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Ends of locally symmetric spaces with maximal bottom spectrum

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Abstract. Let X be a symmetric space of non-compact type and $\Gamma\backslash X$ a locally symmetric space. Then the bottom spectrum $\lambda_1(\Gamma\backslash X)$ satisfies the inequality $\lambda_1(\Gamma\backslash X) \leqq \lambda_1(X)$. We show that if equality $\lambda_1(\Gamma\backslash X) = \lambda_1(X)$ holds, then $\Gamma\backslash X$ has either one end, which is necessarily of infinite volume, or two ends, one of infinite volume and another of finite volume. In the latter case, $\Gamma\backslash X$ is isometric to $\mathbb{R}^1 \times N$ endowed with a multi-warped metric, where N is compact.

§0. Introduction

A Riemannian symmetric space X of noncompact type and its quotients $\Gamma\backslash X$ have been studied from various points of view. One important problem is to study relations between their geometry and spectral theory. The spectral theory of locally symmetric spaces of *finite volume* is part of the vast subject of spectral theory of automorphic forms. The geometry of such finite volume locally symmetric spaces $\Gamma\backslash X$ can be understood by combining the reduction theory for arithmetic subgroups of semisimple Lie groups [B2] and lattices in rank one semisimple Lie groups [GaR], and the arithmeticity of irreducible lattices of higher rank semisimple Lie groups [Ma2]. In fact, the structure at infinity (end structure) of such spaces $\Gamma\backslash X$ can be understood quite well and is connected with the Tits buildings of the associated groups.

For general locally symmetric spaces $\Gamma\backslash X$ of infinite volume, their geometry at infinity is not so well understood. For example, various geometric finiteness conditions have been proposed to give control of the geometry and topology at infinity.

One purpose of this paper is to study the end structure of a particular class of locally symmetric spaces $M = \Gamma\backslash X$ consisting of those satisfying the extremal condition

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$\lambda_1(M) = \lambda_1(X)$, where $\lambda_1(M)$ denotes the greatest lower bound of the L^2 spectrum of the Laplacian acting on functions defined on M.

Let us first recall the following well-known monotonicity property of λ_1.

Proposition 0.1. *If $\tilde{M}$ is a complete Riemannian manifold and Γ is any discrete group acting isometrically on $\tilde{M}$, then*

$$\lambda_1(\Gamma\backslash\tilde{M}) \leqq \lambda_1(\tilde{M}).$$

Proof. Note that $\lambda_1(\tilde{M})$ can be characterized by

$$\lambda_1(M) = \sup\{\lambda \in \mathbb{R} \mid \Delta u = -\lambda u \text{ for some positive function } u \text{ on } \tilde{M}\}.$$

If $u > 0$ is a function on $\Gamma\backslash\tilde{M}$ satisfying $\Delta u = -\lambda u$, then its lift $\tilde{u}$ on $\tilde{M}$ also satisfies the equation $\Delta\tilde{u} = -\lambda\tilde{u}$. This implies the desired inequality. □

If $\Gamma\backslash X$ has finite volume, then $\lambda_1(\Gamma\backslash X) = 0$. Since it is known that $\lambda_1(X) > 0$, it follows that in this case the strict inequality

$$\lambda_1(\Gamma\backslash X) < \lambda_1(X)$$

is valid.

On the other hand, if Γ is an amenable group, then it can be shown (Theorem 5.1) that

$$\lambda_1(\Gamma\backslash X) = \lambda_1(X).$$

So the class of spaces $\Gamma\backslash X$ with $\lambda_1(\Gamma\backslash X) = \lambda_1(X)$ lies at the extreme opposite to the class of finite volume ones.

If Γ is a finite group, then it clearly contains nontrivial torsion elements. On the other hand, many naturally defined infinite groups Γ also contain nontrivial torsion elements. For example, arithmetic subgroups such as $\mathrm{SL}(n,\mathbb{Z})$ and $\mathrm{Sp}(n,\mathbb{Z})$ contain nontrivial torsion elements, and their associated locally symmetric spaces are natural moduli spaces in algebraic geometry. Other infinite groups Γ containing torsion elements with the volume of $\Gamma\backslash X$ being infinite can also be constructed.

If Γ contains nontrivial torsion elements, then the space $\Gamma\backslash X$ is not a manifold, but rather an orbifold. An important point is that the techniques employed in this article can be easily carried over to orbifold setting. In particular, the relationship between the spectral geometry and the end structure of $\Gamma\backslash X$ follows without much effort.

While it is true that if $\Gamma\backslash X$ is an orbifold and Γ is finitely generated, then there exists a finite smooth covering $\Gamma'\backslash X \to \Gamma\backslash X$, by using the so-called Selberg lemma. In this case, some analytical and geometric problems of $\Gamma\backslash X$ can be studied by lifting them to the finite smooth cover $\Gamma'\backslash X$. However, it is, nonetheless, natural to study $\Gamma\backslash X$.

It is also important to point out that it is not known if any orbifold $\Gamma\backslash X$ admits a finite smooth cover. In fact, the following conjecture of Margulis [Ma1] is still open: *If*

X is a complete and simply connected, nonpositively curved Riemannian manifold and Γ is a discrete group acting properly and isometrically on X, then there exists a torsion-free subgroup $\Gamma' \subset \Gamma$ of finite index.

To be precise, in this paper, by an orbifold, we mean an orbifold which is not locally $\mathbb{R}^n$ divided out by a reflection. So the singularities are of the cone type, and the set of singular points is of codimension at least 2. This assumption is natural. Indeed, if an orbifold has singular points of the form $\mathbb{R}^n$ divided out by a reflection, then it is locally a manifold with non-empty boundary. On the other hand, if an orbifold arises from such group actions, then one can exclude the reflections by restricting to orientation preserving actions.

If X is a Riemannian symmetric space as above, and G is the isometry group of X, then G is a semisimple Lie group acting transitively on X. For any point $x \in X$, the stabilizer $G_x = \{g \in G \mid gx = x\}$ is a maximal compact subgroup of G, usually denoted by K, and hence X can be written as $X = G/K$.

While the symmetric space X is arguably the most important Riemannian homogeneous space associated with G, there are other important Riemannian homogeneous spaces associated with G. For example, with respect to a left invariant Riemannian metric, G is a homogeneous Riemannian manifold, but is not a symmetric space (note that G is not a compact Lie group). If $\Gamma \subset G$ is a discrete subgroup, then $\Gamma\backslash G$ is a locally Riemannian homogeneous space. Such spaces are important for various applications. For instance, when $G = \mathrm{SL}(n, \mathbb{R})$ and $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, the quotient $\mathrm{SL}(n, \mathbb{Z})\backslash\mathrm{SL}(n, \mathbb{R})$ is the moduli space of all unimodular lattices in $\mathbb{R}^n$.

Let K be a maximal compact subgroup of G as above. Then K acts on $\Gamma\backslash G$ on the right, and the quotient $\Gamma\backslash G/K$ is the locally symmetric space $\Gamma\backslash X$ discussed earlier. If the left invariant metric of G is right invariant under K, then it descends to a Riemannian metric on G/K, which is proportional to the Riemannian metric associated with the symmetric space X. There is one advantage of $\Gamma\backslash G$ over $\Gamma\backslash X$ in the sense that G acts on the former but not on the latter. This action of G on $\Gamma\backslash G$ gives the regular representation of G on $L^2(\Gamma\backslash G)$, which is the fundamental object in the theory of automorphic representations.

Another important class of homogeneous spaces arising from the above set-up is of the form G/H, where $H \subset K$ is not *necessarily a maximal* compact subgroup of G. Period domains in the theory of variation of Hodge structures is of this form, and arithmetic quotients of the form $\Gamma\backslash G/H$ are important in the study of Hodge structures and their variations. We refer the interested reader to the books [CMP] and [Vl for more details about period domains.

If the invariant metric on G is invariant on the right under K, then it is also invariant under H, and hence it descends to a Riemannian metric on G/H.

As a natural generalization of the results for locally symmetric spaces discussed in this paper, we also study, in §4, the end structure of spaces $\Gamma\backslash G$ satisfying the extremal condition $\lambda_1(\Gamma\backslash G) = \lambda_1(G)$, and also spaces $\Gamma\backslash G/H$ with $\lambda_1(\Gamma\backslash G/H) = \lambda_1(G/H)$.

The idea behind the main techniques in this study was originated in a series of work by the second and the third author [LW1]–[LW5]. To understand the ends of these spaces it is important to separate them into two distinct categories. Recall that an *end* E of a complete manifold (or orbifold) M is simply an unbounded component of $M \backslash D$ for some smooth compact domain $D \subset M$. In particular, E is a manifold (or orbifold) with boundary.

Definition 0.2. Let E be an end of a complete manifold (orbifold) M. It is said to be nonparabolic if it admits a positive Neumann Green's function for the Laplacian acting on functions. Otherwise, E is parabolic.

While the above definition is for an arbitrary complete manifold M, the notion of nonparabolicity has a precise geometric interpretation when the manifold has

$$\lambda_1(M) > 0. \tag{0.1}$$

In this case, nonparabolicity is equivalent to the end having infinite volume. In fact, in [LW1] the authors proved that under the condition of (0.1), a nonparabolic end must have volume growth satisfying

$$V_E(R) \geqq C \exp\left(2\sqrt{\lambda_1(M)}R\right),$$

where $V_E(R)$ denotes the volume of the set given by the intersection of the geodesic ball of radius R centered at some fixed point with the end E. They also proved that a parabolic end must have finite volume with decay rate

$$V(E) - V_E(R) \leqq C \exp\left(-2\sqrt{\lambda_1(M)}R\right),$$

where $V(E)$ is the volume of the end E. This equivalence allows us to bridge from analysis to geometry.

It was proved in [LT2] that the number of nonparabolic ends of M is given by the dimension of some space of bounded harmonic functions with finite Dirichlet integral. It is through this theory that the authors prove that for most locally symmetric spaces $\Gamma \backslash X$ whose associated symmetric spaces X are irreducible, there is exactly one (nonparabolic) end of infinite volume. The exceptional cases occur only when X is a rank one symmetric space. The readers should refer to [LW1], [LW5], [KLZ], and [Lm] for more details.

