The Wild Numbers

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Abstract

This paper studies the integers that belong the multiplicative semigroup $\mathcal{W}$ generated by $\{\frac{3n+2}{2n+1} : n \geq 0\}$ and $\frac{1}{2}$. They form a multiplicative semigroup $\mathcal{W}(\mathbb{Z})$ of integers which we call the wild integer semigroup, and the wild numbers are the generators of $\mathcal{W}(\mathbb{Z})$. It presents convincing evidence that the wild numbers consist of all prime numbers excluding 3.

1. Introduction

This paper studies a problem in multiplicative number theory originating from a weakened form of the $3x+1$ problem. The notorious $3x+1$ problem (see [7], [15]) concerns the iteration of the $3x+1$ function

$$T(x) = \begin{cases} \frac{3x+1}{2} & \text{if } x \equiv 1 \pmod{2} \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod{2} . \end{cases}$$

on the positive integers. The $3x+1$ conjecture asserts that every positive integer n, if iterated long enough, eventually reaches 1. It has now reportedly been verified for all $n \leq 3.24 \times 10^{17}$, (see [9], [10]) but remains unproven, and appears intractible at present.

There is some interest in finding variants of the $3x+1$ problem which incorporate many of its features but which are potentially easier to resolve. The following weakened version of the $3x+1$ problem was recently proposed by Herschel Farkas [6], and states:

Weak $3X+1$ Conjecture. *Let $\mathcal{S}$ be the multiplicative semigroup which is generated by $\{2\}$ together with the elements $\{\frac{2n+1}{3n+2} : n \geq 0\}$. Then $\mathcal{S}$ contains every positive integer.*

The semigroup $\mathcal{S}$ consists of all finite products of generators, allowing repetitions. We call it the $3x+1$ *semigroup*, because it encodes the $3x+1$ iteration using the semigroup generators. That is, the $3x+1$ iteration can be rewritten $T(2m) = m$ and $T(2m+1) = 3m+2$, and the generators show that if $T(n) \in \mathcal{S}$, then $n \in \mathcal{S}$. Since $1 = 2 \cdot \frac{1}{2} \in \mathcal{S}$, it follows that the truth of the $3x+1$ conjecture implies the truth of the weak $3x+1$ conjecture. However the semigroup $\mathcal{S}$ has more flexibility, because there are products of generators giving integers which do not correspond to iterating the $3x+1$ map; so the weak $3x+1$ conjecture above is potentially easier to resolve than the $3x+1$ problem. Indeed this is so, and the weak $3x+1$ conjecture has been proved by D. Applegate and the author [1], based in part on results developed in the present paper.

We study the following two multiplicative semigroups, which were investigated as an aid to understanding the structure of the $3x+1$ semigroup. Let $\mathcal{W} := \mathcal{S}^{-1}$ denote the multiplicative semigroup generated by the inverses of the generators of $\mathcal{S}$, namely $\frac{1}{2}$ together with all rationals $\{\frac{3n+2}{2n+1} : n \geq 0\}$. which is the set $\{\frac{2}{1}, \frac{5}{3}, \frac{8}{5}, \frac{11}{7}, ...\}$. Let $\mathcal{W}_0$ denote the smaller multiplicative semigroup generated by the rationals $\{\frac{3n+2}{2n+1} : n \geq 0\}$ without $\frac{1}{2}$. We call $\mathcal{W}$ the *wild semigroup* and $\mathcal{W}_0$ the *Wooley semigroup*, respectively. The question we address, analogous to the weak $3x+1$ conjecture is: which integers belong to these two semigroups?

The sets of integer elements $\mathcal{W}(\mathbb{Z}) := \mathcal{W} \cap \mathbb{Z}$ and $\mathcal{W}_0(\mathbb{Z}) := \mathcal{W}_0 \cap \mathbb{Z}$ themselves form semigroups, which we term the *wild integer semigroup* and *Wooley integer semigroup*, and their members "wild integers" and "Wooley integers", respecively. The Wooley semigroup $\mathcal{W}_0$ is a semigroup without unit, while the wild semigroup $\mathcal{W}$ is a semigroup with unit, and $\mathcal{W}(\mathbb{Z})$ and $\mathcal{W}_0(\mathbb{Z})$ inherit these properties, respectively. Our particular choice of terminology is explained at the end of the introduction.

