authors use the term *dc equilibrium point* for a solution to the network equations $F(\mathbf{x}) = \mathbf{0}$, and reserve the term dc operating point for a linearly stable equilibrium point. We call the latter a *stable dc operating point*, as in Green [24] and Green and Willson [26].

on a single component, which is the threading homotopy problem.

We call a homotopy $H(x, \lambda)$ *weakly proper over 0* if for every closed interval $[a, b]$ the restriction H to $\mathbb{R}^n \times [a, b]$ is *proper over 0*, i.e. $H^{-1}(\mathbf{0}) \cap (\mathbb{R}^n \times [a, b])$ is compact. For such a homotopy no part of the zero set of $H(\mathbf{x}, \lambda)$ can escape to infinity at a finite value of λ. We consider the following problem.

Weakly Proper Threading Homotopy Problem. *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^r-function $(1 \leq r \leq \infty)$ having a finite set of isolated zeros. Construct if possible a C^r-homotopy*

$$H(\mathbf{x}, \lambda) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n ,$$

with

(i). $H(\mathbf{x}, 0) \equiv F(\mathbf{x})$.

(ii). *Nondegeneracy Condition. The Jacobian $DH(\mathbf{x}, \lambda)$ has rank n whenever $H(\mathbf{x}, \lambda) = \mathbf{0}$.*

(iii). *Connectedness Condition. The zero set $\Gamma(H) = \{(\mathbf{x}, \lambda) : H(\mathbf{x}, \lambda) = \mathbf{0}\}$ is connected.*

(iv). *Weakly Proper over 0 Condition. For every closed interval $[a, b] \subseteq \mathbb{R}$ the restriction $H| : \mathbb{R}^n \times [a, b] \rightarrow \mathbb{R}^n$ is proper over 0.*

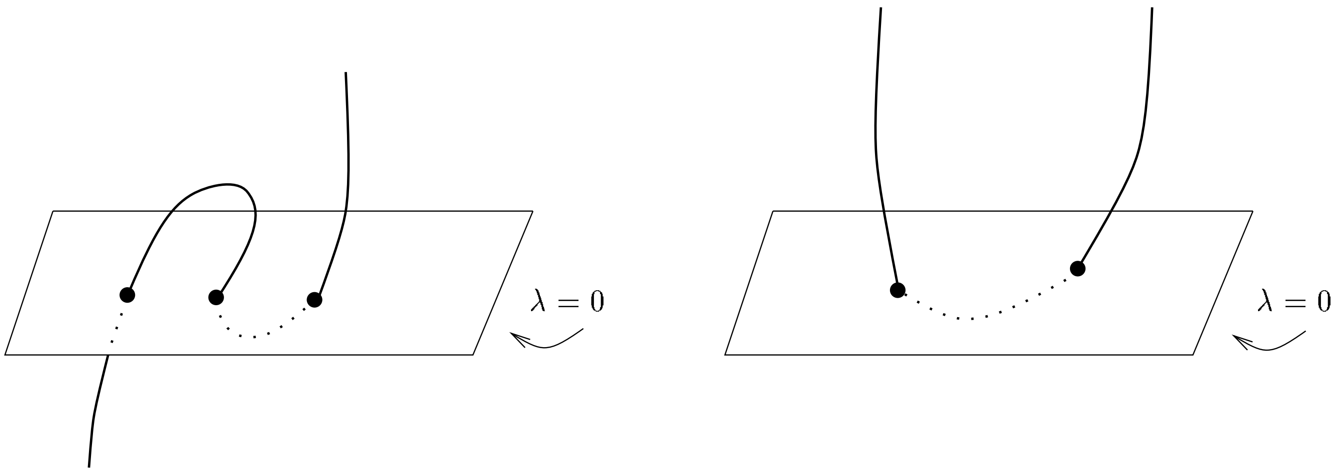
The condition (ii) implies that the zero curves $\{(\mathbf{x}, \lambda) : H(\mathbf{x}, \lambda) = 0\}$ have no bifurcations, and with condition (iii) this implies that the set $H(\mathbf{x}, \lambda) = \mathbf{0}$ is a single curve containing all the zeros of $F(\mathbf{x})$. As mentioned above, condition (iv) prevents the zero set from escaping to infinity at any finite value of λ. The conditions (ii)–(iv) lead to two cases; pictured in Figure 1.1.

Case (a). $F(\mathbf{x}) = \mathbf{0}$ has an odd number of solutions. Then the sets

$$\Gamma_\lambda = \{\mathbf{x} \in \mathbb{R}^n : H(\mathbf{x}, \lambda) = \mathbf{0}\}$$

are nonempty for all $\lambda \in \mathbb{R}$, and $|\Gamma_\lambda| = 1$ for all $|\lambda|$ sufficiently large. (Here, $|\Gamma_\lambda|$ denotes the number of elements in the set Γ_λ.)

Case (b). $F(\mathbf{x}) = \mathbf{0}$ has an even number of solutions. Then $|\Gamma_\lambda| = 0$ for all large λ of one sign, and for large λ of the other sign $|\Gamma_\lambda| = 0$ or 2 according as the zero set $\Gamma(H)$ is bounded or unbounded.



(a) $|\Gamma_\lambda|$ odd (b) $|\Gamma_\lambda|$ even

Figure 1.1. Threading paths

We are particularly interested in case (a). There, $H(\mathbf{x}, \lambda) = \mathbf{0}$ for large fixed λ_0 has a single zero $\mathbf{x}_{\lambda_0}$, which one can use as the starting point for a homotopy method to find all zeros.

In §2 we present necessary conditions and sufficient conditions for existence of threading homotopies. It is clear that, given a finite set of isolated points in $\mathbb{R}^{n+1}$, one can always construct a smooth curve in $\mathbb{R}^{n+1}$ passing through these points. However it is sometimes impossible to extend a map $F : \mathbb{R}^n \to \mathbb{R}^n$ to a weakly proper threading homotopy $H : \mathbb{R}^{n+1} \to \mathbb{R}^n$. For a C^1 mapping F a necessary condition for the existence of a nondegenerate C^1 semi-threading homotopy is that the topological degree of F be 0 or ± 1. (Theorem 2.1). An immediate consequence is that there exists no C^1 semi-threading homotopy for finding all zeros of a complex polynomial $p : \mathbb{C} \to \mathbb{C}$, where $\mathbb{C}$ is identified with $\mathbb{R}^2$, whenever $p(\mathbf{z})$ is nonlinear. (Corollary 2.1). We show for mappings F that are $C^r (2 \leq r \leq \infty)$ with a finite set of isolated nondegenerate zeros that the condition $\deg(F) = 0$, or ± 1 is necessary and sufficient for C^r weakly proper threading homotopies to exist, in all dimensions except possibly dimension $n = 2$. (Theorem 2.2). We then show for mappings F that are C^r $(1 \leq r \leq \infty)$ with a finite set of isolated nondegenerate zeros that the condition $\deg(F) = 0$ or ± 1 is necessary and sufficient for the existence of a weakly proper C^r semi-threading homotopy, in all dimensions $n \geq 1$. (Theorem 2.3).

To design threading homotopies, it is clearly useful to have criteria which verify that the threading property holds. Diener [16] gives a (somewhat restrictive) set of global conditions on a C^2-function $H : \mathbb{R}^{n+1} \to \mathbb{R}^n$ which guarantee that it has the threading property. Diener's condition is that there exists some positive K such that

$$\sup\{||(DH(\mathbf{x})DH(\mathbf{x})^T)^{-1}|| : \mathbf{x} \in \mathbb{R}^{n+1}\} \leq K < \infty \ , \tag{1.2}$$

where the Frobenius norm $||M||$ for the matrix M is $||M||^2 = \sum_{i,j} M_{ij}^2$. He proves that, when (1.2) holds, the Newton's method flow gives a retraction of $\mathbb{R}^{n+1}$ onto the set $\Gamma(H)$, thus establishing that $\Gamma(H)$ is a connected set.