With the number of infinite volume ends under control, our remaining effort is to utilize the assumption

$$\lambda_1(\Gamma \backslash X) = \lambda_1(X),$$

to conclude that either $M = \Gamma \backslash X$ has no finite volume ends, or it must be diffeomorphic to a product manifold $\mathbb{R} \times N$ for some compact manifold N. In the latter case, M has exactly one finite volume and one infinite volume end, respectively.

The paper is structured so that §1 deals with the situation when M is a general Riemannian manifold. In §2, we prove the main theorem for X being an irreducible symmetric space of noncompact type. The case when $X = X_1 \times \cdots \times X_m$ is given by a product of m irreducible symmetric spaces is considered in §3. We also deal with homogeneous spaces in §4 by reducing to the locally symmetric space case. Finally, in §5 we gave some examples of $M = \Gamma \backslash X$ that satisfy

$$\lambda_1(M) = \lambda_1(X).$$

Quotients of *irreducible* symmetric spaces are the most familiar locally symmetric spaces, and include important quotients of the upper half plane (or more generally hyperbolic spaces) and the Siegel upper half spaces. On the other hand, it might be helpful to point out that among locally symmetric spaces, it is natural and important to consider locally symmetric spaces which are irreducible (i.e., no finite covers split as isometric products) but their universal covering spaces are reducible symmetric spaces, i.e., products of irreducible symmetric spaces. For example, the Hilbert modular varieties are such locally symmetric spaces. It is also important to consider *reducible locally symmetric spaces*.

One way to obtain examples of such locally symmetric spaces is a generalization of the construction of the above example of Hilbert modular varieties. Briefly, if we consider a semisimple linear algebraic group $\mathbf{G}$ defined over a number field k which is not equal to the field of rational numbers, and a subgroup Γ of $\mathbf{G}(k)$, then the natural spaces for Γ to act on are nontrivial products of symmetric spaces, and locally symmetric spaces associated with Γ are of the above type.

Since there are infinitely many number fields different from the field of rational numbers, this is really the generic case in some sense.

The second author would like to acknowledge S. T. Yau for raising the question of considering locally symmetric spaces of higher rank.

§1. General situation

We will first discuss the more general situation of an arbitrary complete, noncompact, Riemannian manifold M^n of real dimension n.

Theorem 1.1. *Let M^n be a complete Riemannian orbifold of dimension n. Suppose $f : (0, \infty) \to \mathbb{R}$ is a function with the property that*

$$\lim_{r \to \infty} f(r) = 2a > 0.$$

Assume that for any point $p \in M$, and if $r(x)$ is the distance function to the point p, we have

$$\Delta r(x) \leqq f\big(r(x)\big)$$

in the weak sense. If M has at least one parabolic end, then

$$\lambda_1(M) \leqq a^2.$$

Moreover, if $\lambda_1(M) = a^2$ and let $\gamma(t)$ be a geodesic ray issuing from a fixed point p to infinity of the parabolic end, then the Buseman function

$$\beta(x) = \lim_{t \to \infty} \big(t - r\big(\gamma(t), x\big)\big)$$

with respect to γ must satisfy

$$\Delta\beta = -2a,$$
$$|\nabla\beta| = 1.$$

In particular, M must be homeomorphic to $\mathbb{R} \times N$ for some compact orbifold N given by the level set of β.

Proof. Let us first observe that the singular set in M is of at most codimension 2 hence most analysis involving the Laplace operator, such as integration by parts, is valid due to similar argument provided by [LTi]. Also, following the argument of [BoZ], the splitting of M into products also follows. Hence, our proof reduces to the case when M is smooth, as long as we do not involve any techniques that are sensitive to orbifold singularities, such as the positivity of injectivity radius, in our argument.

We may assume that $\lambda_1(M) > 0$ as otherwise the theorem is trivial. In this case, M must be nonparabolic. By assumption, let E be a parabolic end of M. We can consider $E_1 = M \backslash E$ as a nonparabolic end. Let $\gamma : [0, \infty) \to M$ be a geodesic ray with $\gamma(0) = p$ and $\gamma(t) \to E(\infty)$, where $E(\infty)$ denotes the infinity of the end E. Using the inequality on $\Delta r(x)$, we conclude that

$$\Delta\beta \geqq -2a \tag{1.1}$$

and β is Lipschitz with Lipschitz constant 1 as proved in [LW5], Theorem 1.1. Setting $u = \exp(a\beta)$ and using (1.1), we have

$$\begin{aligned}\Delta u &\geqq au\Delta\beta + a^2 u|\nabla\beta|^2 \\ &= -a^2 u.\end{aligned} \tag{1.2}$$

For any nonnegative cut-off function ϕ, we consider

$$\begin{aligned}\int_M |\nabla(\phi u)|^2 &= -\int_M \phi^2 u\Delta u + \int_M |\nabla\phi|^2 u^2 \\ &\leqq a^2 \int_M \phi^2 u^2 + \int_M |\nabla\phi|^2 u^2.\end{aligned}$$

Combining with the variational principal of $\lambda_1(M)$, we conclude that

$$\big(\lambda_1(M) - a^2\big) \int_M \phi^2 u^2 \leqq \int_M |\nabla\phi|^2 u^2. \tag{1.3}$$

Assuming the contrary that $\lambda_1 > a^2$, we obtain a contradiction if we can justify the right-hand side tends to 0 for a sequence of cut-off functions ϕ unless $\lambda_1(M) = a^2$ and all the above inequalities are equalities.

To estimate the right-hand side of (1.3) on the parabolic end E, we first show that

$$V_E(R) \backslash V_E(R-1) \leqq \exp\big(-2a(R-1)\big), \tag{1.4}$$

where $V_E(R)$ denotes the volume of the set $B_E(R) = B_p(R) \cap E$ given by the intersection of the geodesic ball centered at $p \in M$ of radius R and E. Indeed, this follows from the volume estimate of Theorem 2.1 in [LW3] stating that

$$\begin{aligned} V_E(R) \backslash V_E(R-1) &\leqq \exp\bigl(-2\sqrt{\lambda_1(M)}(R-1)\bigr) \\ &\leqq \exp\bigl(-2a(R-1)\bigr). \end{aligned}$$

Using $|\nabla\beta| = 1$, we observe that

$$u(x) \leqq C \exp\bigl(ar(x)\bigr)$$

hence together with (1.4), we have

$$\int_{B_E(R)\backslash B_E(R-1)} u^2 \leqq C.$$

Choosing the cut-off function on E to be

$$\phi(x) = \begin{cases} 1 & \text{if } r(x) \leqq R, \\ \dfrac{2R - r(x)}{R} & \text{if } R \leqq r(x) \leqq 2R, \\ 0 & \text{if } 2R \leqq r(x), \end{cases}$$

the right-hand side of (1.3) on E can be estimated by

$$\begin{aligned} \int_E |\nabla\phi|^2 u^2 &\leqq R^{-2} \int_{B_E(2R)\backslash B_E(R)} u^2 \\ &= R^{-2} \sum_{i=1}^{[R]} \int_{B_E(R+i)\backslash B_E(R+i-1)} u^2 \\ &\leqq CR^{-1}, \end{aligned}$$

which tends to 0 as $R \to \infty$.

For the non-parabolic end $M\backslash E$, we choose the cut-off function ϕ to be

$$\phi(x) = \begin{cases} 1 & \text{if } -\beta(x) \leqq R, \\ \dfrac{2R + \beta(x)}{R} & \text{if } R \leqq -\beta(x) \leqq 2R, \\ 0 & \text{if } 2R \leqq -\beta(x). \end{cases}$$

Note that since $M\backslash E$ and E can be disconnected by a compact set, it was proved in [LW5], Theorem 1.1, that $-\beta$ is equivalent to the distance function to the compact set $B_p(R_0)$. Indeed, for any point $x \in M$, let us consider the geodesic segment τ_t joining $x = \tau_t(0)$ to $\gamma(t)$. Letting $t \to \infty$, the sequence τ_t converges to a geodesic ray emanating from $x = \tau(0)$ to $E(\infty)$. If x is in $M\backslash B_p(R_0)$ but not in E, say $x \in E_1$, then τ must pass through $B_p(R_0)$.

Let us denote $y = \tau(s)$ to be the first point on τ that intersects $B_p(R_0)$. Then by the triangle inequality,

$$\begin{aligned}\beta\big(\tau(s)\big) - \beta\big(\tau(0)\big) &= \lim_{t\to\infty}\Big(r\big(\tau(0),\gamma(t)\big) - r\big(\tau(s),\gamma(t)\big)\Big)\\ &\geqq \lim_{t\to\infty}\Big(r\big(\tau(0),\gamma(t)\big) - r\big(\tau_t(s),\gamma(t)\big) - r\big(\tau_t(s),\tau(s)\big)\Big)\\ &\geqq \lim_{t\to\infty}\Big(r\big(\tau(0),\gamma(t)\big) - r\big(\tau_t(s),\gamma(t)\big)\Big)\\ &= s.\end{aligned}$$

So

$$\beta(y) - \beta(x) \geqq r(y,x)$$

and

$$-\beta(x) \geqq \inf_{z\in B_p(R_0)} r(z,x) - \sup_{z\in B_p(R_0)} \beta(z).$$

Combining with $|\nabla\beta| = 1$, we conclude that, when restricted on E_1, $-\beta$ is equivalent to the distance function to the set $B_p(R_0)$.

