2. For which values of the parameters $a, b,$ and c the matrix $\begin{bmatrix} a & 1 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ is invertible? Find the inverse when it exists.

Solution. Let us try to invert the matrix, reducing it to rref. We will use the 3×6 matrix of form $[A|I]$ where A is the given matrix. We first notice that if $a = 0$, then the first column of A is all zeros, so there is no leading 1 in this column, the rref cannot be the identity matrix, so the matrix is not invertible. If $a \neq 0$ then we can divide by a, and proceed to reduce the 3×6 matrix to rref, in the following steps.

$$
\left[\begin{array}{ccc|ccc}
a & 1 & b & 1 & 0 & 0 \\
0 & 1 & c & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{array}\right]
\longrightarrow
\left[\begin{array}{ccc|ccc}
1 & 1/a & b/a & 1/a & 0 & 0 \\
0 & 1 & c & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{array}\right]
$$

$$
\longrightarrow
\left[\begin{array}{ccc|ccc}
1 & 0 & (b-c)/a & 1/a & -1/a & 0 \\
0 & 1 & c & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{array}\right]
$$

$$
\longrightarrow
\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & 1/a & -1/a & (c-b)/a \\
0 & 1 & c & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{array}\right]
$$

$$
\longrightarrow
\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & 1/a & -1/a & (c-b)/a \\
0 & 1 & 0 & 0 & 1 & -c \\
0 & 0 & 1 & 0 & 0 & 1
\end{array}\right]
$$

Here the rref of the 3×3 matrix we started from is the identity matrix, and the last three columns give the inverse.

Answer. The matrix is invertible when $a \neq 0$, where b and c can be any numbers. When $a \neq 0$ the inverse is

$$
\begin{bmatrix}
1/a & -1/a & (c-b)/a \\
0 & 1 & -c \\
0 & 0 & 1
\end{bmatrix}.
$$