In §3 we return to our motivating problem, which concerns the possible existence of threading homotopies to find dc operating points of nonlinear circuits. We present theoretical results which indicate that threading homotopies exist for a large class of nonlinear circuits, without exhibiting such homotopies explicitly. More precisely, we show that a large class of circuits can be modelled so as to have operating point equations $F(\mathbf{x}) = \mathbf{0}$ with $\deg(F) = \pm 1$. Results showing that $\deg(F) = 1$ for some classes of circuits were already obtained in the 1970's in work of Wu [52] and Chua and Wang [13], and we describe one such result (Theorem 3.1). This result already applies to a large class of circuits of practical interest. Our main new result of section 3 is a result implying that $\deg(F) = \pm 1$ for operating point equations of circuits in Sandberg-Willson form with nonlinear elements satisfying a suitable passivity condition (Theorem 3.2). This condition is quite general. It applies to circuits using bipolar junction transistors, and may well hold for all other transistor types. In any case it appears that most if not all transistor models can be easily modified outside the "physically relevant" parameter range to satisfy this passivity property. The resulting operating point equations then detect all the "physically relevant" operating points. We thus can construct operating point equations for which threading homotopies exist in principle; it remains an open problem to explicitly construct such homotopies. At the end of §3 we briefly sketch a class of "circuit deformation" homotopies, some of which have been used in circuit simulators (see [35], [43]). These homotopies satisfy a "no-gain" condition which insures properness of the homotopy, as observed in [36], [42]. It may well be that a subclass of these homotopies have the threading property.

The problem of explicitly constructing threading homotopies to find dc operating points seems to warrant further investigation, in view of the lack of reasonable alternative methods to find multiple dc operating points for nonlinear circuits. We are not aware of any existing method that can specify in advance the number of operating points of a given circuit, and this seems to rule out approaches that follow distinct paths to find each zero separately. Other zero-finding methods that proceed by a grid search to find zeros would be prohibitively slow due to the very large dimensionality of the search space for any reasonable sized circuit. Various algorithms have been given to find all operating points for piecewise linear models of circuits, see Chua and Ying [14], Pastore and Premoli [39] and Yamamura [54]. Here the enormous dimensionality of the search space presents difficulties. In contrast, methods that trace a single connected component can be immediately implemented in any software that uses continuation methods. Indeed they are already in use, but at present come with no guarantee of finding all dc operating points (see [35], [36], [43].) Finally we note a recent approach using multi-parameter homotopies, proposed by Wolf and Sanders [51].

This paper presents rigorous results for functions $F(\mathbf{x})$ and homotopies that are continuously differentiable. Similar questions can be raised for piecewise linear functions $F(\mathbf{x})$ using piecewise linear homotopies. Piecewise linear functions and homotopies have been considered in modelling nonlinear circuits, see for example Huang and Liu [30], Ohtsuki et al. [38], and Vandenberghe et al. [45].

2. Existence of Threading Homotopies

We derive necessary conditions and sufficient conditions for the existence of threading homotopies. The basic invariant used is the topological degree of a mapping. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and suppose that F is proper over 0, i.e. that the zero set $\Gamma(F)$ is compact. If $\Gamma(F) \subseteq B(\mathbf{0}, T) = \{\mathbf{x} : ||\mathbf{x}|| < T\}$, and $S^{n-1} = \{\mathbf{x} : ||\mathbf{x}|| = 1\}$ then for $R > T$ the map F induces a mapping $\phi_{F,R} : S^{n-1} \to S^{n-1}$ given by

$$\phi_{F,R}(\mathbf{x}) = \frac{F(R\mathbf{x})}{||F(R\mathbf{x})||} \quad \text{for} \quad \mathbf{x} \in S^{n-1} .$$

The homotopy class of $\phi_{F,R}(\mathbf{x})$ in the homotopy group $\pi_{n-1}(S^{n-1}) \cong \mathbf{Z}$ is independent of $R > T$ and is called the *degree of* F, denoted $\deg(F)$. We identify $\pi_{n-1}(S^{n-1})$ with $\mathbf{Z}$, using the isomorphism in which the degree of the identity map is 1, and henceforth view $\deg(F)$ as an integer.

Now suppose that the zeros of F are isolated and finite in number. The *index* $\mathrm{ind}_{\mathbf{x}_0}(F)$ is defined for any isolated zero $\mathbf{x}_0$ of a continuous function $F(\mathbf{x})$, as the degree of the mapping $F_\epsilon : S^{n-1} \to S^{n-1}$ given by

$$F_\epsilon(\mathbf{x}) := \frac{F(\mathbf{x}_0 + \epsilon\mathbf{x})}{||F(\mathbf{x}_0 + \epsilon\mathbf{x})||} \ , \quad ||\mathbf{x}|| = 1 \ ,$$

for small enough positive ϵ (see Cronin [9], p. 53); any integer can occur as a value of $\mathrm{ind}_{\mathbf{x}_0}(F)$. The *degree of* F is given in terms of the indexes of the zeros of F by

$$\deg(F) = \sum_{F(\mathbf{x}_0)=\mathbf{0}} \mathrm{ind}_{\mathbf{x}_0}(F) \ . \tag{2.1}$$

More generally, for an open set U in $\mathbb{R}^n$ whose closure $\bar{U}$ is compact, and with $F(\mathbf{x}) \neq \mathbf{0}$ everywhere on its boundary ∂U, we set

$$\deg(F;U) := \sum_{\substack{F(\mathbf{x}_0)=\mathbf{0} \\ \mathbf{x}_0 \in U}} \mathrm{ind}_{\mathbf{x}_0}(F) \ .$$

Now suppose that F is C^1. A zero $\mathbf{x}_0$ of $F(\mathbf{x})$ is *nondegenerate* if $\det(DF(\mathbf{x}_0)) \neq 0$. (Nondegenerate zeros are always isolated.) The *index* of a nondegenerate zero $\mathbf{x}_0$ then satisfies

$$\mathrm{ind}_{\mathbf{x}_0}(F) = \mathrm{sign}\ (\det(DF(\mathbf{x}_0)) = \pm 1 \ .$$

The degree is an invariant of homotopies which are weakly proper over 0, in the following sense. Suppose the C^r function $H(\mathbf{x}, \lambda) : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ is proper over 0, where $r \geq 1$, and set $F_\lambda = H(\mathbf{x}, \lambda)$. Assume that $\mathbf{0}$ is a regular value for H, for F_0, and for F_1, i.e. all three Jacobians DH, DF_0 and DF_1 have rank n at all points of the zero set. Then $H^{-1}(\mathbf{0})$ is a one-dimensional "neat" C^r-submanifold of $\mathbb{R}^n \times [0,1]$ (Hirsch [28], 1.4.1). This 1-manifold is compact because H is proper over 0, so the zero set does not "escape to infinity." Then one has $\deg(F_0) = \deg(F_1)$, by an easy adaptation of the proof of Corollary 5.1.3 of Hirsch [28]. Similarly, if U is as above, and $H^{-1}(\mathbf{0})$ is disjoint from $\partial U \times [0,1]$, then $\deg(F_0; U) = \deg(F_1; U)$. The necessity for the assumption "proper over 0" in such a homotopy is shown (for $n = 1$) by:

$$H(\mathbf{x}, \lambda) = \frac{2}{\pi}\ \arctan(x) - \lambda \ ,$$

where "escape to infinity" occurs, and $\deg(F_0) \neq \deg(F_1)$.

We give a necessary condition for the existence of a semi-threading homotopy. Call a C^1 homotopy H *nondegenerate* if its Jacobian $DH(x, \lambda)$ has full rank n at every zero of $H(\mathbf{x}, \lambda)$.

Theorem 2.1. *Suppose that the zero set of a C^1-mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ consists of a finite number of isolated nondegenerate zeros. If the C^1-function $H(\mathbf{x}, \lambda) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a nondegenerate semi-threading homotopy extending $F(\mathbf{x})$, then the degree of F is 1, 0 or -1.*

The simple proof of this result is based on the following well-known fact, which concerns the index of successive zeros encountered in following a continuation method path having no bifurcations. It is essentially 5.1.1 in Hirsch [28], who however assumes all functions are C^∞, see also [3], Corollary 11.5.6. We include a proof for the reader's convenience.