In particular, the function $-\beta$ is proper on $M\backslash E$. Also, because $|\nabla\beta| = 1$, we have

$$\int_{M\backslash E} |\nabla\phi|^2 u^2 = R^{-2} \int_{\bar{B}(R,2R)} \exp(2a\beta) \tag{1.5}$$

where $\bar{B}(R,2R) = \{x \in M\backslash E \mid R < -\beta(x) < 2R\}$. We now claim that the volume of the set $\bar{B}(R,R+1)$, denoted by $\bar{V}(R,R+1)$, is bounded by

$$\bar{V}(R,R+1) \leqq C\exp(2aR)$$

for sufficiently large R. Indeed, integrating (1.1) yields

$$\begin{aligned}2a\bar{V}(R_0,t) &\geqq -\int_{\bar{B}(R_0,t)} \Delta\beta\\ &= -\int_{\bar{\ell}(t)} \frac{\partial\beta}{\partial v} + \int_{\bar{\ell}(R_0)} \frac{\partial\beta}{\partial v}\\ &= \bar{A}(t) - \bar{A}(R_0),\end{aligned}$$

where $\bar{\ell}(R) = \{x \in M\backslash E \mid \beta(x) = -R\}$ and $\bar{A}(R)$ denotes its $(n-1)$-dimensional area. Hence integrating from R_0 to $R+1$, the above inequality and co-area formula yield

$$\begin{aligned}2a(R+1-R_0) &\geqq \int_{R_0}^{R+1} \bar{A}(t)\big(\bar{V}(R_0,t) + (2a)^{-1}\bar{A}(R_0)\big)^{-1}\,dt\\ &= \log\big(\bar{V}(R_0,R+1) + (2a)^{-1}\bar{A}(R_0)\big) - \log\big((2a)^{-1}\bar{A}(R_0)\big),\end{aligned}$$

implying

$$\overline{V}(R_0, R+1) \leqq C \exp(2aR)$$

as claimed. Hence we can estimate

$$\begin{aligned}
\int_{\bar{B}(R,2R)} \exp(2a\beta) &\leqq \sum_{i=1}^{[R]} \int_{\bar{B}(R+i-1,\,R+i)} \exp(2a\beta) \\
&\leqq \sum_{i=1}^{[R]} \overline{V}(R+i-1, R+i) \exp\bigl(-2a(R+i-1)\bigr) \\
&\leqq CR,
\end{aligned}$$

and conclude that the right-hand side of (1.5) tends to 0. In particular, $\lambda_1(M) \leqq a^2$.

In the event if

$$\lambda_1(M) = a^2,$$

then we conclude that

$$\Delta\beta = -2a, \tag{1.6}$$

$$|\nabla\beta| = 1, \tag{1.7}$$

and β has no critical points. In particular, M must be homeomorphic to $\mathbb{R} \times N$ for some compact orbifold N. □

Note that in the case when $\lambda_1(M) = a^2$ in Theorem 1.1, the Bochner formula together with (1.6) and (1.7) implies that

$$\begin{aligned}
0 &= \Delta|\nabla\beta|^2 \\
&= 2\beta_{ij}^2 + 2\langle\nabla\beta, \nabla\Delta\beta\rangle + 2\,\mathrm{Ric}_{11} \\
&= 2\beta_{ij}^2 + 2\,\mathrm{Ric}_{11}
\end{aligned}$$

for the unit vector $e_1 = \nabla\beta$. Using (1.7) again, this implies that $\beta_{1i} = 0$ for all i and the second fundamental form II of a level set of β satisfies

$$|II|^2 = -\mathrm{Ric}_{11}. \tag{1.8}$$

On the other hand, the Gauss curvature equation asserts that for $\alpha \neq \tau$,

$$K_N(e_\alpha, e_\tau) = K_M(e_\alpha, e_\tau) + \lambda_\alpha\lambda_\tau$$

for an orthonormal frame $\{e_\alpha\}_{\alpha=2}^n$ on the level set of β that diagonalizes II with corresponding eigenvalues $\{\lambda_\alpha\}_{\alpha=2}^n$. Since the scalar curvature of M is given by

$$
\begin{aligned}
S_M &= \sum_{i=1}^{n} \mathrm{Ric}_{ii} \\
&= \mathrm{Ric}_{11} + \sum_{\alpha=2}^{n} \mathrm{Ric}_{\alpha\alpha} \\
&= 2\,\mathrm{Ric}_{11} + \sum_{\alpha,\tau \neq 1} K_M(e_\alpha, e_\tau) \\
&= 2\,\mathrm{Ric}_{11} + \sum_{\tau \neq \alpha} K_N(e_\alpha, e_\tau) - \sum_{\tau \neq \alpha} \lambda_\alpha \lambda_\tau \\
&= 2\,\mathrm{Ric}_{11} + S_N - \sum_{\tau \neq \alpha} \lambda_\alpha \lambda_\tau,
\end{aligned}
$$

this implies that

$$
S_N - S_M + 2\,\mathrm{Ric}_{11} = \sum_{\tau \neq \alpha} \lambda_\alpha \lambda_\tau. \tag{1.9}
$$

On the other hand, (1.6) and (1.7) assert that

$$
H = -2a
$$

where H is the mean curvature of the level set of β. Combining with (1.8) and (1.9), we conclude that

$$
\begin{aligned}
4a^2 = H^2 &= |II|^2 + \sum_{\tau \neq \alpha} \lambda_\alpha \lambda_\tau \\
&= S_N - S_M + \mathrm{Ric}_{11}.
\end{aligned}
$$

Hence

$$
S_N = 4a^2 + S_M - \mathrm{Ric}_{11} \tag{1.10}
$$

and

$$
\sum_{\tau \neq \alpha} \lambda_\alpha \lambda_\tau = 4a^2 + \mathrm{Ric}_{11}.
$$

Also note that the inequality

$$
|II|^2 \geqq \frac{H^2}{n-1}
$$

implies that

$$
-(n-1)\,\mathrm{Ric}_{11} \geqq 4a^2 \tag{1.11}
$$

with equality if and only if $\lambda_\alpha = \lambda_\tau$ for all α and τ.

We first observe that the above theorem allows us to recover a theorem proved in [LW2].

Corollary 1.2. *Let M^n be a complete manifold of dimension $n \geqq 2$. Assume that*

$$\mathrm{Ric}_M \geqq -(n-1)$$

and

$$\lambda_1(M) \geqq \frac{(n-1)^2}{4}.$$

Then M must either have no finite volume end or it must be a warped product $M = \mathbb{R} \times N$ with metric given by

$$ds_M^2 = dt^2 + \exp(2t)\, ds_N^2,$$

where N is a compact manifold whose metric ds_N^2 has nonnegative Ricci curvature.

Proof. We first observe that the assumption on the Ricci curvature and Laplacian comparison theorem asserts that

$$\Delta r \leqq (n-1)\coth r,$$

hence one checks readily that the function $f(r) = (n-1)\coth r$ satisfies the hypothesis of Theorem 1.1 with $a = \frac{n-1}{2}$. Therefore we conclude that if M has a parabolic end it must be homeomorphic to $\mathbb{R} \times N$ for some compact manifold. Moreover, since $\lambda_1(M) = \frac{(n-1)^2}{4} > 0$, an end being parabolic is equivalent to having finite volume. Also, (1.11) takes the form

$$\begin{aligned} -(n-1) &\geqq \mathrm{Ric}_{11} \\ &\geqq -(n-1) \end{aligned}$$

of an equality and we conclude that

$$II = -(\delta_{\alpha\tau})$$

is a diagonal matrix. In this case, the metric on M must be of the form

$$ds_M^2 = dt^2 + \exp(-2t)\, ds_N^2.$$

A direct computation shows that the sectional curvature for the section spanned by $\left\{e_1 = \frac{\partial}{\partial t}, e_\alpha\right\}$ is given by

$$K_M(e_1, e_\alpha) = -1.$$

The Gauss curvature equation implies that

$$K_M(e_\alpha, e_\tau) = K_N(e_\alpha, e_\tau) - 1,$$

and hence

$$\mathrm{Ric}_{\alpha\alpha} = \overline{\mathrm{Ric}}_{\alpha\alpha} - (n-1),$$

where $\overline{\mathrm{Ric}}_{\alpha\alpha}$ is the Ricci curvature of N. This implies that

$$\overline{\mathrm{Ric}}_{\alpha\alpha} \geqq 0.$$

The theorem follows by setting t to be $-t$. □

Let us remark a special case of Theorem 1.1 when M is an Einstein manifold with Einstein constant $-C < 0$. In particular, (1.10) and (1.11) become

$$S_N = 4a^2 - (n-1)C$$

and

$$(n-1)C \geqq 4a^2.$$

This implies that $S_N \leqq 0$ with $S_N = 0$ if and only if

$$II = -\frac{2a}{n-1}(\delta_{\alpha\tau}).$$

In the case of equality, using the same argument as in the above corollary, we conclude that

$$ds_M^2 = dt^2 + \exp\left(-\frac{4a}{n-1}t\right) ds_N^2,$$

hence

$$K(e_1, e_\alpha) = -\frac{4a^2}{(n-1)^2}$$

and

$$\begin{aligned} -\frac{4a^2}{n-1} = -C &= \mathrm{Ric}_{\alpha\alpha} \\ &= \overline{\mathrm{Ric}}_{\alpha\alpha} - \frac{4a^2}{n-1}. \end{aligned}$$

We conclude that N must be Ricci flat.

§2. Quotients of irreducible symmetric spaces

This section deals with the case when $M = \Gamma \backslash X$ is a quotient of an irreducible symmetric space X by a discrete group of isometry Γ acting effectively on X. It should be stressed that Γ is not necessarily torsion-free, and hence $\Gamma \backslash G/K$ is in general an orbifold.

Theorem 2.1. *Let $X = G/K$ be an irreducible symmetric space of noncompact type with rank at least 2. Suppose $M = \Gamma\backslash G/K$ is noncompact quotient of G/K by a discrete subgroup Γ. Then M has no nontrivial L^2-harmonic 1-forms. In particular, M has at most one nonparabolic end.*

Proof. The theorem is basically a noncompact version of the Matsushima vanishing theorem [M]. A Bochner type argument for this case was observed by Calabi and generalized by Jost and Yau [JY] to harmonic maps. The vanishing of L^2 harmonic 1-form can be used to rule out the existence of a second nonparabolic end for M. Indeed, if M has two nonparabolic ends, then the Li-Tam [LT2] theory asserts the existence of a nonconstant bounded harmonic function, f, with finite Dirichlet integral. In particular, df is an L^2 harmonic 1-form, contradicting the vanishing theorem. For the sake of completeness, we will give a quick outline of the vanishing theorem.