An *irreducible element* of a (commutative) semigroup is any element of which cannot be written as a product of two non-units in the semigroup. We call the irreducible elements of the wild integer semigroup *wild numbers*; similarly we call the irreducible elements of the Wooley integer semigroup *Wooley numbers.* It follows that the wild numbers are a (strict) subset of the wild integers; similarly the Wooley numbers are a subset of the Wooley integers.

We have the immediate implication that each Wooley integer is a wild integer, but the converse need not hold. It is immediately evident that 2 is both a Wooley number and a wild number. It is also easy to show that 3 is not a wild number, hence not a Wooley number. However the nature of other wild numbers or Wooley numbers is less evident.

The wild numbers and the Wooley numbers differ in some significant ways. An odd integer w is in the wild integer semigroup if and only if there is some $j \geq 0$ such that $2^j w$ is in the Wooley integer semigroup. We infer that if w is a wild number, then some $2^j w$ is a Wooley number, but the converse need not hold. At first glance the Wooley numbers appear the simpler object computationally, in that in §2 we show there is an effectively computable procedure to determine if any given rational r belongs to $\mathcal{W}_0$. This gives an effectively computable procedure to determine if an integer is a Wooley integer, and if so, to determine if it is a Wooley number. This permits computer experiments to determine such integers. In contast it is not at all evident whether there exists an algorithm for recognizing if a given integer is a wild integer. Indeed for a general semigroup of this kind, consisting of the commutative semigroup generated by a given recursive set of rational numbers, it semms plausible that the problem of recognizing which integers are in the semigroup is an undecidable problem, see Dyson [5] and Taitslin [13] for results in this direction.

Nevertheless the following conjecture would completely characterize the wild integer semigroup, in a form which permits a simple decision procedure for membership.

Wild Numbers Conjecture. *The wild numbers comprise the set of all prime numbers excluding* 3. *Equivalently, the wild integer semigroup* $\mathcal{W}(\mathbb{Z})$ *consists of all positive integers not divisible by* 3.

This paper studies properties of wild and Wooley integers that bear on these questions. In particular it establishes a strong connection between the weak $3x+1$ conjecture and the wild numbers conjecture: each implies the other. We also prove there are infinitely many wild numbers.

In §2 we study Wooley integers. We prove that the Wooley semigroup is not a free abelian semigroup, by showing that 5 and 10 are not Wooley numbers, but 20 is a Wooley number. Determining the primitive elements of the Wooley integer semigroup $\mathcal{G}$ may well be a complicated problem. In contrast, in the wild numbers conjecture implies that the wild integer semigroup $\mathcal{W}(\mathbb{Z})$ is a free abelian semigroup.

In §3 we show that there are infinitely many wild numbers. Then we show that the weak $3x + 1$ conjecture implies strong restrictions on wild numbers, concluding by showing that it implies the wild numbers conjecture. We also deduce the converse assertion that the wild numbers conjecture implies the weak $3x+1$ conjecture. This provides a strong conviction that both of these conjectures must be true. We conclude by showing that these two conjectures taken together completely characterize the structure of the wild semigroup, which turns out to be quite tame.

As remarked above, the weak $3x+1$ conjecture, and hence the wild numbers conjecture, were subsequently proved in [1]. In §4 we indicate some features of this proof. We also formulate some open problems about Wooley numbers, which remain mysterious. In retrospect, the additional generator $\frac{1}{2}$ present in the wild semigroup simplifies its global structure compared to that of the Wooley semigroup.

The terms "wild semigroup" and "wild number" introduced here were motivated by the novel "The Wild Numbers" by Philibert Schogt [11]. The novel chronicles the efforts of a mathematics professor to solve the (fictitious) "Beauregard's Wild Numbers Problem," while dealing with the ups and downs of life in a university mathematics department. "Beauregard Wild Numbers" are stated to be certain integers produced at the end of a sequence of elementary operations that involve non-integer rationals at the intermediate steps. The "Beauregard Wild Numbers problem" is the assertion there are infinitely many wild numbers ([11, pp.35]). The novel also supplies [11, pp.34, 37] the "empirical data" that 2, 67 and 4769 are Beauregard wild numbers, while 3 is not a wild number. The semigroup problem posed here has some striking resemblances to the information given above. The semigroup products of $\mathcal{W}$ giving an element of $\mathcal{W}(\mathbb{Z})$ generally consist of rationals, whose partial products typically become integers only at the last step. The "wild numbers" here reproduce nearly [1] all the "empirical data".