Lemma 2.1. *Suppose that the C^1-mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ has a finite zero set with all zeros nondegenerate. If $H(\mathbf{x}, \lambda) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a C^1-function with $H(\mathbf{x}, 0) = F(\mathbf{x})$ and the Jacobian $DH(\mathbf{x}, \lambda)$ has full rank n at every zero of H, then any two consecutive zeros $\mathbf{x}'$, $\mathbf{x}''$ of $F(\mathbf{x})$ found by traversing a solution curve $(\mathbf{x}(t),\ \lambda(t))$ of $H(\mathbf{x}, \lambda) = 0$ have opposite index, i.e.*

$$\det(DF(\mathbf{x}')) \det(DF(\mathbf{x}'')) < 0 \ . \tag{2.2}$$

Proof. By the implicit function theorem $(\mathbf{x}(t),\ \lambda(t))$ is locally defined and C^1 in a neighborhood of every zero $(\mathbf{x}_0,\ \lambda_0)$ of $H(\mathbf{x},\ \lambda)$. When traversing the curve $(\mathbf{x}(t),\ \lambda(t))$, in the zero set $\Gamma(H)$ from $\mathbf{x}'$ to $\mathbf{x}''$, the augmented gradient of H is:

$$\mathbf{J} := \begin{bmatrix} D\tilde{H} & \dfrac{\partial H}{\partial \lambda} \\[2ex] \dfrac{d\mathbf{x}}{dt} & \dfrac{d\lambda}{dt} \end{bmatrix} ,$$

in which $D\tilde{H} = \left[\frac{\partial H}{\partial \mathbf{x}_1} \cdots \frac{\partial H}{\partial \mathbf{x}_n}\right]$ and $\frac{d\mathbf{x}}{dt} = \left(\frac{d\mathbf{x}_1(t)}{dt}, \ldots, \frac{d\mathbf{x}_n(t)}{dt}\right)$. The augmented Jacobian $\det(\mathbf{J})$ is always nonzero, because the tangent vector $\mathbf{v} = \left(\frac{d\mathbf{x}}{dt}\ ,\ \frac{d\lambda(t)}{dt}\right)$ to the curve is perpendicular to the row space of $DH(\mathbf{x}(t),\ \lambda(t))$. Hence $\det(\mathbf{J})$ has constant sign; call this sign $\hat{\epsilon}$. In addition this perpendicularity gives

$$\begin{bmatrix} D\tilde{H} & \dfrac{\partial H}{\partial \lambda} \\[2ex] \dfrac{d\mathbf{x}}{dt} & \dfrac{d\lambda}{dt} \end{bmatrix} \begin{bmatrix} I & \dot{\mathbf{x}}^T \\[2ex] \mathbf{0} & \dfrac{d\lambda}{\partial t} \end{bmatrix} = \begin{bmatrix} D\tilde{H} & \mathbf{0} \\[2ex] \dfrac{d\mathbf{x}}{dt} & 1 \end{bmatrix} ,$$

because $\left(\frac{d\lambda}{dt}\right)^2 + \sum_{i=1}^{n}\left(\frac{dx_i}{dt}\right)^2 = 1$, using the arclength parametrization. Taking determinants, we obtain

$$\det(\mathbf{J})\frac{d\lambda}{dt} = \det(D\tilde{H}) \ .$$

At a point $(\mathbf{x}(t'),\ \lambda(t')) = (\mathbf{x}',\ 0)$ which gives a zero of F, $D\tilde{H}(\mathbf{x}') = DF(\mathbf{x}')$ hence

$$\text{sign } (\det DF(\mathbf{x}')) = \hat{\epsilon} \text{ sign}\left(\frac{d\lambda}{dt}\right) \ . \tag{2.3}$$

If $t' < t''$ are two consecutive zeros of $\lambda(t)$ along the curve, then the sign of $\lambda(t)$ is constant on the interval (t', t''), while

$$\text{sign } \left(\frac{d\lambda}{dt}(t')\right) = -\text{ sign}\left(\frac{d\lambda}{dt}(t'')\right) \ .$$

Then (2.3) shows that $\det DF(\mathbf{x}')$ and $\det(DF(\mathbf{x}''))$ have opposite signs, and (2.2) follows. ∎

Proof of Theorem 2.1. Suppose that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are the zeros of $F(\mathbf{x})$ in the order they are encountered when traversing, in a fixed direction, the curve $\{(\mathbf{x}(t),\ \lambda(t)) : t \in \mathbb{R}\}$ comprising the connected component of the zero set $\Gamma(H)$ that contains the zeros of $F(\mathbf{x})$. By Lemma 2.1,

$$\text{ind}_{\mathbf{x}_i}(F) + \text{ ind}_{\mathbf{x}_{i+1}}(F) = 0 \ .$$

Applying this in pairs, we have $\deg(F) = 0$ if $F(\mathbf{x})$ has an even number of zeros, and

$$\deg(F) = \text{ ind}_{\mathbf{x}_m}(F) = \pm 1$$

if $F(\mathbf{x})$ has an odd number of zeros. ∎

This degree constraint of Theorem 2.1 is automatically satisfied in dimension $n = 1$, and in that case the homotopy

$$H(x, \lambda) = F(x) - \lambda \tag{2.4}$$

is always a threading homotopy. However in dimensions $n \geq 2$ the degree constraint is a nontrivial obstruction:

Corollary 2.1. *Let $p(z) = \sum_{j=0}^{d} a_j z^j$ be a polynomial of degree $d \geq 2$ with distinct roots. If $p(z)$ is regarded as a mapping $p : \mathbb{C} \to \mathbb{C}$, then there exists no semi-threading homotopy $H(z, \lambda) : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ for p.*

Proof. The index of each simple zero of a polynomial $p(z)$ is 1. To see this, translate the zero to $z = 0$, and by simplicity of the zero only linear terms in $p(z)$ contribute to the index, so without loss of generality suppose that $p(z) = \alpha z$, with $\alpha \neq 0$. Write $\alpha = a + bi$ and $z = x + yi$, and viewing $p(z) = (Re(p(z)), Im(p(z))$ in $\mathbb{R}^2$ one finds

$$Dp(\mathbf{0}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

hence $\det(Dp(0)) = a^2 + b^2 > 0$ since $\alpha \neq 0$. Thus $\deg(p)$ is just the algebraic degree of $p(z)$, and is at least 2 if $p(z)$ is not linear, hence Theorem 2.1 gives the result. ∎

Corollary 2.1 also holds for polynomials $p(z)$ having multiple zeros, using the general definition of index, cf. Milnor [37], p. 32, but it does not apply to general multivariate polynomial maps $P : \mathbb{C}^n \to \mathbb{C}^n$. For $n \geq 2$ such a polynomial map can have an isolated zero with index -1. However one can show that if the map $P(\mathbf{z})$ has real coefficients, then all nondegenerate real zeros of $P(\mathbf{z})$ have index 1, see Cronin [9], Lemma 9.3.2. In particular, if such a map has at least two zeros, with all zeros real and nondegenerate, then $\deg(P) \geq 2$, so that Theorem 2.1 applies to show that no threading homotopy exists.

We next establish sufficiency of the condition $\deg(F) = 0$, or ± 1 for the existence of a threading homotopy for C^2 mappings in dimensions $n \neq 2$. For this we introduce a condition stronger than "weakly proper over 0." Call a function $H(\mathbf{x}, \lambda)$ *$\mathbb{R}$-proper over 0* if there is a compact set $B \subseteq \mathbb{R}^n$ such that $H^{-1}(\mathbf{0}) \subseteq B \times \mathbb{R}$. This is "weakly proper over 0" with an additional uniformity condition in the $\mathbb{R}$-direction.

Theorem 2.2. *For any $n \neq 2$, let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^r-mapping $(2 \leq r \leq \infty)$ whose zero set consists of a finite number of isolated nondegenerate zeros. If $\deg(F)$ is $1, 0$ or -1 then there exists a nondegenerate threading homotopy $H(\mathbf{x}, \lambda) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ for F, such that H is a C^r mapping which is $\mathbb{R}$-proper over 0.*

We do not know if Theorem 2.2 is true when $n = 2$.

The main part of the proof is:

Lemma 2.2. *Suppose that $n \geq 3$ and that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^r-mapping $(2 \leq r \leq \infty)$ which has exactly two zeros $\mathbf{x}^{\pm} = (\pm 1, 0, \ldots 0)$ with $ind_{\mathbf{x}^+}(F) = 1$ and $ind_{\mathbf{x}^-}(F) = -1$. Then there exists a C^r-homotopy $H(\mathbf{x}, \lambda) : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ with $H(\mathbf{x}, 0) = F(\mathbf{x})$, stationary outside a preassigned neighborhood of the line segment connecting $\mathbf{x}^+$ and $\mathbf{x}^-$ such that $H^{-1}(\mathbf{0})$ is a C^r*

embedded "neat" arc in $\mathbb{R}^n \times [0,1]$.

Here, as in Hirsch [28], p. 30, "neat" means that the arc meets the boundary at $(\mathbf{x}^+, 0)$ and at $(\mathbf{x}^-, 0)$ in a C^r-manner.