We first assume M is a smooth manifold. Let $\Omega = \sum_{i=1}^{n} a_i\omega_i$ be an L^2 harmonic 1-form, with $\{\omega_i\}$ being an orthonormal coframe near a point in M. Since Ω is L^2 and harmonic, it is known that it satisfies the conditions

$$\sum_{i=1}^{n} a_{i,i} = 0$$

and

$$a_{i,j} = a_{j,i} \quad \text{for} \quad 1 \leqq i, j \leqq n,$$

where the subscripts denote covariant derivatives of Ω. The Bochner formula for harmonic 1-form asserts that

$$\Delta|\Omega|^2 = 2\,\mathrm{Ric}_{ij}\, a_i a_j + 2\sum_{i,j} a_{i,j}^2. \tag{2.1}$$

In particular, using a computation (see [LW1], Theorem 2.1) originated by Yau [Y], we have

$$\begin{aligned} |\nabla\Omega|^2 &= \sum_{i,j} a_{i,j}^2 \\ &\geqq \frac{n}{n-1}\big|\nabla|\Omega|\big|^2, \end{aligned} \tag{2.2}$$

and hence

$$\Delta|\Omega| \geqq |\Omega|^{-1}\,\mathrm{Ric}_{ij}\, a_i a_j + \frac{1}{n}(|\Omega|)^{-1}\sum_{i=1}^{n} a_{i,j}^2. \tag{2.3}$$

We first claim that $\sum_{i,j} a_{i,j}^2$ is integrable. Indeed, let ϕ be the cut-off function defined by

$$\phi = \begin{cases} 1 & \text{on } B_p(R), \\ \dfrac{2R-r}{R} & \text{on } B_p(2R)\backslash B_p(R), \\ 0 & \text{on } M\backslash B_p(2R). \end{cases} \tag{2.4}$$

Then (2.3) implies that

$$\int_M \phi^2|\Omega|\Delta|\Omega| \geqq \int_M \phi^2 \operatorname{Ric}_{ij} a_i a_j + \frac{1}{n}\int_M \phi^2 \sum_{i=1}^n a_{i,j}^2$$
$$\geqq -C\int_M \phi^2|\Omega|^2 + \frac{1}{n}\int_M \phi^2 \sum_{i=1}^n a_{i,j}^2, \tag{2.5}$$

where we have used the fact that M is Einstein with Einstein constant $-C$. However,

$$\int_M \phi^2|\Omega|\Delta|\Omega| = -2\int_M \phi|\Omega|\langle\nabla\phi, \nabla|\Omega|\rangle - \int_M \phi^2\bigl|\nabla|\Omega|\bigr|^2$$
$$\leqq \frac{1}{\varepsilon}\int_M |\nabla\phi|^2|\Omega|^2 + (\varepsilon - 1)\int_M \phi^2\bigl|\nabla|\Omega|\bigr|^2.$$

Hence together with (2.5), we conclude that

$$\frac{1}{\varepsilon}\int_M |\nabla\phi|^2 + (C - 1 + \varepsilon)\int_M \phi^2|\Omega|^2 \geqq \frac{1}{n}\int_M \phi^2 \sum_{i=1}^n a_{i,j}^2.$$

First letting $R \to \infty$ in the definition of ϕ and using the assumption that Ω is L^2, then by letting $\varepsilon \to 0$, we obtain

$$n(C-1)\int_M |\Omega|^2 \geqq \int_M \sum_{i=1}^n |\nabla\Omega|^2. \tag{2.6}$$

Taking the same cut-off function ϕ as in (2.4) and using (2.1), we derive

$$\int_M \phi^2 \operatorname{Ric}_{ij} a_i a_j + \int_M \phi^2 \sum_{i,j} a_{i,j}^2 = \frac{1}{2}\int_M \phi^2\Delta|\Omega|^2$$
$$= -2\int_M \phi|\Omega|\langle\nabla\phi, \nabla|\Omega|\rangle.$$

However, using Schwarz inequality and (2.2), the last term can be estimated by

$$\left|2\int_M \phi|\Omega|\langle\nabla\phi, \nabla|\Omega|\rangle\right| \leqq \varepsilon\int_M \phi^2|\nabla\Omega|^2 + \frac{1}{\varepsilon}\int_M |\nabla\phi|^2|\Omega|^2.$$

Hence, we obtain

$$\left|\int_M \phi^2 \operatorname{Ric}_{ij} a_i a_j + \int_M \phi^2 \sum_{i,j} a_{i,j}^2\right| - \varepsilon\int_M \phi^2 \sum_{i,j} a_{i,j}^2 \leqq \frac{1}{\varepsilon}\int_M |\nabla\phi|^2|\Omega|^2.$$

The assumption that Ω is L^2 implies that the right-hand side tends to 0 as $R \to \infty$. After letting $\varepsilon \to 0$, we conclude that

$$-\int_M \operatorname{Ric}_{ij} a_i a_j = \int_M \sum_{i,j} a_{i,j}^2. \tag{2.7}$$

Now for any 1-form Ω (not necessarily harmonic), we consider the commutation formula

$$a_{i,jk} - a_{i,kj} = -R_{\ell ijk} a_\ell,$$

which implies

$$(2.8) \qquad -2\int_M \phi^2 \sum_{i,j,k,\ell} R_{\ell ijk} a_\ell a_{i,jk} = -\int_M \phi^2 \sum_{i,j,k,\ell} R_{\ell ijk} a_\ell (a_{i,jk} - a_{i,kj})$$
$$= \int_M \phi^2 \sum_{i,j,k,\ell,m} R_{\ell ijk} R_{mijk} a_\ell a_m.$$

On the other hand, integration by parts and using the assumption that M is locally symmetric imply

$$(2.9) \qquad -2\int_M \phi^2 \sum_{i,j,k,\ell} R_{\ell ijk} a_\ell a_{i,jk} = 4\int_M \phi \sum_{i,j,k,\ell} R_{\ell ijk} a_\ell a_{i,j}\phi_k + 2\int_M \phi^2 \sum_{i,j,k,\ell} R_{\ell ijk} a_{\ell,k} a_{i,j}.$$

However, since both $|\Omega|^2$ and $\sum_{i,j} a_{i,j}^2 = |\nabla\Omega|^2$ are integrable, by choosing ϕ as given by (2.4) and letting $R \to \infty$, we conclude that

$$\left|4\int_M \phi \sum_{i,j,k,\ell} R_{\ell ijk} a_\ell a_{i,j}\phi_k\right| \leqq C\int_M |\phi|\,|\nabla\phi|\,|\Omega|\,|\nabla\Omega|$$
$$\leqq CR^{-1}\int_M |\Omega|\,|\nabla\Omega|$$

tends to 0 as $R \to \infty$. Hence combining with (2.8) and (2.9) yields the identity

$$(2.10) \qquad \int_M \sum_{i,j,k,\ell,m} R_{\ell ijk} R_{mijk} a_\ell a_m = 2\int_M \sum_{i,j,k,\ell} R_{\ell ijk} a_{\ell,k} a_{i,j}.$$

Note that we have only used the facts that Ω and $|\nabla\Omega|$ are L^2.

We now follow the argument as in Jost-Yau [JY]. For irreducible symmetric spaces, it was computed by Calabi-Vesentini [CV], Borel [B1] and Kaneyuki-Nagano [KN] that there exists λ depending on G/K, such that

$$\sum_{i,j,k,\ell} R_{\ell ijk} a_{\ell,k} a_{i,j} = -\lambda \sum_{i,j} a_{i,j}^2.$$

Moreover, there is also a constant μ depending on G/K, such that

$$\sum_{i,j,k,\ell,m} R_{\ell ijk} R_{mijk} a_\ell a_m = -\mu \sum_{ij} \mathrm{Ric}_{ij}\, a_i a_j.$$

Hence (2.10) becomes

$$\mu \int_M \sum_{i,j} \mathrm{Ric}_{ij}\, a_i a_j = 2\lambda \int_M \sum_{i,j} a_{i,j}^2.$$

Substituting (2.7), we conclude that

$$0 = (\mu + 2\lambda) \int_M \sum_{i,j} a_{i,j}^2.$$

However, it was shown in [KN] that if the rank of G/K is at least 2, then $\mu > -2\lambda$ and we conclude that $a_{i,j} = 0$ for all $1 \leqq i, j \leqq n$, hence Ω is parallel. This contradicts (2.1) since the Einstein constant is negative.

In the case M is an orbifold, the singular set of M is of co-dimension at least two. So the cut-off argument in [LTi] can be applied to justify the preceding argument. □

Theorem 2.2. *Let $M = \Gamma \backslash X$, where X is an irreducible symmetric space of noncompact type $X = G/K$ and Γ is a not necessarily torsion-free discrete group acting effectively and isometrically on X. Suppose $\lambda_1(M) = \lambda_1(X)$. Then either*

(1) *M has only one end*; *or*

(2) *M is isometric to $\mathbb{R} \times N$ with metric*

$$ds_M^2 = dt^2 + \sum_{\alpha=2}^{m} \exp(-2b_\alpha t)\omega_\alpha^2,$$

where $\{\omega_2, \ldots, \omega_n\}$ is an orthonormal basis at the smooth points of N. Moreover, N is given by a compact quotient of some horosphere of X and b_α are the nonnegative constants such that $\{-b_\alpha^2\}$ are the eigenvalues of the symmetric tensor

$$A_{\alpha\gamma} = R_{1\alpha 1\gamma}$$

for a fixed direction e_1.

Proof. We first deal with the case that M is smooth. If X is of rank one, then it must be either the real hyperbolic space, the complex hyperbolic space, the quaternionic hyperbolic space, or the Cayley plane. For the case when X is the real hyperbolic space, the theorem follows from the previous work of the authors [LW1] and [LW2] as indicated by Corollary 1.2. In this case, the cross section is a flat manifold since M is assumed to have constant -1 curvature and the horosphere of $\mathbb{H}^n$ is simply the Euclidean space $\mathbb{R}^{n-1}$. In the case when X is the complex hyperbolic space, this was covered by [LW5], Theorem 1.1. The remaining two rank one cases given by the quaternionic hyperbolic space and the Cayley plane are separately studied in [KLZ] and [Lm].

Assume now that M is an irreducible locally symmetric space of rank at least 2. The assumption and Theorem 2.1 assert that M has exactly one nonparabolic end. Let us now assume that M has at least one parabolic end. We first observe that Theorem 1.1 is applicable here. Indeed, for the geodesic distance function $r(x)$ to a fixed point p on X, we may choose a parallel orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ along the normal geodesic $\gamma(t)$ from p to x such that $e_1 = \gamma'(t)$ and $\{e_2, \ldots, e_n\}$ diagonalizes the curvature tensor $R_{1\alpha 1\mu}$ with corresponding eigenvalues $-b_\alpha^2$, $\alpha = 2, \ldots, n$. Then it is easy to see that

$$\Delta r = \sum_{\alpha=2}^{n} b_\alpha \coth(b_\alpha r).$$

In this formula, when $b_\alpha = 0$, the term $b_\alpha \coth(b_\alpha r)$ is interpreted as r^{-1}.