The terms "Wooley semigroup" and "Wooley number" introduced here are named after Trevor D. Wooley, in honor of his contributions in related areas of number theory, e.g. [3].

There have been various other candidates proposed for "wild numbers", inspired by the novel, which have some of the properties above. Examples are given by Sequence A58883 in the Encyclopedia of Integer Sequences maintained by Neil Sloane [12], as well as six versions of "pseudo-wild numbers" referenced there. An iteration problem with a flavor like that of the "wild numbers' problem concerns the "approximate multiplication" maps studied in Lagarias and Sloane [8], of which a typical example is the map $f(n) = \frac{4}{3}\lceil n \rceil$. The question studied in [8] concerns whether it is true, for each positive integer n, that some iterate $f^{(k)}(n)$ is again an integer. This problem is open, and seems likely to be difficult.

2. Wooley Numbers

We show that the Wooley semigroup $\mathcal{W}_0$ is a recursive semigroup.

Theorem 2.1. *There is an effectively computable procedure which, when given any positive*

[1]The one exception is $4769 = 169 \cdot 253$, which belongs to the wild integer semigroup $\mathcal{W}(\mathbb{Z})$ but is not a wild number as we define it. Perhaps the novel has a misprint for 4759 or 4789 or 4967, all primes.

rational r, determines whether or not r belongs to the Wooley semigroup $\mathcal{W}_0$, and if so, exhibits it as a product of generators.

Proof. We cannot represent r unless it is a positive rational number having an odd denominator(in lowest terms). Let $g(n) = \frac{3n+2}{2n+1}$ denote the n-th generator of the semigroup $\mathcal{W}_0$, and suppose $r = \prod_{i=1}^{m} g(n_i)$, with $n_1 \le n_2 \le \cdots \le n_m$. We first upper bound m. Since each $g(n) > \frac{3}{2}$ we must have $r > (\frac{3}{2})^m$, which upper bounds m.

Now let m be fixed, we can upper bound n_1. For any $\epsilon > 0$ there is a bound $N_0(\epsilon)$ such that if $n_1 > N_0$ then $n_1 < \frac{3}{2} + \epsilon$, so that $(\frac{3}{2} + \epsilon)^m > r > (\frac{3}{2})^m$. Since $r \neq (\frac{3}{2})^m$, for sufficiently small small ϵ it cannot fall in this interval. Thus we obtain an upper bound for n_1. Now we can divide out $g(n_1)$ and a create a new problem of the same kind with a smaller value $r' = r(g(n_1))^{-1} < \frac{2}{3}r$, asking for a representation using a product of exactly $m-1$ generators. We can now show there a finite set of choices of n_2, obtaining an explicit upper bound for n_2 as a function of r, m and n_1. Proceeding by induction on m, the total allowable set of choices is finite, with an effectively computable upper bound. Searching all of them yields either a relation certifying r belongs to $\mathcal{W}_0$ or a proof that r does not belong to $\mathcal{W}_0$. ∎

We can carry out this procedure in simple cases.

Theorem 2.2. *The integers 5 and 10 are not Wooley integers, while 20 is a Wooley integer. It follows that 20 is a Wooley number.*

Proof. Suppose 5 were a product of generators on $\mathcal{W}_0$. Since 2 cannot be cancelled from the numerator of any product, or 3 from the denominator, any representation of 5 cannot use the generators $g(0) = \frac{2}{1}, g(1) = \frac{5}{3}$ or $g(2) = \frac{8}{5}$. The largest number it can use is then $g(3) = \frac{11}{7}$ and $(\frac{11}{7})^3 = \frac{1331}{243} < 5$ so the representation uses at least four generators of $\mathcal{W}_0$. But any such product is larger than $(\frac{3}{2})^4 = \frac{81}{16} > 5$.