A form of Lemma 2.2 is essentially to be found in Whitney [46]; topologists call all of its variants "The Whitney Lemma." The condition $n \geq 3$ arises from Whitney's need to approximate a singular disk in $\mathbb{R}^{2n}$ by an embedded disk. It is not clear to us however that the proof in [46] avoids introducing extra circle components in H^{-1} $(\mathbf{0})$ $\cap(\mathbb{R}^n \times [0,1])$: compare the difference between semi-threading and threading above. However this problem is avoided in a rather elementary proof of Lemma 2.2 appearing in Jezierski [27, Lemma 2.2]. The proof of Jezierski makes no mention of embedded disks. Rather, it uses advanced calculus and the fact that spheres of dimension ≥ 2 are simply connected. Like Whitney's proof, it is presented for the C^∞-case; however the proof requires only the hypothesis C^2, hence our restriction $r \geq 2$. Jezierski uses $n \geq 3$ for the property of $(n-1)$-spheres mentioned above.

Proof of Theorem 2.2. The necessary degree condition was already shown to be sufficient in dimension 1, see (2.4), so suppose that $n \geq 3$. Lemma 2.2 shows how to "remove" a pair of zeros of opposite degree. Now suppose $\deg(F) = \pm 1$. Then one can arrange the zeros in an order $x_1,\ x_2, \ldots, x_{2m+1}$ so that consecutive zeros have opposite degree. One can find arcs connecting them in pairs $(x_i,\ x_{i+1})$ so that all tubular neighborhoods of disjoint pairs are disjoint. One can then combine the homotopies above for the pairs (x_1, x_2), $(x_3, x_4), \ldots$ (x_{2m-1}, x_{2m}), with homotopy parameter $1 \geq \lambda \geq 0$ and those for (x_2, x_3), $(x_4, x_5), \ldots, (x_{2m}, x_{2m+1})$, with homotopy parameter $0 \geq \lambda \geq -1$, to obtain a threading homotopy, which is $\mathbb{R}$-proper over 0. With care, one can ensure that the "combined" homotopy is still C^r; see Jezierski [31] for a discussion of similar matters. A slight and obvious modification handles the case $\deg(F) = 0$. ∎

Finally we establish the sufficiency of the condition $\deg(F) = 0$ or ± 1 for the existence of a semi-threading homotopy for C^1 mappings in all dimensions $n \geq 1$. We include this result because it is the best we can do when $n = 2$, and, while it obtains a weaker conclusion than Theorem 2.2, it has a more elementary proof.

Theorem 2.3. *For all* $n \geq 1$, *let* $F : \mathbb{R}^n \to \mathbb{R}^n$ *be a* C^r*-mapping* $(1 \leq r \leq \infty)$ *whose zero set consists of a finite number of isolated nondegenerate zeros. If* $\deg(F) = 1,\ 0$ *or* -1, *then there* *exists a nondegenerate C^r homotopy $H(\mathbf{x},\lambda):\mathbb{R}^{n+1}\to\mathbb{R}^n$ extending F, which is weakly proper over 0, such that:*

(i). *All zeros of $F(\mathbf{x})$ lie on a single connected component of the zero set of $H(\mathbf{x},\lambda)$.*

(ii). *All other components of the zero set of $H(\mathbf{x},\lambda)$ are closed loops on which $0<|\lambda|<1$.*

Proof. The necessary degree condition was already shown to be sufficient in dimension 1, see (2.4). For $n \geq 2$ we use an approach which starts from the proof of Lemma 5.2.9 of Hirsch [28]. That lemma essentially shows that given two zeros of degree 1 and -1 respectively, together with an arc connecting them, and a tubular neighborhood U_2 of the arc, then there is a continuous function G agreeing with F outside U_2 which has no zeros in U_2. That lemma is stated for C^∞ maps, but the cited proof, and all other proofs cited below, go through without change for C^r maps, with $r \geq 1$.

Now pick a nested collection of tubular neighborhoods $U_2 \subset U_1 \subset U_0 \subset U$, where the closure of each lies in the next; to find these, use the Tubular Neighborhood Theorem, Theorem 4.5.2 of Hirsch [28]. We have already used U_2. Note that $G = F$ outside U_1 so G is C^r outside U_1. We use the Relative Approximation Theorem 2.2.5 of Hirsch [28] to obtain a C^r map $\tilde{G}$ agreeing with $G \equiv F$ outside U_1, which is close enough to G inside U_2 that it has no zeros there. (To be specific: use that theorem with Hirsch's U and K being U_0 and with his W being U_1.)

Let $\phi:\mathbb{R}\to\mathbb{R}$ be a C^r-function satisfying $0 \leq \phi(\lambda) \leq 1$, with $\phi(\lambda) = 0$ for $\lambda \leq 0$, $\phi(\lambda) = 1$ for $\lambda \geq 1$, and define $J:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$ by

$$J(\mathbf{x},\lambda) = \phi(1-\lambda)F(\mathbf{x}) \ + \ \phi(\lambda)\tilde{G}(\mathbf{x}) \ .$$

Now J is C^r. Let $N = \overline{U_0 \times [0,1]}$. N has "corners" at $\partial U_0 \times \{0,1\}$, so N is a C^0-manifold but not a differentiable manifold. The set $(J|_{\partial N})^{-1}(\mathbf{0}) = \{\mathbf{x}', \mathbf{x}''\}$, where $\mathbf{x}'$ and $\mathbf{x}''$ are the two zeros of F that we are trying to remove. By "smoothing out the corners" (see Kirby and Siebenmann [32], pp. 8 and 119) we can find a differentiable manifold M lying in the interior of N with respect to $\mathbb{R}^n\times[0,\infty)$ — interior in the topological sense — such that $U_1\times\{0\}\subseteq M$, see Figure 2.1.

There is a relative transversality theorem, stated as "Corollary" on p. 73 of Guillemin and Pollack [27], which says that $J|_{\partial M}$ can be extended to a map $\tilde{J}: M\to\mathbb{R}^n$ which is transverse to

$\{\mathbf{0}\}$. Extend $\tilde{J}$ to $\mathbb{R}^n \times \mathbb{R}$ so as to be C^r and agree with J outside M. Then $(\tilde{J}|_M)^{-1}(\mathbf{0})$ includes an arc of zeros of the type desired. However $(\tilde{J}|_M)^{-1}(\mathbf{0})$ may also contain extra components, which are closed loops in the interior of M. If $\mathbf{x} \notin U_0$, then $\tilde{J}(\mathbf{x}, \lambda) = F(\mathbf{x}, \lambda)$ for all $\lambda \in \mathbb{R}$. Thus we have shown how to connect and "remove" a pair of zeros of opposite degree.

Now if $\deg(F) = -1,\ 0$ or 1 one proceeds exactly as in the proof of Theorem 2.2, to thread all the zeros together. ■