Also, one computes that

$$\lambda_1(X) = \frac{\left(\sum_{\alpha=2}^{n} b_\alpha\right)^2}{4}.$$

Now it is not difficult to see that Theorem 1.1 can be applied to M with

$$f(r) = \sum_{\alpha=2}^{n} b_\alpha \coth(b_\alpha r)$$

and

$$2a = \sum_{\alpha=2}^{n} b_\alpha.$$

Thus, M has no finite volume end or

$$\Delta\beta = a^2,$$
$$|\nabla\beta| = 1.$$

In the latter case, following the argument as in [LW5], Theorem 1.1, we fix a level set N_0 of β and consider a geodesic τ given by $\tau' = \nabla\beta$ with $\tau(0) \in N_0$. At the point $\tau(0)$, let us consider the curvature $R_{1\alpha 1\mu}$ as a bilinear form restricted on the tangent space of N_0. In particular, there exists an orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ with $e_1 = \tau'$ and $e_\alpha \in TN_0$ for all $2 \leqq \alpha \leqq n$ such that $\{e_2, \ldots, e_n\}$ diagonalizes $R_{1\alpha 1\mu}$. Since M is an irreducible locally symmetric manifold of noncompact type, the sectional curvature of M must be nonpositive, hence

$$R_{1\alpha 1\mu} = -b_\alpha^2 \delta_{\alpha\mu}$$

for some $b_\alpha \geqq 0$. We extend the orthonormal frame along τ by parallel translating the basis $\{e_1, e_2, \ldots, e_n\}$. Since M is locally symmetric, the curvature satisfies

$$\frac{\partial R_{1\alpha 1\mu}}{\partial t} = R_{1\alpha 1\mu, 1} = 0.$$

So

$$R_{1\alpha 1\mu} = -b_\alpha^2 \delta_{\alpha\mu}$$

along τ. We consider the vector field

$$V_\alpha(t) = \exp(-b_\alpha t) e_\alpha$$

and verify that

$$\nabla_{\tau'} \nabla_{\tau'} V_\alpha = b_\alpha^2 V_\alpha.$$

On the other hand,

$$R_{\tau' V_\alpha}\tau' = \exp(-b_\alpha t) R_{1\alpha 1\alpha} e_\alpha = -b_\alpha^2 V_\alpha.$$

Hence, V_α is a Jacobi field along τ. Since this is true for all $2 \leqq \alpha \leqq n$, we conclude that the metric on N_t must be of the form

$$ds_t^2 = \sum_{\alpha=2}^{n} \exp(-2b_\alpha t)\omega_\alpha^2,$$

where $\{\omega_\alpha\}_{\alpha=2}^n$ is the dual coframe to $\{e_\alpha\}_{\alpha=2}^n$ at N_0. In particular, the second fundamental form on N_t must be a diagonal matrix when written in terms of the basis $\{e_\alpha\}_{\alpha=2}^n$ with eigenvalues given by $\{-b_\alpha\}_{\alpha=2}^n$. Moreover, since the Buseman function β has no critical points, and for any $x_0 \in N$, the curve (t, x_0) is a geodesic line which can be used to define β, the level sets N_t must be a compact quotient of some horosphere on X.

For the case that M has orbifold singularities, according to [BoZ], the same Laplacian comparison theorem for the distance function still holds globally on M in the weak sense. So the preceding argument works also, again by noticing that the singular set is at least of codimension two, hence posing no problem for the cut-off argument (see [LTi]). $\square$

§3. Quotients of products

In this section, we consider the case when X is an arbitrary symmetric space that is not necessarily irreducible. In particular, let us assume that $X = X_1 \times \cdots \times X_m$ are products of m irreducible factors, where each X_i is of noncompact type. We assume that $M = \Gamma\backslash X$ is a quotient of X by a discrete subgroup Γ, which is a not-necessarily torsion-free discrete group acting isometrically and properly on X. Hence, $M = \Gamma\backslash X$ is an orbifold.

First we will establish a vanishing theorem similar to that of Theorem 2.1 for this situation. Our proof combines the argument of Theorem 2.1 and an argument of Mok, Siu, and Yeung [MSY].

Theorem 3.1. *Let M be a noncompact quotient of X by a discrete group Γ. Suppose all the irreducible factors X_i are of rank at least* 2, *then M has no non-trivial L^2 harmonic* 1*-form. In particular, M has at most* 1 *nonparabolic end. If X has some rank*-1 *factors, then any L^2 harmonic* 1*-form Ω can be lifted to a harmonic* 1*-form on those factors alone.*

Proof. Let us first assume that all isometries $\gamma \in \Gamma$ are of the form $\gamma = (\gamma_1, \dots, \gamma_m)$ where γ_i is an isometry of X_i for all $1 \leqq i \leqq m$.

For each factor X_i of dimension n_i, let $\{u_\alpha^i \,|\, 1 \leqq \alpha \leqq n_i\}$ be local orthonormal frames for the tangent bundle. Let us choose local orthonormal frames $\{e_\alpha^i \,|\, 1 \leqq i \leqq m, 1 \leqq \alpha \leqq n_i\}$ on X so that for a fixed i if $\pi_p : TX \to TX_i$ is the projection on the i-th factor, then

$\pi_i(e^j_\alpha) = \delta_{ij} u^j_\alpha$. Note that since the metric on X is the product metric, the curvature tensor of X splits into the form

$$\langle \tilde{R}_{e^i_\alpha, e^i_\beta} e^i_\gamma, e^i_\tau \rangle = \langle R^i_{u^i_\alpha, u^i_\beta} u^i_\gamma, u^i_\tau \rangle$$

for all $1 \leqq \alpha, \beta, \gamma, \tau \leqq n_i$, where R^i denotes the curvature tensor of the factor X_i. Moreover, $\tilde{R}$ vanishes for all other indices. Suppose Ω is an L^2 harmonic 1-form on M given by

$$\Omega = \sum_{i=1}^{m} \sum_{\alpha=1}^{n_i} a^i_\alpha \omega^i_\alpha,$$

where $\{\omega^i_\alpha\}$ is the dual coframe to $\{e^i_\alpha\}$. Using Ω being harmonic, (2.7) implies that

$$(3.1) \qquad -\sum_{i=1}^{m} \sum_{\alpha,\beta} \int_M \mathrm{Ric}^i_{\alpha\beta}\, a^i_\alpha a^i_\beta = \sum_{i=1}^{m} \int_M \sum_{\alpha,\beta=1}^{n_i} (a^i_{\alpha,\beta})^2 + \sum_{i=1}^{m} \sum_{\alpha=1}^{n_i} \sum_p \int_M (a^i_{\alpha,p})^2,$$

where p denotes the index for those directions e^j_β with $j \neq i$.

For $1 \leqq i \leqq m$, define the 1-form Ω^i on M by $\Omega^i = \sum_{\alpha=1}^{n_i} a^i_\alpha \omega^i_\alpha$. This can be viewed as lifting Ω to a Γ-invariant harmonic 1-form $\tilde{\Omega}$ on X and then restrict to X_i to give a Γ-invariant 1-form locally given by

$$\tilde{\Omega}^i = \sum_{\alpha=1}^{n_i} a^i_\alpha \eta^i_\alpha,$$

where $\{\eta^i_\alpha\}$ is the dual coframe to $\{u^i_\alpha\}$. The 1-form $\tilde{\Omega}^i$ is Γ-invariant because of the assumption on Γ. Globally speaking $\tilde{\Omega}_i$ is a 1-form on X when restricted to the sub-bundle spanned by the tangent vectors of TX_i.

Obviously $\tilde{\Omega}_i$ defines a 1-form Ω^i on M. The same argument as in the proof of Theorem 2.1 asserts that since $|\Omega|$ is L^2, the norm of its covariant derivative $|\nabla\Omega|$ is also L^2. This implies that $|\Omega^i|^2$ and $\sum_{\alpha,\beta=1}^{n_i} (a^i_{\alpha,\beta})^2$ are integrable on M. Since the metric is the product metric, the same commutation formula and integration by parts argument as in the proof of (2.10) applying to Ω^i asserts that

$$(3.2) \qquad \int_M \sum_{\alpha,\beta,\gamma,\tau,\eta} R^i_{\alpha\gamma\tau\eta} R^i_{\beta\gamma\tau\eta} a^i_\alpha a^i_\beta = 2 \int_M \sum_{\alpha\beta\gamma\tau} R^i_{\alpha\beta\gamma\tau} a^i_{\alpha,\tau} a^i_{\beta,\gamma},$$

for all $1 \leqq i \leqq m$. Again, as noted in the proof of Theorem 2.1, there exists λ^i depending on X_i, such that

$$\sum_{\alpha\beta\gamma\tau} R^i_{\alpha\beta\gamma\tau} a^i_{\alpha,\tau} a^i_{\beta,\gamma} = -\lambda^i \sum_{\alpha,\beta} (a^i_{\alpha,\beta})^2.$$

Moreover, there is also a constant μ^i depending on X_i, such that

$$\sum_{\alpha,\beta,\gamma,\tau,\eta} R^i_{\alpha\gamma\tau\eta} R^i_{\beta\gamma\tau\eta} a^i_\alpha a^i_\beta = -\mu^i \sum_{\alpha,\beta} \mathrm{Ric}^i_{\alpha\beta}\, a^i_\alpha a^i_\beta.$$

Hence (3.2) becomes

$$\mu^i \int_M \sum_{\alpha,\beta} \mathrm{Ric}^i_{\alpha\beta}\, a^i_\alpha a^i_\beta = 2\lambda^i \int_M \sum_{\alpha,\beta} (a^i_{\alpha,\beta})^2. \tag{3.3}$$

Notice that all λ^i and μ^i are nonzero constants satisfying $\mu^i \geqq -2\lambda^i$ with equality if and only if X_i is of rank 1. After combining with (3.1), we conclude that

$$a^i_{\alpha,p} = 0 \tag{3.4}$$

for all $1 \leqq i \leqq m$ and

$$a^i_{\alpha,\beta} = 0$$

for those factors X_i that are of rank at least 2. As before, the first part of the theorem follows.

