1. Let $\vec{a}$ and $\vec{b}$ be vectors such that $\|\vec{a}\| = 2$, $\|\vec{b}\| = 3$ and the cosine of the angle between $\vec{a}$ and $\vec{b}$ is $1/3$. Find the cosine of the angle between $\vec{a}$ and $\vec{a} + \vec{b}$.

Solution. We have

$$\vec{a} \cdot \vec{a} = \|\vec{a}\|^2 = 4, \quad \vec{b} \cdot \vec{b} = \|\vec{b}\|^2 = 9, \quad \text{and} \quad \vec{a} \cdot \vec{b} = 2 \cdot 3 \cdot \frac{1}{3} = 2.$$

Now, let ϕ denote the angle between $\vec{a}$ and $\vec{a} + \vec{b}$, and we have

$$\vec{a} \cdot (\vec{a} + \vec{b}) = \|\vec{a}\| \|\vec{a} + \vec{b}\| \cos \phi,$$

so

$$\cos \phi = \frac{\vec{a} \cdot (\vec{a} + \vec{b})}{|\vec{a}| \|\vec{a} + \vec{b}\|}.$$

We know $|\vec{a}|$, so if we determine $\vec{a} \cdot (\vec{a} + \vec{b})$ and $\|\vec{a} + \vec{b}\|$ then we can find $\cos \phi$.

Now

$$\vec{a} \cdot (\vec{a} + \vec{b}) = \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} = \|\vec{a}\|^2 + \vec{a} \cdot \vec{b},$$

and

$$\|\vec{a} + \vec{b}\| = \sqrt{(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})} = \sqrt{\vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}}.$$

Substituting the values of $\vec{a} \cdot \vec{a}$, $\vec{b} \cdot \vec{b}$, and $\vec{a} \cdot \vec{b}$, we get

$$\|\vec{a} + \vec{b}\| = \sqrt{4 + 2 + 2 + 9} = \sqrt{17} \quad \text{and}$$
$$\vec{a} \cdot (\vec{a} + \vec{b}) = 4 + 2 = 6.$$

Thus $\cos \phi = \frac{6}{2\sqrt{17}} = \frac{3}{\sqrt{17}}$.

Answer. The cosine of the angle between $\vec{a}$ and $\vec{a} + \vec{b}$ is $\dfrac{3}{\sqrt{17}}$.