Suppose that 10 were a product of generators on $\mathcal{W}_0$. Then any representation of 10 cannot use the generator $\frac{2}{1}$, for if it did this fraction could be removed, giving a representation of 5, a contradiction. Also $\frac{5}{3}$ and $\frac{8}{5}$ cannot be used for same reasons as above, so the largest possible fraction used in any product is again $\frac{11}{7}$. Any representation uses at most 5 generators, since $(\frac{3}{2})^6 = \frac{729}{64} > 10$. However $(\frac{11}{7})^5 < 10$, so there can be no solution.

The number 20 satisfies

$$20 = g(3)^2 \cdot g(5) \cdot g(8) \cdot g(27) \cdot g(32) \cdot g(41) = (\frac{11}{7})^2(\frac{17}{11})(\frac{26}{17})(\frac{83}{55})(\frac{98}{65})(\frac{125}{83}),$$

which shows it belongs to $\mathcal{W}_0(\mathbb{Z})$. ∎

Theorem 2.2 implies that $\mathcal{W}_0(\mathbb{Z})$ is not a free semigroup, since both $\{2\}$ and $\{2^2 \cdot 5\}$ are generators.

Table 1 gives products of 2 times all primes $5 \le p < 50$ in $\mathcal{W}_0(\mathbb{Z})$, which were found by A. Wilks by computer search. There are interesting questions concerning the design of an efficient algorithm to search for such identities, which we do not address here. The computer search above used certain heuristics; it does not decide whether these products give the minimal power of 2 possible, so we can only say that the entries of the table are Wooley integers, not necessarily Wooley numbers.

3. Wild Numbers

We begin by showing that there are infinitely many wild numbers.