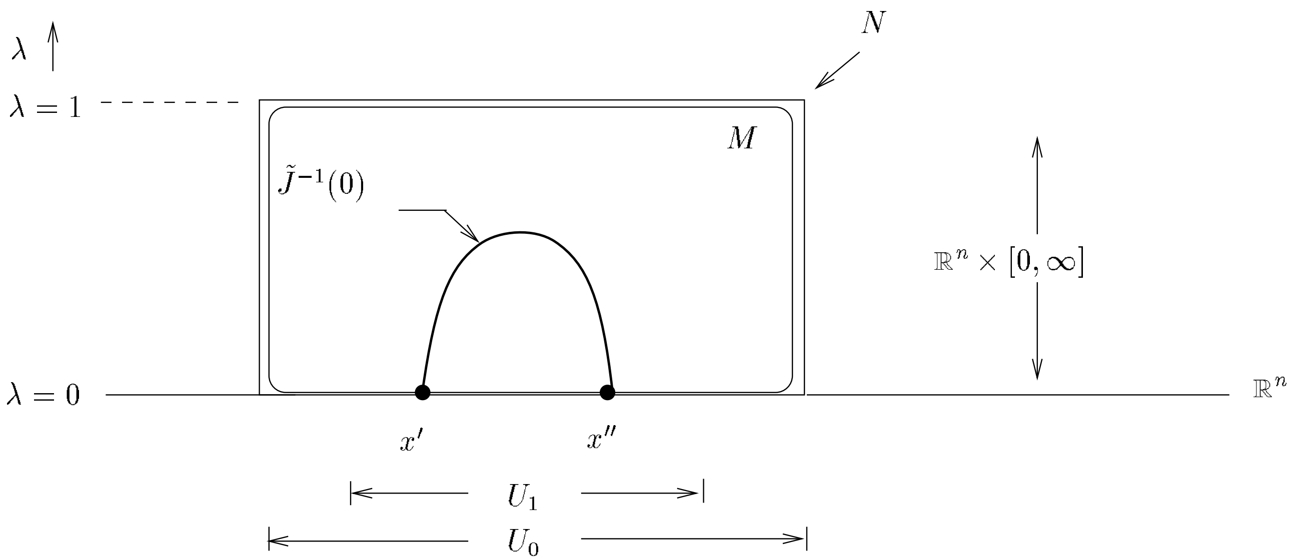


Figure 2.1. Smoothing out the corners

3. DC Operating Points of Nonlinear Resistive Circuits

The theory of dc operating points of transistor circuits is surveyed in Trajković and Willson [44] and, for work before 1974, in Willson [48]. In this section we make the theoretical observation that threading homotopy methods potentially apply to the dc operating point problem by showing that most circuits can be modelled with operating point equations $F(\mathbf{x}) = \mathbf{0}$ such that $\deg(F) = \pm 1$. It follows that there is no topological obstruction to the existence of threading homotopies for such equations, and they certainly exist whenever Theorem 2.2 applies, i.e. when F is C^2; see also Theorem 2.3.

There is a long history of results on topological degree applied to nonlinear networks. These methods were developed to prove the existence of dc operating points, for which it suffices to prove that $\deg(F)$ is odd, see Chua and Wang [12, Property 7]. The original method of Wu [52] uses passivity properties of the circuits to prove $\deg(F) = \pm 1$, and we follow this approach here.

We consider nonlinear circuits made up of transistors and nonlinear diodes, and driven by active sources which are current sources or voltage sources. A nonlinear resistive network is *passive* if

$$P(\mathbf{v}, \mathbf{i}) := \langle \mathbf{v}, \mathbf{i} \rangle := v_1 i_1 + v_2 i_2 + \ldots + v_n i_n \geq 0 \ , \tag{3.1}$$

for any allowed set of voltages $\mathbf{v}$ and currents $\mathbf{i}$. Here $P(\mathbf{v}, \mathbf{i})$ measures the *power* consumed by the network, and the passivity condition[2] asserts that a network never generates power internally, but may consume power. Circuits composed solely of nonlinear passive resistors and transistors with no voltage or current sources are passive.

For a general circuit we extract a set of n independent variables $\mathbf{x} = (x_1, \ldots, x_n)$ among the $2n$ variables $\{v_1, \ldots, v_n, i_1, \ldots, i_n\}$, one from each pair (v_j, i_j), and solve for the remaining variables $\mathbf{y} = (y_1, \ldots, y_n)$ using Kirchhoff's voltage and current laws, to obtain

$$\mathbf{y} = F(\mathbf{x}) \ .$$

That is, the variables $\mathbf{y}$ are uniquely determined as functions of $\mathbf{x}$, and we call $\mathbf{x}$ the *controlling variables*. The simplest case consists of *voltage-controlled circuits*, in which the controlling variables $\mathbf{v} = (v_1, \ldots, v_n)$ are the node voltages, giving potentials measured from a reference node ("ground") in the network, and the remaining variables $\mathbf{i} = (i_i, \ldots i_n)$ give the currents at each node. (There are no voltage or current variables for the reference node.) We may force the node voltage at node k to be v_k by attaching a new branch from the reference node to node k which either contains a voltage source with potential v_k or a current source with current i_k, with the branch oriented towards node k. We define the column vector

$$F(\mathbf{v}) := (F_1(\mathbf{v}), \ldots, F_n(\mathbf{v}))^T \ , \tag{3.2}$$

where $i_k = F_k(\mathbf{v})$ denotes the current at node k entering from the branch containing the voltage source v_k. The operating point equations for a voltage-controlled circuit with offered currents

[2] More general definitions of passivity are discussed in Chua et al. [10] and Wyatt et al. [53].

$\mathbf{i} = (i_1, i_2, \ldots, i_n)$ is

$$F(\mathbf{v}) = \mathbf{i} \ . \tag{3.3}$$

For fixed $\mathbf{i} \in \mathbb{R}^n$ this equation may have zero, one, or many solutions in $\mathbf{v}$. The power $P(\mathbf{v}, \mathbf{i})$ drawn by the circuit from the voltage sources is

$$P(\mathbf{v}, \mathbf{i}) := \langle \mathbf{v}, \mathbf{i} \rangle = \langle \mathbf{v}, F(\mathbf{v}) \rangle \ , \tag{3.4}$$

and the passivity condition asserts that $P(\mathbf{v}, \mathbf{i}) \geq 0$.

The relevance of a passivity condition to the topological degree of $F(\mathbf{v}) - \mathbf{i}$ is the following well-known fact.

Lemma 3.1. *If a function* $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ *satisfies a strict coercivity condition*

$$\langle \mathbf{x}, G(\mathbf{x}) \rangle > 0 \quad \text{if} \quad ||\mathbf{x}|| \geq R \ , \tag{3.5}$$

then $\deg(G) = 1$.

Proof. The condition (3.5) shows that all zeros of $G(\mathbf{x})$ lie in the compact set $||\mathbf{x}|| \leq R$. The map $\phi_{G,R}(\mathbf{x}) = \frac{G(R\mathbf{x})}{||G(R\mathbf{x})||}$ is homotopic to the identity map on S^{n-1} using radial projection of the map

$$G_\lambda(\mathbf{x}) = \lambda G(R\mathbf{x}) + (1 - \lambda)\mathbf{x}, \qquad \mathbf{x} \in S^{n-1} \ , \qquad 0 \leq \lambda \leq 1 \ .$$

onto S^{n-1}, which is well-defined since the strict coercivity condition gives $\langle \mathbf{x}, G_\lambda(\mathbf{x}) \rangle > 0$, hence $G_\lambda(\mathbf{x}) \neq \mathbf{0}$. Now $\deg(G) = 1$ by the invariance of degree for homotopies proper over 0, as explained in §2. ∎

If we set

$$F_{\mathbf{i}}(\mathbf{v}) := F(\mathbf{v}) - \mathbf{i} \tag{3.6}$$

then a sufficient condition for $F_{\mathbf{i}}(\mathbf{v})$ to satisfy a strict coercivity condition (3.5) can be given in terms of the power drawn by the circuit. We say that function $F(\mathbf{x})$ is *eventually strongly passive* if there exist $c > 0$ and $R > 0$, such that

$$\langle \mathbf{x}, F(\mathbf{x}) \rangle \geq c||\mathbf{x}||^2 \ , \quad \text{for } ||\mathbf{x}|| > R \ . \tag{3.7}$$

A positive linear resistor has this property. Eventual strong passivity of $F(\mathbf{x})$ implies eventual strong passivity of $F_{\mathbf{c}}(\mathbf{x})$ for each $\mathbf{c} \in \mathbb{R}^n$, since

$$\begin{aligned}
\langle \mathbf{x}, F_{\mathbf{c}}(\mathbf{x}) \rangle &= \langle \mathbf{x}, F(\mathbf{x}) \rangle - \langle \mathbf{x}, \mathbf{c} \rangle \\
&\geq c||\mathbf{x}||^2 - ||\mathbf{x}|| \ \ ||\mathbf{c}|| \ , \quad \text{if } ||\mathbf{x}|| > R \ , \\
&\geq \frac{1}{2} c||\mathbf{x}||^2 \ , \qquad \text{if } ||\mathbf{x}|| > R' \ ,
\end{aligned} \tag{3.8}$$

with $R' = \max(\frac{2}{c}||\mathbf{c}||, R)$.

We now present results which show that $\deg(F) = \pm 1$ for two large classes of circuit equations. One reasonably large class of transistor circuits has dc operating point equations that are of the form

$$\tilde{F}(\mathbf{x}) + P\mathbf{x} = \mathbf{s} \ , \tag{3.9}$$

in which $\tilde{F}(\mathbf{x})$ is a vector of n functions of $\mathbf{x}$ describing the effect of nonlinear elements of the circuit, assumed eventually passive (defined below), the conductance matrix P is assumed positive definite but not necessarily symmetric, and $\mathbf{s}$ is a constant describing the active sources in the circuit. Chua and Wang [13, theorem 2] prove the following result.

