

2. For which values of the parameters  $a, b,$  and  $c$  the matrix  $\begin{bmatrix} a & 1 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$  is invertible? Find the inverse when it exists.

**Solution.** Let us try to invert the matrix:

$$\begin{aligned} & \begin{bmatrix} a & 1 & b & | & 1 & 0 & 0 \\ 0 & 1 & c & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} a & 1 & 0 & | & 1 & 0 & -b \\ 0 & 1 & 0 & | & 0 & 1 & -c \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix} \\ \longrightarrow & \begin{bmatrix} a & 0 & 0 & | & 1 & -1 & c-b \\ 0 & 1 & 0 & | & 0 & 1 & -c \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

We notice that if  $a = 0$ , the reduced row echelon form of the matrix will have a row of zeros, so the matrix will not be invertible. If  $a \neq 0$ , we can make the final step:

$$\begin{aligned} & \begin{bmatrix} a & 0 & 0 & | & 1 & -1 & c-b \\ 0 & 1 & 0 & | & 0 & 1 & -c \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix} \longrightarrow \\ & \begin{bmatrix} 1 & 0 & 0 & | & 1/a & -1/a & (c-b)/a \\ 0 & 1 & 0 & | & 0 & 1 & -c \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

**Answer.** The matrix is invertible if  $a \neq 0$ , in which case the inverse is

$$\begin{bmatrix} 1/a & -1/a & (c-b)/a \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}.$$