$$
\begin{array}{rcl}
2^2 \cdot 5 & = & (\frac{11}{7})^2 \cdot \frac{17}{11} \cdot \frac{26}{17} \cdot \frac{83}{55} \cdot \frac{98}{65} \cdot \frac{125}{83} \\
 & = & g(3)^2 \cdot g(5) \cdot g(8) \cdot g(27) \cdot g(32) \cdot g(41) \\
 & & \\
2^2 \cdot 7 & = & \frac{11}{7} \cdot \frac{26}{17} \cdot \frac{35}{23} \cdot \frac{215}{143} \cdot \frac{299}{199} \cdot \frac{323}{215} \cdot \frac{371}{247} \cdot \frac{398}{265} \\
 & = & g(3) \cdot g(8) \cdot g(11) \cdot g(71) \cdot g(99) \cdot g(107) \cdot g(123) \cdot g(132) \\
 & & \\
2^2 \cdot 11 & = & (\frac{11}{7})^2 \cdot \frac{26}{17} \cdot \frac{35}{23} \cdot \frac{215}{143} \cdot \frac{299}{199} \cdot \frac{323}{215} \cdot \frac{371}{247} \cdot \frac{398}{265} \\
 & = & g(3)^2 \cdot g(8) \cdot g(11) \cdot g(71) \cdot g(99) \cdot g(107) \cdot g(123) \cdot g(132) \\
 & & \\
2^3 \cdot 13 & = & (\frac{11}{7})^2 \cdot (\frac{17}{11})^3 \cdot (\frac{26}{17})^2 \cdot \frac{35}{23} \cdot \frac{215}{143} \cdot \frac{299}{199} \cdot \frac{323}{215} \cdot \frac{371}{247} \cdot \frac{398}{265} \\
 & = & g(3)^2 \cdot g(5)^3 \cdot g(8)^2 \cdot g(11) \cdot g(71) \cdot g(99) g(107) \cdot g(123) \cdot g(132) \\
 & & \\
2^2 \cdot 17 & = & (\frac{11}{7})^2 \cdot \frac{17}{11} \cdot \frac{26}{17} \cdot \frac{83}{55} \cdot \frac{98}{65} \cdot \frac{125}{83} \cdot \frac{143}{95} \cdot \frac{215}{143} \cdot \frac{323}{215} \\
 & = & g(3)^2 \cdot g(5) \cdot g(8) \cdot g(27) \cdot g(32) \cdot g(41) \cdot g(47) \cdot g(71) \cdot g(107) \\
 & & \\
2^5 \cdot 19 & = & (\frac{11}{7})^4 \cdot (\frac{17}{11})^2 \cdot (\frac{26}{17})^2 \cdot \frac{38}{25} \cdot (\frac{83}{55})^2 \cdot (\frac{98}{65})^2 \cdot (\frac{125}{83})^2 \\
 & = & g(3)^4 \cdot g(5)^2 \cdot g(8)^2 \cdot g(12) \cdot g(27)^2 \cdot g(32)^2 \cdot g(41)^2 \\
 & & \\
2^5 \cdot 23 & = & \frac{11}{7} \cdot \frac{26}{17} \cdot \frac{35}{23} \cdot \frac{47}{31} \cdot \frac{137}{91} \cdot \frac{155}{103} \cdot \frac{206}{137} \cdot \frac{215}{143} \cdot (\frac{299}{199})^2 \cdot \frac{323}{215} \cdot \frac{353}{235} \cdot \frac{371}{247} \cdot (\frac{398}{265})^2 \cdot \frac{530}{353} \\
 & = & g(3) \cdot g(8) \cdot g(11) \cdot g(15) \cdot g(45) \cdot g(51) \cdot g(68) \cdot g(71) \\
 & & \quad \cdot g(99)^2 \cdot g(107) \cdot g(117) \cdot g(123) \cdot g(132)^2 \cdot g(176) \\
 & & \\
2^5 \cdot 29 & = & (\frac{11}{7})^4 \cdot (\frac{17}{11})^2 \cdot (\frac{26}{17})^2 \cdot \frac{29}{19} \cdot \frac{38}{25} \cdot (\frac{83}{55})^2 \cdot (\frac{98}{65})^2 \cdot (\frac{125}{83})^2 \\
 & = & g(3)^4 \cdot g(5)^2 \cdot g(8)^2 \cdot g(9) \cdot g(12) \cdot g(27)^2 \cdot g(32)^2 \cdot g(41)^2 \\
 & & \\
2^{11} \cdot 31 & = & (\frac{11}{7})^6 \cdot (\frac{17}{11})^3 \cdot \frac{29}{19} \cdot \frac{38}{25} \cdot \frac{62}{41} \cdot (\frac{83}{55})^3 \cdot (\frac{98}{65})^3 \cdot (\frac{125}{83})^3 \cdot \frac{164}{109} \cdot \frac{218}{145} \\
 & = & g(3)^6 \cdot g(5)^3 \cdot g(8)^3 \cdot g(9) \cdot g(12) \cdot g(20) \\
 & & \quad \cdot g(27)^3 \cdot g(32)^3 \cdot g(41)^3 \cdot g(54) \cdot g(72) \\
 & & \\
2^5 \cdot 37 & = & (\frac{11}{7})^2 \cdot (\frac{26}{17})^2 \cdot (\frac{35}{23})^2 \cdot \frac{74}{49} \cdot (\frac{215}{143})^2 \cdot (\frac{299}{199})^2 \cdot (\frac{323}{215})^2 \cdot (\frac{371}{247})^2 \cdot (\frac{398}{265})^2 \\
 & = & g(3)^2 \cdot g(8)^2 \cdot g(11)^2 \cdot g(24) \cdot g(71)^2 \cdot g(99)^2 \cdot g(107)^2 \cdot g(123)^2 \cdot g(132)^2 \\
 & & \\
2^{10} \cdot 41 & = & (\frac{11}{7})^6 \cdot (\frac{17}{11})^3 \cdot (\frac{26}{17})^3 \cdot \frac{29}{19} \cdot \frac{38}{25} \cdot (\frac{83}{55})^3 \cdot (\frac{98}{65})^3 \cdot (\frac{125}{83})^3 \cdot \frac{164}{109} \cdot \frac{218}{145} \\
 & = & g(3)^6 \cdot g(5)^3 \cdot g(8)^3 \cdot g(9) \cdot g(12) \cdot g(27)^3 \cdot g(32)^3 \cdot g(41)^3 \cdot g(54) \cdot g(72) \\
 & & \\
2^{11} \cdot 43 & = & (\frac{11}{7})^5 \cdot (\frac{17}{11})^2 \cdot (\frac{26}{17})^3 \cdot \frac{29}{19} \cdot \frac{35}{23} \cdot \frac{38}{25} \cdot (\frac{83}{55})^2 \cdot (\frac{98}{65})^2 \cdot (\frac{125}{87})^2 \cdot \frac{215}{143} \\
 & & \quad \cdot \frac{299}{199} \cdot \frac{305}{203} \cdot \frac{323}{215} \cdot \frac{344}{229} \cdot \frac{371}{247} \cdot \frac{398}{265} \cdot \frac{458}{305} \\
 & = & g(3)^5 \cdot g(5)^2 \cdot g(8)^3 \cdot g(9) \cdot g(11) \cdot g(12) \cdot g(27)^2 \cdot g(32)^2 \cdot g(41)^2 \cdot g(71) \\
 & & \quad \cdot g(99) \cdot g(101) \cdot g(107) \cdot g(114) \cdot g(123) \cdot g(132) \cdot g(152) \\
 & & \\
2^{11} \cdot 47 & = & (\frac{11}{7})^6 \cdot (\frac{17}{11})^3 \cdot \frac{29}{19} \cdot \frac{38}{25} \cdot \frac{47}{31} \cdot \frac{62}{41} \cdot (\frac{83}{55})^3 \cdot (\frac{98}{65})^3 \cdot (\frac{125}{83})^3 \cdot \frac{164}{109} \cdot \frac{218}{145} \\
 & = & g(3)^6 \cdot g(5)^3 \cdot g(8)^3 \cdot g(9) \cdot g(12) \cdot g(15) \cdot g(20) \\
 & & \quad \cdot g(27)^3 \cdot g(32)^3 \cdot g(41)^3 \cdot g(54) \cdot g(72)
\end{array}
$$

