**Problem # 2.** We note that $A$ defines a linear transformation $\mathbf{R}^3 \longrightarrow \mathbf{R}^5$, $B$ defines a linear transformation $\mathbf{R}^4 \longrightarrow \mathbf{R}^3$, and $AB$ is a $5 \times 4$ matrix which defines a linear transformation $\mathbf{R}^4 \longrightarrow \mathbf{R}^5$, which consists of applying first $B$ and then $A$: $\mathbf{R}^4 \xrightarrow{B} \mathbf{R}^3 \xrightarrow{A} \mathbf{R}^5$.

a) Since the rank of $B$ is the dimension of the image of $B$ and the sum of the dimensions of the image of $B$ and of the kernel of $B$ is 4, we conclude that the dimension of the kernel of $B$ is 2. **Answer:** The dimension of the kernel of $B$ is 2.

b) Since the sum of the dimensions of the kernel of $A$ and of the image of $A$ is 3, possible dimensions of the image of $A$ are 0,1,2,3. Examples show that all these dimensions may indeed occur. **Answer:** The possible dimensions of the image of $A$ are 0,1,2,3.

c) The rank of $AB$ is equal to the dimension of the image of $AB$. Since the image of $B$ is obtained by applying a linear transformation with matrix $A$ to the image of $B$, the dimension of the image of $AB$ cannot be greater than the dimension of the image of $B$. Hence possible ranks of $AB$ are $0, 1, 2$. Examples show that all these ranks may indeed occur. **Answer:** The possible values of rank $AB$ are 0,1,2.

d) Since the rank of $A$ is the dimension of the image of $A$ and the sum of the dimensions of the kernel of $A$ and of the image of $A$ is 3, we conclude that the dimension of the kernel of $A$ is 1. The rank of $AB$ is the dimension of the image of $AB$, which is obtained by applying the transformation with matrix $A$ to the image of $B$. The image of $B$ is a plane in $\mathbf{R}^3$ and the kernel of $A$ is a line in $\mathbf{R}^3$. It all depends now how is the kernel of $A$ positioned with respect to the image of $B$. If the kernel of $A$ (line) lies in the image of $B$ (plane), then the image of $AB$ shrinks to a line, so rank $AB = 1$. If the kernel of $A$ (line) intersects the image of $B$ (plane) in the origin, the image of $AB$ is a plane, so rank $AB = 2$. Thus possible values of rank $AB$ are 1 and 2. Examples show that these ranks may indeed occur. **Answer:** The possible values of rank $AB$ are 1 and 2.