2. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ (same as in Problem 1) and let $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find a closed form expression for $A^t\vec{v}$, where t is a positive integer.

Solution. From Problem 1, the eigenvalues of A are 2 and 3 and we choose $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ as the corresponding eigenvectors. Now we write $\vec{v}$ as a linear combination of eigenvectors:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = x \begin{bmatrix} -1 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \text{or, in the matrix form,}$$

$$\begin{array}{cccc} \underline{x} & \underline{y} & & \\ \end{array} \qquad \begin{array}{cccc} \underline{x} & \underline{y} & & \end{array}$$

$$\begin{bmatrix} -1 & -1 & | & 1 \\ 1 & 2 & | & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & | & -3 \\ 0 & 1 & | & 2 \end{bmatrix}, \quad \text{so} \quad x = -3, y = 2.$$

Thus

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix},$$

where $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$ is an eigenvector with the eigenvalue 2 and $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$ is an eigenvector with the eigenvalue 3. Therefore,

$$\begin{aligned} A^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= A^t \left(\begin{bmatrix} 3 \\ -3 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix} \right) = A^t \begin{bmatrix} 3 \\ -3 \end{bmatrix} + A^t \begin{bmatrix} -2 \\ 4 \end{bmatrix} \\ &= 2^t \begin{bmatrix} 3 \\ -3 \end{bmatrix} + 3^t \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2^t - 2 \cdot 3^t \\ -3 \cdot 2^t + 4 \cdot 3^t \end{bmatrix}. \end{aligned}$$

Answer. $A^t\vec{v} = \begin{bmatrix} 3 \cdot 2^t - 2 \cdot 3^t \\ -3 \cdot 2^t + 4 \cdot 3^t \end{bmatrix}$.