Table 1: Some members of the Wooley integer semigroup $\mathcal{W}_0(\mathbb{Z})$.

Theorem 3.1. *The semigroup of wild integers is infinitely generated. That is, there are infinitely many wild numbers.*

Proof. For $n = \frac{5^k-1}{2}$ we have $g(n) = \frac{\frac{1}{2}(3 \cdot 5^k+1))}{5^k}$. Theorem 2.2 implies that 5 is a wild number so the denominator 5^k is a wild integer. We conclude that $h(k) := \frac{1}{2}(3 \cdot 5^k + 1) \in \mathcal{W}(\mathbb{Z})$, for all $k \geq 1$.

The sequence $\{h(k) : k \geq 1\}$ satisfies a homogeneous second order linear recurrence

$$h(k) = 6h(k-1) - 5h(k-2).$$

This sequence $h(k)$ is nondegenerate in the sense of Ward [14] (i.e. it does not satisfy a first order linear recurrence), so by appeal to the main result of Ward [14] the sequence $h(k)$ has an infinite number of distinct prime divisors, i.e. the set of primes p that divide some $\{h(k) : k \geq 0\}$ is infinite.

We now argue by contradiction that the wild integer semigroup $\mathcal{W}(\mathbb{Z})$ is infinitely generated. If it were finitely generated, then we could find some prime p in the infinite set above that does not occur in any generator. But this prime divides some $h(k)$, whence $h(k)$ could not be represented as a product of the generators, a contradiction. ∎

One can obtain much stronger results about the structure of the wild integer semigroup if we assume the truth of the weak $3x + 1$ conjecture.

Theorem 3.2. *Suppose that the weak $3x+1$ conjecture holds. Then the wild integer semigroup is a free commutative semigroup whose set of generators $\mathcal{P}$ consists entirely of primes. That is, all wild numbers are prime numbers.*

Proof. The wild integer semigroup $\mathcal{W}(\mathbb{Z})$ contains all powers of 2. Also, since 2 is invertible in $\mathcal{W}$ it can be cancelled from all other generators, which therefore must be odd integers. However the weak $3x + 1$ conjecture says that if $n \in \mathcal{W}$ then so is $\frac{n}{k} \in \mathcal{W}$ for any positive integer k. Thus if a composite number $n \in \mathcal{W}(\mathbb{Z})$ then so are all its prime divisors. It follows that all generators of $\mathcal{W}(\mathbb{Z})$ are prime numbers. ∎

In general we can certify that a given prime number p is a wild number by finding some $2^j p$ that is a Wooley number. Thus 67 is a wild number since $2^{12} \cdot 67$ is a Wooley number using

$$\frac{2^5 \cdot 67}{5 \cdot 37} = g(29) \cdot g(44) \cdot g(69) \cdot g(78) \cdot g(92) \cdot g(104)$$

and the already-proved facts that $2^2 \cdot 5$ and $2^5 \cdot 37$ are Wooley numbers.