In general, for any higher rank factor X_i, we conclude that the 1-form $\tilde{\Omega}^i$, treated as a form on X_i when X_i is embedded into X as a submanifold, is parallel and after applying (2.1) on X_i, we conclude that $\tilde{\Omega}^i = 0$. Therefore $\tilde{\Omega}$ has no components in the higher rank factors. Moreover, if X_j is a factor of higher rank, then (3.4) implies that Ω^j is parallel along any direction in X_i for $i \neq j$. Let us write

$$X = X' \times X'',$$

where X' is the product of all the rank-1 factors and X'' is the product of all the higher rank factors. For any $g \in \Gamma$, we write $g = (g', g'')$ with g' and g'' being isometries of X' and X'', respectively. Since Ω has no components involving X'' and Ω is parallel in the directions of X'', Ω is invariant under the isometry $\big(\mathrm{id}, (g'')^{-1}\big)$. In particular, since Ω is invariant under $g = (g', g'')$, we conclude that Ω is invariant under (g', id). Let us define the discrete subgroup of isometries acting on X' by $\Gamma' = \{g' \mid (g', g'') \in \Gamma\}$. The 1-form Ω is obviously defined on X', which is invariant under Γ'. Moreover, it is harmonic and hence defines a harmonic 1-form on $\Gamma' \backslash X'$.

In general, the isometry group Γ might contain elements not of the form $(\gamma_1, \ldots, \gamma_m)$. In this case, we consider the subgroup $\overline{\Gamma} = \{\gamma \in \Gamma \mid \gamma = (\gamma_1, \ldots, \gamma_m)\}$. It can be seen easily that $\overline{\Gamma}$ has index at most $m!$ in Γ. Hence, after lifting to a finite covering $\overline{M}$ of M, $\overline{M} = \overline{\Gamma} \backslash X$. Any L^2 harmonic 1-form defined on M can be lifted to an L^2 harmonic 1-form defined on $\overline{M}$. We now apply the above argument on $\overline{M}$. □

Let us now recall that when X is an irreducible rank one symmetric space, vanishing of L^2 harmonic 1-forms is no longer automatic. In fact, there are examples of $M = \Gamma \backslash X$ that have two infinite volume ends. In paricular, it was proved [LW1] that if X is the real hyperbolic space of dimension $n \geqq 3$ and if $\lambda_1(M) \geqq n - 2$ then M either has one infinite volume end or it must be a warped product with $\lambda_1(M) = n - 2$. In [LW5], Li-Wang considered the case when $X = \mathbb{CH}^m$ is the complex hyperbolic space with complex dimension m and Ricci curvature normalized to be $-2(m+1)$, if $\lambda_1(M) > \dfrac{m+1}{2}$ then M has only one infinite volume end. In [KLZ], the authors treated the case when $X = \mathbb{QH}^m$ is the quaternionic hyperbolic space of real dimension $2m$. They showed that if the Ricci curvature is normalized to be $-4(m+2)$ and if $\lambda_1(M) \geqq \frac{8(m+2)}{3}$, then M must have only one infinite volume end. Similarly, when $X = \mathbb{O}\mathbb{H}$ is the Cayley hyperbolic space of dimension 16, it was proved by [Lm] that by normalizing the Ricci curvature to be -36 and if $\lambda_1(M) \geqq \frac{216}{7}$, then M must have only one infinite volume end. Note that after using the fact that $\mathbb{H}^2 = \mathbb{C}\mathbb{H}^1$, the above list covered all the rank one cases. For the sake of bookkeeping, it is convenient to unify the normalization. Let us assume X^n is any of the rank one symmetric space of noncompact type. Let us define the constant

$$A = \begin{cases} \frac{n-1}{n-2} & \text{if } X = \mathbb{H}^n \text{ and } n \geqq 3, \\ 4 & \text{if } X = \mathbb{C}\mathbb{H}^m, \\ \frac{3}{2} & \text{if } X = \mathbb{Q}\mathbb{H}^m, \\ \frac{7}{6} & \text{if } X = \mathbb{O}\mathbb{H}. \end{cases} \tag{3.5}$$

If $M = \Gamma \backslash X$ and

$$\mathrm{Ric}_M > -A\lambda_1(M)$$

then M must have only one infinite volume end. If

$$\mathrm{Ric}_M = -A\lambda_1(M)$$

and M has more than one infinite volume end, then either $X = \mathbb{H}^n$ with $n \geqq 3$ and M is a warped product, or $X = \mathbb{C}\mathbb{H}^m$.

We can now state the theorems for quotients of products of rank one symmetric spaces. The first theorem deals with products of real hyperbolic spaces of dimension at least 3.

Theorem 3.2. *Let $X = X_1 \times X_2 \times \cdots \times X_m$ be the product of m irreducible, rank one, symmetric spaces of non-compact type. Assume that each of the factors X_i is a scalar multiple of some real hyperbolic space $\mathbb{H}^{n_i}$ of dimension $n_i \geqq 3$. Let us denote $-C_i$ to be the Ricci curvature of X_i. Suppose Γ is a discrete subgroup of the isometry group acting on X and $M = \Gamma \backslash X$ is its quotient space. Assume that M is noncompact with $\lambda_1(M) \geqq \max_{i=1}^m \left\{\frac{n_i - 2}{n_i - 1} C_i\right\}$. Then M must either have only one infinite volume end, or it is diffeomorphic to a product manifold $\mathbb{R} \times N$ where N is compact. In the latter case, $\lambda_1(M) = \max_{i=1}^m \left\{\frac{n_i - 2}{n_i - 1} C_i\right\}$.*

Proof. As in the argument of the proof of Theorem 3.1, if necessary by lifting to a finite covering, we may assume that all $\gamma \in \Gamma$ are of the form $\gamma = (\gamma_1, \ldots, \gamma_m)$. Since we assume $\lambda_1(M) > 0$, an end of M is nonparabolic if and only if its volume is infinite by [LW1]. Now suppose that M has more than one infinite volume end. Then by [LT2] there exists a bounded nonconstant harmonic function f on M with finite total Dirichlet integral. Applying Theorem 3.1 to $\Omega = df$, we conclude from (3.4) that the Hessian of f satisfies $f_{pq} = 0$ for all p and q, where p denotes the index for directions e^i_α and q for directions e^j_β with $i \neq j$. Lifting the harmonic function f to the covering X and applying a result of Freire [F], we conclude that f is harmonic on each factor X_i. Let us decompose the exterior derivative

$$d = \sum_{i=1}^m d_i$$

with respect to the decomposition of the tangent space $TM = \bigoplus_{i=1}^m TX_i$. In particular,

$$d_i f = \Omega^i = \sum_{\alpha=1}^{n_i} f^i_\alpha \omega^i_\alpha.$$

Using the product structure, d_i commutes with the Laplacian Δ, hence $d_i f = \Omega^i$ is also harmonic on M and

$$\sum_{\alpha=1}^{n_i} f^i_{\alpha\alpha} = 0. \tag{3.6}$$

One readily checks that the Bochner formula yields

$$\Delta |d_i f|^2 = 2\,\mathrm{Ric}_{M_i}(d_i f, d_i f) + 2\sum_{\alpha,p=1}^{n_i} f^i_{\alpha p}.$$

Using (3.6), we conclude that

$$\Delta |d_i f|^2 \geqq -2C_i |d_i f|^2 + \frac{2n_i}{n_i - 1} \big|d_i |d_i f|\big|^2. \tag{3.7}$$

Let us define $h_i = |\Omega^i|^{\frac{n_i-2}{n_i-1}}$, and (3.7) can be expressed as

$$\Delta h_i \geqq -\frac{n_i - 2}{n_i - 1} C_i h_i. \tag{3.9}$$

We now claim that using the assumption $\lambda_1(M) \geqq \max_{i=1}^m \left\{ \frac{n_i - 2}{n_i - 1} C_i \right\}$ implies that either $h_i = 0$ or (3.9) becomes equality with h_i being nonzero everywhere (see [LW4]). Indeed this follows from a similar argument as in [LW4] by justifying

$$\int_M |\nabla \phi|^2 h_i^2 \to 0$$

for a sequence of appropriate cut-off functions ϕ. Since $h_i = |d_i f|^{\frac{n_i-2}{n_i-1}} \leqq |\nabla f|^{\frac{n_i-2}{n_i-1}}$, it suffices to show that

$$\int_M |\nabla \phi|^2 |\nabla f|^{\frac{2(n_i-2)}{n_i-1}} \to 0,$$

which is exactly what was shown in [LW4]. We can conclude that either $df = 0$ or it is nonvanishing everywhere on M. Since f is nonconstant, we must have df nonzero everywhere and M splits into $\mathbb{R} \times N$, where N is any fixed level set of f. As M is assumed to have at least two ends, N must be compact. Note that on those X_i such that $d_i f$ is not identically zero, we must have $\lambda_1(M) = \max_{j=1}^m \left\{ \frac{n_j - 2}{n_j - 1} C_j \right\} = \frac{n_i - 2}{n_i - 1} C_i$. At this point, we should also point out that in case we need to argue on the finite covering $\Gamma' \backslash X$ of $M = \Gamma \backslash X$, the product $\mathbb{R} \times N$ is diffeomorphic to $\Gamma' \backslash X$. Since N is given by a level set of f and f is invariant under the deck transformations $\overline{\Gamma} = \Gamma / \Gamma'$, N is invariant under $\overline{\Gamma}$ also. In particular, $\mathbb{R} \times N / \overline{\Gamma}$ is diffeomorphic to M. We can now conclude that $N / \overline{\Gamma}$ is compact. This completes the proof. □

Theorem 3.3. *Let $X = X_1 \times X_2 \times \cdots \times X_m$ be the product of m irreducible, rank one, symmetric spaces of non-compact type. Let us denote $-C_i$ to be the Ricci curvature of X_i. Suppose Γ is a discrete subgroup of the isometry group acting on X and $M = \Gamma \backslash X$ is its quotient space. Assume that M is noncompact with $\lambda_1(M) > \max_{i=1}^m \{A_i^{-1} C_i\}$, where A_i is the corresponding constant of X_i defined by* (3.5). *Then M must have only one infinite volume end.*

Proof. We essentially follow a similar argument as in the proof of Theorem 3.2 and combine with the vanishing theorem of Siu and Corlette. The main issue is to deal with the case when X_i is either $\mathbb{CH}^m$, $\mathbb{QH}^m$, or $\mathbb{OH}$. The technique is similar for all these cases, so we will just assume X_i is a scalar multiple of some $\mathbb{CH}^m$. Following the proof of the previous theorem, we claim that the Bochner formular (3.7) can be improved to

$$\Delta |d_i f|^2 \geqq -2C_i |d_i f|^2 + 4\big|d_i |d_i f|\big|^2. \tag{3.10}$$

Once the claim is verified, the proof of [LW5], Theorem 2.1 will imply that $d_i f = 0$.