Theorem 3.1. *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function*

$$F(\mathbf{x}) := \tilde{F}(\mathbf{x}) + P\mathbf{x} \ , \tag{3.10}$$

in which P is a positive definite matrix and $\tilde{F}(\mathbf{x})$ is eventually passive, that is,

$$\langle \mathbf{x}, \tilde{F}(\mathbf{x}) \rangle \geq 0 \quad \textit{for all } ||\mathbf{x}|| \geq R \ . \tag{3.11}$$

Then $F(\mathbf{x})$ is eventually strongly passive, hence if $F_{\mathbf{c}}(\mathbf{x}) = F(\mathbf{x}) - \mathbf{c}$, then $\deg(F_{\mathbf{c}}) = 1$ for all $\mathbf{c} \in \mathbb{R}^n$.

Proof. Positive definiteness of P gives

$$\langle \mathbf{x}, P\mathbf{x} \rangle \geq c||\mathbf{x}||^2 \ , \quad \text{all} \quad \mathbf{x} \in \mathbb{R}^n \ ,$$

for some positive constant c. The eventual passivity condition for $\tilde{F}(\mathbf{x})$ yields

$$\langle \mathbf{x}, F(\mathbf{x}) \rangle \geq \langle \mathbf{x}, P\mathbf{x} \rangle \geq c||\mathbf{x}||^2 \quad \text{if } ||\mathbf{x}|| \geq R \ , \tag{3.12}$$

i.e. $F(\mathbf{x})$ is eventually strongly passive. Now $\deg(F_{\mathbf{c}}) = 1$ follows from (3.8) and Lemma 3.1. ∎

Theorem 3.1 is readily applicable to a wide class of practical circuits. Consider for the moment voltage-controlled circuits using bipolar junction transistors. These circuits are modelled using variants of the Ebers-Moll transistor model as a two-port in the common base configuration, see Appendix A. The resulting circuit equation has the form (3.9) except that P is positive semidefinite rather than positive definite. Existing circuit simulators, such as SPICE, add small shunt conductances to the Ebers-Moll model, see for example [4, p.14, 44, 45] where the variable is denoted GMIN. These conductances are modelled as two resistors with resistances $(GMIN)^{-1}$ between the base and the other two nodes of the transistor. If these resistors are migrated to the linear part of the circuit, this will change the matrix P to $P + \mathrm{diag}(GMIN)$, which is positive definite, and Theorem 3.1 applies. Green and Willson [26] give a detailed description of circuits satisfying Theorem 3.1.

We next prove a general result which applies to nonlinear circuits in the Sandberg-Willson form that separates linear and nonlinear parts of the circuit, see [40], [41], [47] and which assumes a weaker passivity condition than Theorem 3.1. This result applies to circuits with Ebers-Moll transistors without shunt conductances added. The nonlinear elements are treated as voltage-controlled, with response function

$$F(\mathbf{v}) = -\mathbf{i} \ , \quad \text{with} \quad \mathbf{i} = \begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} . \tag{3.13}$$

The linear part of the circuit has response

$$Q\mathbf{i} = P(\mathbf{v} - \mathbf{c}) \ , \tag{3.14}$$

in which (P, Q) are a *passive pair* of $n \times n$ matrices, i.e.

$$Q\mathbf{i} = P\mathbf{v} \quad \text{implies } \langle \mathbf{v}, \mathbf{i} \rangle = \mathbf{v}^T\mathbf{i} \geq 0 \ , \tag{3.15}$$

and $\mathbf{c}$ is a vector of constants representing independent sources. Any linear circuit consisting of positive linear resistors and independent voltage sources can be put in the form (3.14), as well as many linear circuits containing current sources, see Sandberg and Willson [41, Theorem 1 ff].

This set of equations is converted to circuit equations in Sandberg-Willson form by eliminating the current variables $\mathbf{i}$, to obtain the nonlinear system of equations

$$QF(\mathbf{v}) + P(\mathbf{v} - \mathbf{c}) = \mathbf{0} \ . \tag{3.16}$$

We establish the following result.

Theorem 3.2. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-mapping and consider the mapping $\tilde{G} : \mathbb{R}^n \to \mathbb{R}^n$ given by*

$$\tilde{G}(\mathbf{x}) := QF(\mathbf{x}) + P(\mathbf{x} - \mathbf{c}) \ , \tag{3.17}$$

in which (P, Q) is a passive pair of $n \times n$ matrices, and $\mathbf{c}$ is given. If there exists $R > 0$ such that $F(\mathbf{x})$ satisfies

$$\langle \mathbf{x} - \mathbf{c}, \ F(\mathbf{x}) \rangle > 0 \quad for \ \ ||\mathbf{x}|| > R \ , \tag{3.18}$$

then $\deg(\tilde{G}) = \pm 1$.

Remark. The condition (3.18) is a passivity condition that is much less stringent than the eventually strong passivity condition (3.7). Note also that the form of $\tilde{G}(\mathbf{x})$ can apply to operating point equations using any set of controlling variables (hybrid variables) rather than voltages.

Proof. We first study the $2n \times 2n$ system $G = (G_1, G_2)$ given by

$$\begin{aligned} G_1(\mathbf{x}, \mathbf{y}) &:= F(\mathbf{x}) + \mathbf{y} \\ G_2(\mathbf{x}, \mathbf{y}) &:= Q\mathbf{y} - P(\mathbf{x} - \mathbf{c}) \ . \end{aligned}$$

We consider the homotopy $H : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ given by $H = (H_1, H_2)$ with

$$\begin{aligned} H_1(\mathbf{x}, \mathbf{y}, \lambda) &:= (1 - \lambda) F(\mathbf{x}) + \lambda(\mathbf{x} - \mathbf{c}) + \mathbf{y} \ , \\ H_2(\mathbf{x}, \mathbf{y}, \lambda) &= Q\mathbf{y} - P(\mathbf{x} - \mathbf{c}) \ . \end{aligned} \tag{3.19}$$

We will show that H is a homotopy proper over $\mathbf{0}$ and that $\deg(H(\mathbf{x}, \mathbf{y}; 1)) = \pm 1$. This will imply that $\deg(G) = \deg(H(\mathbf{x}, \mathbf{y}, 0)) = \pm 1$ by invariance of degree for proper homotopies.

To see that H is a proper homotopy, we show that all zeros of $H(\mathbf{x}, \mathbf{y}, \lambda)$ for $0 \le \lambda \le 1$ lie in a compact set. Any such zero satisfies

$$(1 - \lambda) F(\mathbf{x}) + \lambda(\mathbf{x} - \mathbf{c}) + \mathbf{y} = \mathbf{0} \tag{3.20}$$

$$Q\mathbf{y} = P(\mathbf{x} - \mathbf{c}) \ .$$

Now

$$\langle \mathbf{x}-\mathbf{c}, H_1(\mathbf{x},\mathbf{y},\lambda)\rangle = (1-\lambda)\langle \mathbf{x}-\mathbf{c}, F(\mathbf{x})\rangle + \lambda||\mathbf{x}-\mathbf{c}||^2 + \langle \mathbf{x}-\mathbf{c}, \mathbf{y}\rangle \ .$$

The passive pair condition gives

$$\langle \mathbf{x}-\mathbf{c}, \mathbf{y}\rangle \geq 0 \ ,$$

which with (3.18) gives for $0 \leq \lambda \leq 1$ that

$$\langle \mathbf{x}-\mathbf{c}, H_1(\mathbf{x},\mathbf{y},\lambda)\rangle > 0 \quad if \quad ||\mathbf{x}|| > R' \ ,$$

where we define $R' = \max(R, ||\mathbf{c}||)$.