We next show that the weak $3x + 1$ conjecture implies the wild numbers conjecture.

Theorem 3.3. *Suppose that the weak $3x + 1$ conjecture holds. Then the wild numbers conjecture also holds.*

Proof. Under the hypothesis we will prove the wild numbers conjecture by induction on the n-th prime p_n, call it q.

Suppose that the wild numbers conjecture is true for all primes $p < q$. Then it necessarily holds for all "smooth numbers" consisting of products of primes smaller than q. It suffices to find some multiple mq that is a wild number, for the weak $3x + 1$ conjecture implies that $\frac{1}{m}$ belongs to the wild semigroup $\mathcal{W}$, so that $q = \frac{1}{m}(mq)$ would then be a wild number. To show $mq \in \mathcal{W}$, we wish to find $mq = 3n + 2$, such that $2n + 1$ is a "smooth number". If so then

$mq = \frac{3n+2}{2n+1}(2n+1) \in \mathcal{W}$ and the desired result will follow. The congruence restriction puts m in a certain residue class (mod 3), and by imposing a condition (mod 9) we can guarantee that $2n+1 \not\equiv 0 (\text{mod } 3)$. The result integers $2n+1$ will fall in an arithmetic progression $r \pmod{6q}$ with $gcd(r, 6q) = 1$, and we arrive at the well-studied arithmetic question of finding smooth numbers in an arithmetic progression. Complicated results [2] in this area (Balog and Pomerance [2]) already suffice to indicate the result will hold for $q > C_0$, an effectively computable constant.

However our argument above requires only the existence of at least one smooth number in the arithmetic progression, not a positive density of such numbers, and we can establish this by the following combinatorial argument, which morevoer gives an explicit bound for C_0. It rests on the observation that if more than half the invertible residue classes (mod N) contain a smooth number, then (by the pigeonhole principle) every invertible residue class (mod N) will contain some product of two of them, and hence contain a smooth number. To see this, if the set of invertible residue classes with a smooth representative s is Σ, and we wish to find a representative in the class $r \pmod{N}$ then define $s' \equiv r \cdot s^{-1} (\text{mod } N)$, where $s \in \Sigma$. There are now $|\Sigma|$ choices for s' and since $|\Sigma|$ exceeds half the invertible residue classes there must be some $s' \in \Sigma$ and then $s \cdot s'$ is a smooth number in the class $r \pmod{N}$. In our case we have $N = 6q$, which has $2(q-1)$ invertible residue classes, and it suffices to show that more than $q-1$ of these classes contain a smooth number. We show that for all q sufficiently large the set of such classes whose least residue is smooth exceeds $q-1$. Now every integer below $6q$ relatively prime to $6q$ is smooth, except for primes $q < p' < 6q$ and integers of form $5p'$ with $q \le p' < \frac{6}{5}q$. There are $O(\frac{q}{\log q})$ such integers, hence at least $2(q-1) - O(\frac{q}{\log q})$ invertible classes have least residue being smooth. With explicit numerical bounds for the remainder term one can show more than half the classes are smooth for $q > 10^4$.

Now one can prove the wild numbers conjecture by induction (under the assumption that the weak $3x+1$ conjecture is valid), with the base case consisting of checking all $q < 10^4$. This can be done by computer. In fact it suffices to use the table in §2 for $q < 50$, and then for $50 < q < 10^4$ have the computer find a smooth number in a suitable arithmetic progression $\pmod{6q}$. ∎.

Since the $3x+1$ conjecture appears to be true, Theorem 3.3 provides a powerful argument in favor of the wild numbers conjecture. On the other hand, we have a reverse implication, by a similar argument.

Theorem 3.4. *Suppose that the wild numbers conjecture holds. Then the weak $3x+1$ conjecture also holds.*

Proof. This implication is proved in a similar fashion to Theorem 3.3.

