Problem # 4.

a) Examples show that a system of 4 linear equations in 3 variables can have no solutions, a unique solution, or infinitely many solutions. Consider, for example, systems

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= 1 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}, \qquad \begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= 1 \\ x_1 + x_2 + x_3 &= 3 \end{aligned}, \qquad \text{and} \qquad \begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_1 + 2x_2 + 2x_3 &= 2 \\ 3x_1 + 3x_2 + 3x_3 &= 3 \\ 4x_1 + 4x_2 + 4x_3 &= 4. \end{aligned}$$

b) A system of 3 linear equations in 4 variables cannot have a unique solution, since once we reduce the system to the reduced row-echelon form, there has to be at least one free variable. The following examples show that such a system can have no solutions or infinitely many solutions:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ 2x_1 + 2x_2 + 2x_3 + 2x_4 &= 1 \\ 3x_1 + 3x_2 + 3x_3 + 3x_4 &= 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ 2x_1 + 2x_2 + 2x_3 + 2x_4 &= 2 \\ 3x_1 + 3x_2 + 3x_3 + 3x_4 &= 3. \end{aligned}$$

c) Let $\vec{x}$ be a solution of the system $C\vec{x} = \vec{c}$ for some vector $\vec{c}$ and $C = AB$. Let $B\vec{x} = \vec{y}$, so $A\vec{y} = \vec{c}$. Since B is a 3×4 matrix, by Part b) there exists infinitely many vectors $\vec{z}$ such that $B\vec{z} = \vec{y}$. Then, for every such a vector $\vec{z}$, we have $C\vec{z} = (AB)\vec{z} = A(B\vec{z}) = A\vec{y} = \vec{c}$. This proves that the system $C\vec{x} = \vec{c}$ cannot have a unique solution. The system can have no or infinitely many solutions as the following examples show: let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{so } C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consider the systems:

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= 1 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= 1 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0. \end{aligned}$$

Answer: a) Yes, Yes, Yes; b) Yes, No, Yes; c) Yes, No, Yes.