To see that $\deg(H(\mathbf{x},\mathbf{y},1)) = \pm 1$, we observe that $G^*(\mathbf{x},\mathbf{y}) = H(\mathbf{x},\mathbf{y},1)$ has

$$\begin{aligned} G_1^*(\mathbf{x},\mathbf{y}) &= \mathbf{x}-\mathbf{c}+\mathbf{y} \ . \\ G_2^*(\mathbf{x},\mathbf{y}) &= G_2(\mathbf{x},\mathbf{y}) \ = \ Q\mathbf{y} - P(\mathbf{x}-\mathbf{c}) \ . \end{aligned}$$

Thus any zero of G^* has $\mathbf{y} = -(\mathbf{x}-\mathbf{c})$ and the equation $G_2^*(\mathbf{x},\mathbf{y}) = \mathbf{0}$ becomes

$$Q(-(\mathbf{x}-\mathbf{c})) = P(\mathbf{x}-\mathbf{c}) \ .$$

Since (P,Q) is a passive pair, this gives

$$-(\mathbf{x}-\mathbf{c})^T(\mathbf{x}-\mathbf{c}) = -||\mathbf{x}-\mathbf{c}||^2 \geq 0 \ .$$

This forces $\mathbf{x} = \mathbf{c}$, hence G^* has a unique zero $(\mathbf{c}, \mathbf{0})$. Since G^* is an affine map that has a unique zero, it is invertible, hence its Jacobian $\det(DG^*)$ does not vanish. Thus

$$\deg(H(\mathbf{x},\mathbf{y},1)) = \deg(G^*) = sgn\left(\det\left|\begin{array}{cc} I & I \\ -P & Q \end{array}\right|\right) = \pm 1 \ . \tag{3.21}$$

We have now established that

$$\deg(G) = \deg(H(\mathbf{x},\mathbf{y},0)) = \deg(H(\mathbf{x},\mathbf{y},1)) = \pm 1 \ .$$

To complete the proof, we show that $|\deg(\tilde{G})| = |\deg(G)|$. To do this let $K : \mathbb{R}^{2n+1} \to \mathbb{R}^n$ be the homotopy $K = (K_1, K_2)$ with

$$\begin{aligned} K_1(\mathbf{x},\mathbf{y},\lambda) &:= G_1(\mathbf{x}, \mathbf{y} - \lambda F(\mathbf{x})) \ , \\ K_2(\mathbf{x},\mathbf{y},\lambda) &:= G_2(\mathbf{x}, (1-\lambda)\mathbf{y} - \lambda F(\mathbf{x})) \ . \end{aligned} \tag{3.22}$$

Now K is proper over $\mathbf{0}$ by an argument similar to that given for H. Also

$$K(\mathbf{x}, \mathbf{y}, 0) = G(\mathbf{x}, \mathbf{y}) \quad \text{and} \quad K(\mathbf{x}, \mathbf{y}, 1) = (\mathbf{y}, -\tilde{G}(\mathbf{x})) \ , \tag{3.23}$$

so we have $\deg(G) = \deg(K(\mathbf{x}, \mathbf{y}, 1))$. Interchanging coordinates in $K(\mathbf{x}, \mathbf{y}, 1)$ and multiplying by -1 does not change the absolute value of the degree, hence

$$|\deg(G)| = |\deg(K(\mathbf{x}, \mathbf{y}, 1)| = |\deg \tilde{K}(\mathbf{x}, \mathbf{y})| \ , \tag{3.24}$$

where

$$\tilde{K}(\mathbf{x}, \mathbf{y}) = (\tilde{G}(\mathbf{x}), \mathbf{y}) = (\tilde{G} \times I)(\mathbf{x}, \mathbf{y}) \ . \tag{3.25}$$

Now $\deg(\tilde{K}(\mathbf{x}, \mathbf{y})) = \deg(\tilde{G}(\mathbf{x}))$, which proves that $|\deg(\tilde{G})| = |\deg(G)| = 1$. ∎

Theorem 3.2 applies to nearly all transistor circuits of practical interest. To verify the passivity condition (3.18), it suffices to check it on each nonlinear circuit element separately. For example, it holds for the Ebers-Moll model for bipolar junction transistors for all $\mathbf{c} \in \mathbb{R}^2$ as is shown in Sandberg and Willson [41, Theorem 5], see Theorem A.1 in the Appendix. If there are nonlinear elements for which (3.18) does not hold, we may modify their responses for large $||\mathbf{x}||$ to force (3.18) to hold. In this way we obtain modified operating point equations that detect all the "physically relevant" dc operating points. We propose such model modifications purely as artificial adjustments to the transistor model, but actual transistors exhibit breakdown behavior which is roughly equivalent to a passivity property like (3.18).

The degree results show that for most circuits there exist in principle network equations having threading homotopies. It remains an open problem to find explicit threading homotopies for particular classes of network equations.

One of the difficulties in using homotopy methods in circuit simulators to find all zeros is to force properness of the homotopy, to prevent zeros "escaping to infinity." Trajković et al. [42] and Melville et al. [36] noted that this can be achieved for various circuits that have the "no-gain" property defined in Willson [49] and Chua et al. [11]. A circuit has the *no-gain property* if for any set of attached independent sources (either voltage or current sources), the voltage difference between any two nodes of the circuit does not exceed the absolute values of voltages across all the independent sources, and the current flowing into any node does not exceed the sum of the magnitudes of currents flowing through all the independent sources. In

[49] it is shown that all connected networks composed of two-terminal and three-terminal no-gain elements have the no-gain property, and that linear resistors, bipolar junction transistors and MOSFETS all have the no-gain property. Suppose that one can find a homotopy $H(\mathbf{x}, \lambda)$ for $0 \leq \lambda \leq 1$ with the two properties:

(i). $H(\mathbf{x}, \lambda) = F_\lambda(\mathbf{x})$, where each $F_\lambda(\mathbf{x})$ is the operating point equation for a circuit C_λ that has the no-gain property, for $0 \leq \lambda \leq 1$.

(ii). $H(\mathbf{x}, 0) = F(\mathbf{x})$ and $H(\mathbf{x}, 1) = F_1(\mathbf{x})$ corresponds to a circuit with a unique operating point.

The no-gain property of all circuits C_λ then implies that the homotopy is proper. In this case it directly follows that $\deg(F) = 1$ from the invariance of degree for proper homotopies, because $\deg F_0(\mathbf{x})) = 1$ by (ii). Such "no-gain" homotopies can often be found by varying the parameters of the circuit elements, as described in [36]. The particular usefulness of such "no-gain" homotopies is to give a priori bounds on a region containing all zeros of such homotopies, see Trajković et al [42]. These bounds provide a simple error check on correctness of homotopy computations.

There is a natural class of candidate homotopies to consider for use in circuit simulators, which we may call *sandwich homotopies*, that may well include threading homotopies. These homotopies are constructed using circuit deformation homotopies $\{H(\mathbf{x}, \lambda) : 0 \leq \lambda \leq 1\}$, which deform the circuit parameters of a no-gain circuit to obtain a circuit having a unique operating point. A *sandwich homotopy* consists of combining two circuit deformation homotopies which vary the circuit parameters in different ways, with one used on $0 \leq \lambda \leq 1$, and the other on $-1 \leq \lambda \leq 0$, and then we set $H(\mathbf{x}, \lambda) \equiv H(\mathbf{x}, 1)$ for $\lambda \geq 1$, and $H(\mathbf{x}, \lambda) = H(\mathbf{x}, -1)$ for $\lambda \leq -1$. Some care is needed to make such a homotopy C^2 at the boundary values $\lambda = 1$, 0 and -1.

We describe one kind of *circuit-deformation* homotopy, following the approach of Melville et al. [36], for circuits consisting of linear resistors and bipolar junction transistors. First, the coupling elements in the bipolar junction transistors, the forward and reverse gains,[3] are each reduced monotonically to zero. By results of Willson [49], the transistors produced during this process retain the no-gain property throughout. Now one has a network of uncoupled diodes

[3] These are the gain parameters α_F and α_R appearing in the Appendix.

whose $v - i$ curves are eventually monotone, i.e. either $f'(x) > 0$ for $|x| > R$. The second part of the homotopy is to deform the voltage-current curves of the diodes to make them all monotone, by a C^2-homotopy applied to each voltage-current curve on a bounded region. (The diode $v - i$ curves must satisfy some mild conditions for this to be possible. If $f'(x) > 0$ for $|x| \geq R$ then $f(-R) < 0 < f(R)$ suffices.) The resulting circuit of strictly monotone diodes has a unique operating point, by a well-known result of Duffin [14, Theorem 3]. One wants such homotopies $H(\mathbf{x}, \lambda)$ to be bifurcation-free, i.e. for the rank n condition (iii) above to be satisfied. This can be done by allowing a space of small C^2 deformations around the homotopy described above, using the approach of Chow, Mallet-Paret and Yorke [7]. These homotopies certify that $\deg(F_0) = 1$, because $\deg(F_1) = 1$ by the result of Duffin [18] and the homotopy can be shown to be proper using the no-gain condition.