To prove (3.10), we apply the noncompact version of Corlette's vanishing theorem stated in [KLZ]. Note that since X_i has a parallel Kähler form, Ω, it can be viewed as a parallel form on X. [KLZ], Theorem 3.1 asserts that

$$d * (df \wedge \Omega) = 0.$$

Following the computation and the notation of [KLZ], Lemma 3.1, this implies that

$$\sum_{i,j=1}^{2} f_{ij} \omega_j \wedge \ell(e_i) \Omega = 0,$$

where $\ell(e_i)$ is the interior product by e_i. In particular, this implies that f is pluri-harmonic on X_i. The improved Bochner formula (3.10) now follows from [L]. The cases of $\mathbb{QH}^m$ and $\mathbb{OH}$ follow the same way as in [KLZ] and [Lm]. □

Theorem 3.4. *Let $X = X_1 \times X_2 \times \cdots \times X_m$ be the product of m irreducible symmetric spaces of non-compact type. Suppose Γ is a discrete subgroup of the isometry group acting on X and $M = \Gamma \backslash X$ is its quotient space. Assume that M is noncompact with $\lambda_1(M) = \lambda_1(X)$. Then M either has no finite volume ends, or it must be diffeomorphic to the product $\mathbb{R} \times N$, where N is a compact quotient of the level set of the Buseman function with respect to some* *geodesic line* $\gamma(s) = (\gamma_1(s), \gamma_2(s), \ldots, \gamma_m(s))$ *with* γ_i *being geodesics in* X_i *for* $i = 1, \ldots, m$, *respectively. Moreover,* $|\gamma_i'| = \dfrac{B_i}{\sqrt{\sum_{i=1}^m B_i^2}}$ *with* $B_i = \sum_{\alpha=2}^{n_i} b_\alpha^i$ *where* $\{-(b_\alpha^i)^2\}$ *is the set of sectional curvatures of* X_i.

Proof. First let us observe that by the uniqueness of geodesic, if $\gamma(s)$ is a normal geodesic in X, then γ must be given by $\gamma(s) = (\gamma_1(s), \gamma_2(s), \ldots, \gamma_m(s))$ where γ_i are geodesics in X_i for $i = 1, \ldots, m$, respectively. Let us denote the tangent vector $v = \gamma'(s)$ at $\gamma(s)$. It has the decomposition $v = (v_1, v_2, \ldots, v_m)$ with $v_i = \gamma_i'(s)$. Let us choose an orthonormal frame $\{e_1, e_2, \ldots, e_N\}$ with $N = \sum_{i=1}^m n_i$ at the point $\gamma(s)$ such that $e_1 = v$. We choose $\{u_1^i, \ldots, u_{n_i}^i\}$ to be orthonormal frames of X_i, so that $v_i = \theta_i u_1^i$ for $0 \leqq \theta_i \leqq 1$ satisfying $\sum_{i=1}^m \theta_i^2 = 1$. Let us denote R^i to be the curvature tensors of X_i for $i = 1, \ldots, m$, respectively. We now choose the set $\{u_\alpha^i\}$ for $\alpha = 2, \ldots, n_i$ so that

$$\langle R^i_{u_1^i u_\alpha^i} u_\alpha^i, u_1^i \rangle = -(b_\alpha^i)^2$$

as indicated in the proof of Theorem 2.2. We now parallel translate the frames $\{u_1^i, \ldots, u_{n_i}^i\}$ along γ_i for each $i = 1, \ldots, m$. Note that for any tangent vector of the form $r = (a_1 u_1^1, a_2 u_1^2, \ldots, a_m u_1^m)$, we have

$$\langle R_{vr} r, v \rangle = \sum_{i=1}^m a_i^2 \langle R^i_{v_i, u_1^i} u_1^i, v_i \rangle = 0$$

by the property of the curvature tensor R of X and the fact that it is a product metric. Hence, if we choose $\{e_{\sum_{j=1}^{i-1} n_j + 1} \mid i = 1, \ldots, m\}$ to be the set of orthonormal basis spanning the subspace defined by $\{(a_1 u_1^1, \ldots, a_m u_1^m) \mid a_i \in \mathbb{R}\}$, then

$$\langle R_{e_1 e_{\sum_{j=1}^{i-1} n_j + 1}} e_{\sum_{j=1}^{i-1} n_j + 1}, e_1 \rangle = 0$$

for $i = 2, \ldots, m$. We also choose $e_{\sum_{j=1}^{i-1} n_j + \alpha} = (0, \ldots, 0, u_\alpha^i, 0, \ldots, 0)$ for $2 \leqq \alpha \leqq n$ and $1 \leqq i \leqq m$, where u_α^i is in the $\left(\sum_{j=1}^{i-1} n_j + \alpha\right)$-th slot, then

$$\langle R_{e_1 e_{\sum_{j=1}^{i-1} n_j + \alpha}} e_{\sum_{j=1}^{i-1} n_j + \alpha}, e_1 \rangle = -\theta_i^2 (b_\alpha^i)^2.$$

Following the argument as in [LW3], we conclude that

$$\Delta r(p, x) \leqq \sum_{i=1}^m \sum_{\alpha=2}^{n_i} \theta_i b_\alpha^i \coth(\theta_i b_\alpha^i r(x)) + r^{-1}(x) \tag{3.11}$$

where $x = \gamma(s)$ and $p = \gamma(0)$. Once again, when $b_\alpha^i = 0$ we adopt the convention that

$$\theta_i b_\alpha^i \coth(\theta_i b_\alpha^i r(x)) = r^{-1}.$$

Let us now assume M has at least one finite volume end and that $\lambda_1(M) = \sum_{i=1}^{m} \lambda_1(X_i)$. Note that since

$$\lambda_1(X_i) = \frac{B_i^2}{4} \quad \text{with} \quad B_i = \sum_{\alpha} b_\alpha^i$$

for $i = 1, \dots, m$, we conclude that

$$\lambda_1(X) = \frac{|B|^2}{4},$$

where $|B|^2 = \sum_{i=1}^{m} B_i^2$. In particular, M has maximal λ_1 among all quotients of X. Let E be the finite volume end, and $M \backslash E$ can be viewed as an infinite volume end. If τ is a geodesic line in M with $\tau(t) \to E(\infty)$ and $\tau(-t) \to (M \backslash E)(\infty)$ as $t \to \infty$, where $E(\infty)$ denotes the infinity of the end E. We define

$$\beta_\tau(x) = \lim_{t \to \infty} \bigl(t - r\bigl(\tau(t), x\bigr)\bigr)$$

to be the Buseman function with respect to the geodesic ray τ. For a fixed $x \in M$ and $\varepsilon > 0$, then by taking t sufficiently large, (3.11) asserts that

$$\Delta r\bigl(\tau(t), x\bigr) \leqq (1 + \varepsilon) \sum_{i=1}^{m} \theta_i B_i + \varepsilon.$$

Hence,

$$\Delta \beta_\tau(x) \geqq -(1 + \varepsilon) \sum_{i=1}^{m} \theta_i B_i - \varepsilon$$

and using the fact that ε is arbitrary, we conclude

$$\Delta \beta_\tau \geqq -\sum_{i=1}^{m} \theta_i(x) B_i \geqq -|B|.$$

The last inequality follows from the fact that the function

$$f(\theta) = \sum_{i=1}^{m} \theta_i B_i$$

maximizes at $\theta_i = \dfrac{B_i}{|B|}$ for $0 \leqq \theta_i \leqq 1$ satisfying $\sum_{i=1}^{m} \theta_i^2 = 1$.

Note that due to the presence of the cut-locus created by taking the quotient, the inequality

$$\Delta \beta_\tau \geqq -|B|$$

is valid in the weak sense. Let us now define the function

$$u = \exp\left(\frac{|B|}{2}\beta_\tau\right).$$

It satisfies the differential inequality

$$\begin{aligned}\Delta u &= \frac{|B|}{2}u\Delta\beta_\tau + \frac{|B|^2}{4}u|\nabla\beta_\tau|^2\\ &\geqq -\frac{|B|^2}{4}u,\end{aligned}$$

where we have used the fact that $|\nabla\beta| = 1$ as argued in [W-L5]. A similar argument as in the proof of Theorem 1.1 implies that

$$\Delta\beta_\tau = -|B| \tag{3.12}$$

and M must be diffeomorphic to $\mathbb{R} \times N$ where N is a compact orbifold given by the level set of β_τ.

We now claim that a lift of the geodesic τ to X, denoted by $\tilde{\tau}$ must be of the form

$$\tilde{\tau} = (\gamma_1, \gamma_2, \ldots, \gamma_m)$$

with

$$\gamma_i' = \frac{B_i}{|B|}u_i$$

for some unit vectors $u_i \in T(M_i)$. Indeed, if $x = \tau(s)$, then

$$\Delta r\big(\tau(t), x\big) \leqq \sum_{i=1}^{m} \theta_i \sum_\alpha b_\alpha^i \coth\big(\theta_i b_\alpha^i r\big(\tau(t), x\big)\big) + \frac{1}{r\big(\tau(t), x\big)},$$

where θ_i is determined by

$$\tilde{\tau}' = \left(\theta_1\frac{\gamma_1'(s)}{|\gamma_1'(s)|}, \theta_2\frac{\gamma_2'(s)}{|\gamma_2'(s)|}, \ldots, \theta_m\frac{\gamma_m'(s)}{|\gamma_m'(s)|}\right).$$

Hence

$$\Delta\beta(x) \geqq -\sum_{i=1}^{m}\theta_i B_i.$$

Compared with (3.12), we conclude that $\theta_i = \frac{B_i}{|B|}$. It is also obvious that N is given by a compact quotient of the level set of the Buseman function with respect to the geodesic $\tilde{\tau}$. □

The following corollary can be derived by combining Theorem 3.2, Theorem 3.3 and Theorem 3.4.

