Problem # 5. We get the system of linear equations

$$\begin{aligned} c_0 &= 0 \\ c_0 + c_1 &= 1 \\ c_0 + 2c_1 &= 3 \end{aligned}$$

or, in the matrix form, $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \quad \text{and} \quad \vec{x} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}.$$

To find the least squares solution, we solve the normal equation $A^T A\vec{x} = A^T\vec{b}$, where

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}, \quad \text{and}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

Solving the system

$$\left[\begin{array}{cc|c} \underline{c_0} & \underline{c_1} & \\ 3 & 3 & 4 \\ 3 & 5 & 7 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} \underline{c_0} & \underline{c_1} & \\ 1 & 1 & 4/3 \\ 0 & 1 & 3/2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} \underline{c_0} & \underline{c_1} & \\ 1 & 0 & -1/6 \\ 0 & 1 & 3/2 \end{array}\right],$$

we get $c_0 = -1/6$ and $c_1 = 3/2$.

Answer: The least squares solution is $f(t) = -1/6 + (3/2)t$ with $c_0 = -1/6$ and $c_1 = 3/2$.