Sandwich homotopies come with no guarantee of being threading homotopies. However they have successfully been used to find more than one operating point, see Green and Melville [25]. In particular Melville et al. [36] describe a variable-gain homotopy which seems to work well in practice, and which has been implemented in `Sframe`, a circuit simulation platform, see [35]. Some of these homotopies have been observed empirically to have the threading property. Perhaps a subclass of them can be proved to have the threading property, using Diener's condition (1.2) or an analogous criterion.

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Appendix. Ebers-Moll model for bipolar junction transistors

The Ebers-Moll large signal model ([4], [19], [22]) for a bipolar junction transistor is pictured in Figure A.1. This is the injection version of the Ebers-Moll model given in Getreu [22] , p. 12. A *node* in a circuit designates a connected set of points in the circuit which are all at the same voltage with respect to a reference point, usually called ground. There are only three nodes in the Ebers-Moll model- *collector*, *base*, and *emitter*. In Figure A.1 the base node has been drawn as two terminals (b_1, b_2) in order to treat the transistor as a two-port; this arrangement is conventionally called the *common base configuration* for the Ebers-Moll model. This circuit element contains two nonlinear diodes with (different) response curves of the form

$$f(v) = m(e^{nv} - 1) \ , \tag{A.1}$$

where m and n are both positive parameters. The *exponential diodes* (A.1) are sometimes called Ebers-Moll diodes. It also contains two current-controlled current sources with *current gains* α_F, α_R that satisfy $0 \leq \alpha_F, \alpha_R < 1$. The current flowing through a current-controlled current source is equal to a fixed current gain α times a controlling current I flowing on a branch somewhere else in the circuit. Thus a current-controlled current source is a linear element that produces coupling between different parts of the circuit. Figure A.1 models specifically an *npn* transistor; the model for a *pnp* transistor is obtained by systematically reversing the current flow throughout this model.

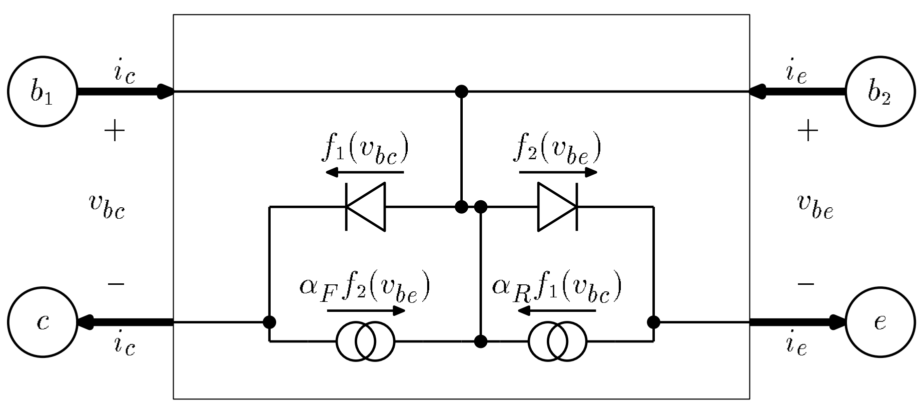

Figure A.1 Ebers-Moll model (common base configuration)

This two-port can be viewed as a voltage-controlled two-port with the current responses

$$\begin{bmatrix} i_c \\ i_e \end{bmatrix} = \begin{bmatrix} 1 & -\alpha_F \\ -\alpha_R & 1 \end{bmatrix} \begin{bmatrix} f_1(v_{bc}) \\ f_2(v_{be}) \end{bmatrix} , \tag{A.2}$$

where v_{be} and v_{bc} are the branch voltages. For example in Figure A.1 the current i_c flowing out of the two-port into the collector terminal is the sum of two components: a current $f_1(v_{bc})$ flowing in the same direction as i_c and a current $\alpha_F f_2(v_{bc})$ flowing in the opposite direction as i_c, in accordance with the minus sign in (A.2).

In (A.2) the exponential diodes are

$$f_1(v_1) = \tilde{I}_{cs}(e^{n_1 v_1} - 1) , \tag{A.3}$$

and

$$f_2(v_2) = \tilde{I}_{es}(e^{n_2 v_2} - 1) , \tag{A.4}$$

where $\tilde{I}_{cs}$ is a parameter called the collector-base saturation current, and $\tilde{I}_{es}$ is a parameter called the emitter-base saturation current. The quantities $n_1 = \frac{q}{\kappa T_1}$, and $n_2 = \frac{q}{\kappa T_2}$, in which q is the electron charge, κ is Boltzmann's constant and T_1 and T_2 are the temperatures at the collector and emitter nodes, respectively. The temperatures are usually equal under normal operating conditions. The power consumed by the transistor is

$$P = i_c f_1(v_{bc}) + i_e f_2(v_{be}) . \tag{A.5}$$

Sufficient conditions for such a transistor to be *passive* ([23]) are that

$$\alpha_F \le \frac{\tilde{I}_{cs}}{\tilde{I}_{es}} \le \frac{1}{\alpha_R} \quad \text{and} \quad \alpha_F \le \frac{n_1}{n_2} \le \frac{1}{\alpha_R} , \tag{A.6}$$

Sufficient conditions for such a transistor to satisfy the *no-gain condition* ([47]) are that

$$\alpha_F \le \frac{\tilde{I}_{cs}}{\tilde{I}_{es}} \le \frac{1}{\alpha_R} \quad \text{and} \quad n_1 = n_2 . \tag{A.7}$$

The conditions (A.7) hold under normal operation.

Sandberg and Willson [41, Theorem 5 and footnote 5] establish the following passivity property of Ebers-Moll bipolar junction transistors.

Theorem A.1. (Sandberg and Willson) Let $0 < \alpha_1 < 1$ and $0 < \alpha_2 < 1$ be given. *Suppose that*

$$f_k(v_k) = m_k(\exp(n_k v_k) - 1) \quad \mathit{for} \quad 1 = 1, 2 \ , \tag{A.8}$$

with $m_k n_k > 0$ *and with*

$$\alpha_1 \leq \frac{m_1}{m_2} \leq \frac{1}{\alpha_2} \quad \mathit{and} \quad \alpha_1 \leq \frac{n_1}{n_2} \leq \frac{1}{\alpha_2} \ . \tag{A.9}$$

Then for any $(c_1, c_2) \in \mathbb{R}^2$ *the quantity*

$$P(v_1, v_2) = [v_1 \ v_2] \begin{bmatrix} 1 & -\alpha_1 \\ -\alpha_2 & 1 \end{bmatrix} \begin{bmatrix} f_1(v_1 + c_1) \\ f_2(v_2 + c_2) \end{bmatrix} \ , \tag{A.10}$$

satisfies

$$\lim_{||\mathbf{v}|| \to \infty} P(v_1, v_2) = +\infty \ . \tag{A.11}$$

Detailed models for bipolar junction transistors (see ([4], [12], [22]) elaborate on the Ebers-Moll large signal model. In SPICE additional conductances are added for stability in solving the algorithms, which amount to adding linear resistors with large resistances $R = (GMIN)^{-1}$, as pictured in Figure A.2.

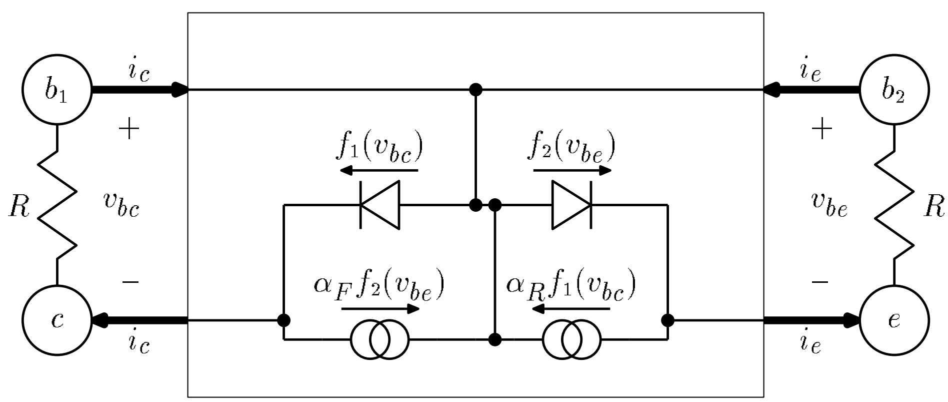

Figure A.2. Added shunt conductances (resistors)

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`email addresses:` `ross@math.binghamton.edu`
`jcl@research.att.com`
`rcm@research.bell-labs.com`