2. Let A be a 4×3 matrix and let B be a 3×4 matrix. Suppose that rank $B = 2$ and rank $A = 3$.

a) What is the dimension of the kernel of A?

b) What is the dimension of the kernel of B?

c) What is the rank of AB?

Solution.

a) We have

$$\dim(\text{kernel of } A) + \dim(\text{image of } A) = 3 \quad \text{and}$$

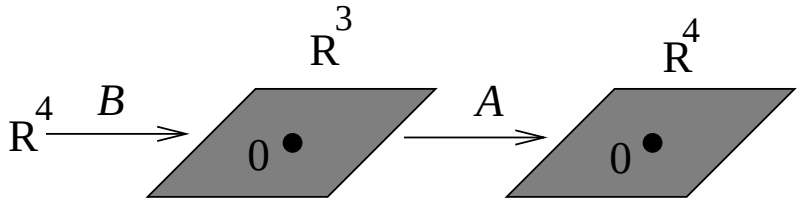
$\dim(\text{image of } A) = \text{rank of } A = 3$, from which $\dim(\text{kernel of } A) = 0$.

b) We have

$$\dim(\text{kernel of } B) + \dim(\text{image of } B) = 4 \quad \text{and}$$

$\dim(\text{image of } B) = \text{rank of } B = 2$, from which $\dim(\text{kernel of } B) = 2$.

c) Let us consider the linear transformation $\vec{x} \longmapsto (AB)\vec{x} = A\,(B\vec{x})$. The transformation $\vec{x} \longmapsto B\vec{x}$ is a linear transformation $\mathbf{R}^4 \longrightarrow \mathbf{R}^3$ and the image of this transformation is a plane in $\mathbf{R}^3$, since rank $B = 2$. The transformation $\vec{y} \longmapsto A\vec{y}$ is a linear transformation $\mathbf{R}^3 \longrightarrow \mathbf{R}^4$. This transformation transforms the plane that is the image of B into a plane, since the kernel of A is $\vec{0}$.



Hence the image of AB is the plane and rank $AB = 2$.

Answer. The dimension of the kernel of A is 0, the dimension of the kernel of B is 2, and the rank of AB is 2.