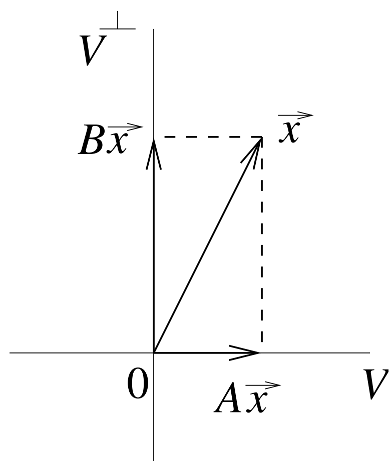
Problem # 7. We note that V is the image of A and the kernel of B while $V^{\perp}$ is the image of B and the kernel of A. In particular, $\dim V^{\perp} = 2$.

a) The rank of A is equal to the dimension of the image of A, that is, to the dimension of V. Hence rank $A = 3$. The rank of B is equal to the dimension of the image of B, that is, to the dimension of $V^{\perp}$. Hence rank $B = 2$. **Answer:** rank $A = 3$, rank $B = 2$.

b) Let us pick a vector $\vec{x}$ in $\mathbf{R}^5$. Then $B\vec{x}$ is the orthogonal projection of $\vec{x}$ onto $V^{\perp}$. In particular, $B\vec{x}$ lies in $V^{\perp}$. Now, $(AB)\vec{x} = A(B\vec{x})$ is the orthogonal projection of $B\vec{x}$ onto V. Since $B\vec{x}$ lies in $V^{\perp}$, we must have $A(B\vec{x}) = \vec{0}$. Thus $AB\vec{x} = \vec{0}$ for all vectors $\vec{x}$, from which AB must be the 5×5 zero matrix. Similarly, BA must be the 5×5 zero matrix.



Let us pick a vector $\vec{x}$ in $\mathbf{R}^5$. Then $A\vec{x}$ is the orthogonal projection of $\vec{x}$ onto V, so the difference $\vec{x} - A\vec{x}$ is orthogonal to every vector in V. Therefore, $\vec{x} - A\vec{x}$ lies in $V^{\perp}$. Moreover, the difference $\vec{x} - (\vec{x} - A\vec{x}) = A\vec{x}$ lies in V and so is orthogonal to every vector from $V^{\perp}$. Thus $\vec{x} - A\vec{x}$ is the orthogonal projection of $\vec{x}$ onto $V^{\perp}$, so $\vec{x} - A\vec{x} = B\vec{x}$. Thus

$$A\vec{x} + B\vec{x} = \vec{x} \quad \text{for all} \quad \vec{x},$$

from which $A + B$ must be the 5×5 identity matrix. **Answer:** AB and BA are the 5×5 zero matrices and $A + B$ is the 5×5 identity matrix.