We proceed by induction on the n-th prime q. We consider now multiples mq where $m \equiv 1$ (mod6). The wild numbers conjecture implies that all such integers are in the wild semigroup. Writing $mp = 2n+1$, we look for a case where $3n+2$ is a smooth number with all prime factors smaller than q. Writing $m = 6k+1$ we have $3n+2 = 9kq + \frac{3q+1}{2}$, which is an arithmetic progression $\pmod{9q}$. As in the earlier result it suffices to show that for sufficiently large q more than half the invertible residue classes $\pmod{9q}$ in the interval

[2]It is known that smooth numbers below a cutoff value Y having all prime factors smaller than Y^α for any fixed α asymptotically have a positive density, namely $\rho(\alpha)Y$, where $\rho(u)$ is a strictly positive function, the Dickman rho-function, given by the solution to a difference-differential equation. The results of Balog and Pomerance [2] carry these bounds over to arithmetic progressions, and their results give an asymptotic formula valid over some range. In particular, choosing $\alpha = \frac{1}{2}$, that is $Y \approx q^2$, leads to the needed result.

$[1, q-1]$ contain a smooth integer, which then implies that all invertible arithmetic progressions (mod $9q$) contain a smooth integer smaller than $81q^2$. This holds when $q > 10^5$. Since the $3x+1$ conjecture has been verified to 10^5, the base case of the induction is already done. ■.

Our final result deduces that the weak $3x+1$ conjecture completely determines the structure of the wild semigroup $\mathcal{W}$.

Theorem 3.5. *If the weak $3x + 1$ conjecture is true, then the wild semigroup $\mathcal{W}$ consists of all positive rational numbers $\frac{a}{b}$ with $gcd(a, 3b) = 1$.*

Proof. The weak $3x+1$ conjecture implies that $\mathcal{W}$ contains all fractions $\frac{1}{p}$, where p is prime. It implies the wild numbers conjecture, which asserts that it contains all primes p with $p \neq 3$. We observed above that any rational $r = \frac{a}{b} \in \mathcal{W}$ in lowest terms has a relatively prime to (3). This gives the result. ■

4. Concluding Remarks

The results of §3 show that the wild number conjecture and the weak $3x + 1$ conjecture are intertwined: each is implied by the other. D. Applegate and the author [1] were subsequently able to prove both conjectures by a bootstrap induction procedure that used the truth of one of the conjectures on an interval to extend the truth of the other to a larger interval, and vice versa. The argument of Theorem 3.3 (resp. Theorem 3.5) provides a way to extend the truth of the conjecture in one direction, provided it holds on a sufficient initial interval in the other. However the methods above do not seem to allow applying this argument simultaneously in both directions. The arguments in [1] use iteration of the $3x + 1$ map starting with n, which (sometimes) find a representation of a given integer n in the $3x + 1$ semigroup $\mathcal{S} := \mathcal{W}^{-1}$, in constructing the other direction of the bootstrap induction. There is an apparent asymmetry between the $3x + 1$ semigroup and the wild semigroup, in that in the latter case we do not know of an associated dynamical system which produces relations generating all the integers in $\mathcal{W}(\mathbb{Z})$.

Our motivation for studying wild integers came from the weak $3x + 1$ conjecture, but the Wooley semigroup introduced in the process seems interesting in its own right, and leads to various unsolved problems. The inversion by 2 used in defining the wild semigroup simplifies its structure, compared to the Wooley semigroup. Determining the generators of the Wooley integer semigroup $\mathcal{W}_0(\mathbb{Z})$ seems a subtle question, for which there may well be no simple description. One interesting question is whether all generators of $\mathcal{W}_0(\mathbb{Z})$ are a power of 2 times a prime. Do there exist any Wooley numbers having three or more prime factors? Another interesting question concerns the rate of growth in the power of 2 necessary so that $2^{e(p)}p$ belongs to the Wooley semigroup, as a function of p. It seems plausible that $e(p)$ is unbounded. The truth of the wild numbers conjecture implies that each number $e(p)$ is finite, so this question is well posed by the results of [1].

The wild numbers conjecture was named after the (fictitious) mathematical problem in the novel of Philibert Schogt, "The Wild Numbers." The "Beauregard Wild Numbers Problem" was described as a famous unsolved problem, with a long and illustrious history. Its namesake problem here is neither famous nor unsolved. since the infinitude of wild numbers is established in Theorem 3.1. Our terminology nevertheless seems fitting, in that the "wild numbers" of this paper coincide with the prime numbers, and the novel asserts "a fundamental relationship between wild numbers and prime numbers" [11, p. 36]. Understanding the behavior of prime numbers is one of the great mysteries of mathematics, with a history as long and impressive as one may hope for, see Derbyshire [4].

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