2. I have a system of 5 linear equations in 5 variables which has no solutions. Somebody changed the right hand side of the system, leaving the coefficient matrix intact. Is it possible that the new system

a) has no solutions?

b) has a unique solution?

c) has infinitely many solutions?

Solution. Let us see what happens if we write the augmented matrix of the old system and bring it to the reduced row-echelon form. Since the old system had no solutions, in the reduced row-echelon form there must be a row of 0's followed by 1: $[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad | \quad 1]$. Therefore, since the coefficient matrix of the system is 5×5 and at least one of the rows of the coefficient matrix has all 0's, we must have at most 4 leading variables and hence at least one free variable in the coefficient matrix.

Let us see now what happens if we write the augmented matrix of the new system and bring it to the reduced row-echelon form. Since the coefficient matrix didn't change, the coefficient part of the reduced matrix doesn't change either and we must have at least one free variable there Hence the new system cannot have a unique solution.

However, it can have no solutions or infinitely many solutions as the following examples demonstrate

$$\begin{aligned} x_1 &= 0 \\ x_1 &= 1 \\ x_2 + x_3 + x_4 + x_5 &= 0 \\ x_2 + x_3 + x_4 + x_5 &= 0 \\ x_2 + x_3 + x_4 + x_5 &= 0 \end{aligned} \quad \text{changes to} \quad \begin{aligned} x_1 &= 0 \\ x_1 &= 2 \\ x_2 + x_3 + x_4 + x_5 &= 0 \\ x_2 + x_3 + x_4 + x_5 &= 0 \\ x_2 + x_3 + x_4 + x_5 &= 0; \end{aligned}$$

$$\begin{aligned} x_1 &= 0 \\ x_1 &= 1 \\ x_2 + x_3 + x_4 + x_5 &= 0 \\ x_2 + x_3 + x_4 + x_5 &= 0 \\ x_2 + x_3 + x_4 + x_5 &= 0 \end{aligned} \quad \text{changes to} \quad \begin{aligned} x_1 &= 1 \\ x_1 &= 1 \\ x_2 + x_3 + x_4 + x_5 &= 0 \\ x_2 + x_3 + x_4 + x_5 &= 0 \\ x_2 + x_3 + x_4 + x_5 &= 0. \end{aligned}$$

Answer. The new system can have no solutions or it can have infinitely many solutions. It cannot have a unique solution.