Corollary 3.5. *Let $X = X_1 \times X_2 \times \cdots \times X_m$ be the product of m irreducible symmetric spaces of non-compact type, with $m \geqq 2$. Suppose Γ is a discrete subgroup of the isometry group acting on X and $M = \Gamma \backslash X$ is its quotient space. Assume that M is noncompact with $\lambda_1(M) = \lambda_1(X)$. Then M either has one end, or it must be diffeomorphic to the product manifold $\mathbb{R} \times N$, where N is a compact orbifold.*

Proof. In view of Theorem 3.4, we only need to rule out the case that M has at least two ends of infinite volume. Assuming this is the case, following the notation and the argument of Theorem 3.2, we obtained a nonconstant, bounded harmonic function f with finite Dirichlet integral. The Bochner identity (3.7) asserts that

$$\Delta |d_i f| \geqq -C_i |d_i f|. \tag{3.13}$$

To get a contradiction, it suffices to show that $d_i f = 0$ for all $1 \leqq i \leqq m$. Note that Theorem 3.1 asserts that $d_i f = 0$ if X_i is of rank at least 2. Hence we only need to consider those X_i that are of rank one. Our assumption that $\lambda_1(M) = \sum_{i=1}^{m} \lambda_1(X_i)$, with $m \geqq 2$, asserts that $\lambda_1(M) > \lambda_1(X_i) \geqq A_i^{-1} C_i$. So by Theorem 3.2, we conclude that $|d_i f| = 0$ if $X_i = \mathbb{H}^n$ for $n \geqq 3$. The remaining case can be handled by applying Theorem 3.3. □

§4. Homogeneous spaces

Let G be a connected noncompact semisimple Lie group with finite center, and $K \subset G$ a maximal compact subgroup. Let us denote their Lie algebras by $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ as in the Cartan decomposition, which is clearly invariant under the adjoint action of K. The Killing form B of $\mathfrak{g}$ defines a non-degenerate bilinear form on $\mathfrak{g}$ satisfying the following conditions:

(1) $\mathfrak{p}$ and $\mathfrak{k}$ are perpendicular with respect to B.

(2) $B|_{\mathfrak{p}}$ is positive definite, and $B|_{\mathfrak{k}}$ is negative definite.

One defines a new inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ by reversing the negative sign on $\mathfrak{k}$ and preserving other conditions. Under the left translation by elements of G, it defines a left-invariant Riemannian metric on G. Since the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ is invariant under the adjoint action of K, the Riemannian metric on G is invariant under the right action of K. It is important to note that it is not invariant under the right action of G. Hence this Riemannian metric induces an invariant metric on $X = G/K$ and X is a symmetric space with respect to it. For any compact subgroup $H \subset K$, the homogeneous space G/H also inherits an invariant Riemannian.

Remark 4.1. Instead of the Killing form B on $\mathfrak{g}$ and the modified inner product $\langle \cdot, \cdot \rangle$, we can also use any inner product on $\mathfrak{g}$ which is invariant under the adjoint action of K, and then the induced left invariant Riemannian metric on G, G/H, etc. If $X = G/K$ is an irreducible symmetric space, then such an inner product is unique up to different scalings on the subspaces $\mathfrak{p}$ and $\mathfrak{k}$. If X is reducible, there are further choices of scaling constants in different irreducible factors of X. Therefore, the inner product $\langle \cdot, \cdot \rangle$ induced from the Killing form B is a natural and general choice. See also the paper [BG] for related discussion.

In the following, the spaces G and G/H are given the Riemannian metric discussed above unless indicated otherwise.

Theorem 4.2. *Let G be a connected semisimple non-compact Lie group as above and $\Gamma \subset G$ be any discrete subgroup, not necessarily torsion-free. If $\lambda_1(\Gamma\backslash G) = \lambda_1(G)$, then one of the following alternatives holds*:

(1) *$\Gamma\backslash G$ has exactly one end, and this end has infinite volume.*

(2) *$\Gamma\backslash G$ is diffeomorphic to a product $\mathbb{R} \times N$, where N is a compact orbifold, and has exactly one infinite volume end and a finite volume end.*

Theorem 4.3. *Let G be a connected semisimple non-compact Lie group, $\Gamma \subset G$ be any discrete subgroup, and $H \subset K$ a compact subgroup of G. If $\lambda_1(\Gamma\backslash G/H) = \lambda_1(G/H)$, then the same alternative as in the previous theorem holds.*

Though the first result is a special case of the second more general one, the proof of the second one depends on the first. This is one of the reasons for stating them separately.

Proof of Theorem 4.2. The basic idea is to reduce it to the result for locally symmetric spaces $\Gamma\backslash X$ with $\lambda_1(\Gamma\backslash X) = \lambda_1(X)$. Briefly, K acts on $\Gamma\backslash G$ on the right and hence also acts on $L^2(\Gamma\backslash G)$. This gives a decomposition

$$L^2(\Gamma\backslash G) = \bigoplus_{\sigma \in \hat{K}} L^2(\Gamma\backslash X, E_\sigma), \tag{4.1}$$

where $\hat{K}$ is the set of all irreducible unitary representations of K, and for each $\sigma \in \hat{K}$ acting on its representation space V_σ, $E_\sigma = G \otimes_K V_\sigma$ is the associated homogeneous vector bundle over X.

When σ is the trivial representation, then

$$L^2(\Gamma\backslash X, E_\sigma) = L^2(\Gamma\backslash G/K) = L^2(\Gamma\backslash G)^K, \tag{4.2}$$

the subspace of functions invariant under K.

The simple but crucial observation here is that the bottom of the spectrum of the Laplace operator of $\Gamma\backslash G$ is achieved on the subspace $L^2(\Gamma\backslash X) = L^2(\Gamma\backslash G/K)$.

A closely related fact is that

$$\lambda_1(G) = \lambda_1(G/K). \tag{4.3}$$

Then the equality

$$\lambda_1(\Gamma\backslash G) = \lambda_1(G)$$

implies the following equality for locally symmetric spaces:

$$\lambda_1(\Gamma\backslash X) = \lambda_1(X).$$

The previous result for locally symmetric spaces in Corollary 3.5 applies and shows that either $\Gamma\backslash X$ has exactly one end, which has infinite volume, or is a product and hence has exactly one infinite volume end and one finite volume end. Since $\Gamma\backslash G$ fibers over $\Gamma\backslash X$ and G is connected (and hence K is connected), the number of ends of $\Gamma\backslash G$ is the same as the number of ends of $\Gamma\backslash X$, and the result of Theorem 4.2 follows.

Now we need to justify the above observation on the bottom of the spectrum by using the characterization

$$\lambda_1(M) = \sup\{\lambda \in \mathbb{R} \mid \Delta u = -\lambda u \text{ for some positive function } u \text{ on } M\}. \tag{4.4}$$

Suppose u is a positive function on $\Gamma\backslash G$ satisfying $\Delta u = -\lambda u$, then its average over K under the right action

$$\bar{u}(x) = \int_{k\in K} u(xk)\,dk,$$

where dk is the Haar measure on K of total measure 1, defines a positive function on $\Gamma\backslash X$. Since the Laplace operator Δ of G commutes with the right action of K, the function $\bar{u}$ also satisfies the equation

$$\Delta\bar{u} = -\lambda\bar{u}.$$

When $\bar{u}$ is considered as a function on $\Gamma\backslash X$, the above equation reduces to an eigen-equation on $\Gamma\backslash X$. This implies that

$$\lambda_1(\Gamma\backslash X) \geqq \lambda_1(\Gamma\backslash G).$$

But the reverse inequality follows from decomposition in (4.1) and the identification in (4.2). Therefore, the following equality holds:

$$\lambda_1(\Gamma\backslash X) = \lambda_1(\Gamma\backslash G). \tag{4.5}$$

The equation (4.3) can be proved in the same way. The proof of Theorem 4.2 is complete. □

Proof of Theorem 4.3. The basic idea is to prove that if

$$\lambda_1(\Gamma\backslash G/H) = \lambda_1(G/H),$$

then

$$\lambda_1(\Gamma\backslash G) = \lambda_1(G),$$

and hence Theorem 4.2 can be applied.

First we prove that

$$\lambda_1(G/H) = \lambda_1(G/K). \tag{4.6}$$

For this purpose, note that the inclusions

$$L^2(G)^K = L^2(G/K) \subset L^2(G)^H = L^2(G/H) \subset L^2(G)$$

imply

$$\lambda_1(G/K) \leqq \lambda_1(G/H) \leqq \lambda_1(G).$$

Hence combining with equality (4.3) yields

$$\lambda_1(G/H) = \lambda_1(G).$$

On the other hand, using the inclusions

$$L^2(\Gamma\backslash G)^K = L^2(\Gamma\backslash G/K) \subset L^2(\Gamma\backslash G)^H = L^2(\Gamma\backslash G/H) \subset L^2(\Gamma\backslash G),$$

and equality (4.5), we obtain

$$\lambda_1(\Gamma\backslash G/H) = \lambda_1(\Gamma\backslash G/K). \tag{4.7}$$

The assumption $\lambda_1(\Gamma\backslash G/H) = \lambda_1(G/H)$ together with (4.6) and (4.7) imply that $\lambda_1(\Gamma\backslash G) = \lambda_1(G)$. Therefore Theorem 4.3 follows from Theorem 4.2. □

Remark 4.4. In the above discussions, the Riemannian metric on G and the homogeneous space G/H is somewhat restrictive. Specifically, on G, we could start with any inner product on $\mathfrak{g}$ without requiring invariance under the adjoint action of K. But one difficulty is that the induced left invariant metric on G is not necessarily invariant under the right K-action. This prevents us from the reduction to locally symmetric spaces $\Gamma\backslash G/K$, which was used crucially in the preceding proofs.

Similarly, when H is a non-maximal compact subgroup, then an invariant Riemannian metric on the homogeneous space G/H comes from an inner product on $\mathfrak{g}$ which is invariant under the adjoint action of H, but *not* of K. If the metric on G lifted from G/H is not invariant under the right action of K, we could not appeal to the reduction to results on locally symmetric spaces.

Remark 4.5. Besides homogeneous spaces associated with semisimple Lie groups, it is also natural to consider homogeneous spaces of other Lie groups. If G is a connected reductive Lie group endowed with an invariant metric induced from an inner product which is invariant under the adjoint action of a maximal compact subgroup K, then the same results hold for G/H etc.

If G is a nilpotent Lie group, then for any Riemannian homogeneous space G/H of G, its bottom of the spectrum $\lambda_1(G/H) = 0$, and hence for any discrete subgroup $\Gamma \subset G$, the equality

$$\lambda_1(\Gamma\backslash G/H) = \lambda_1(G/H)$$

is automatically satisfied.

For more general (linear) Lie groups, it seems natural to try to use the Levi decomposition to reduce them to semisimple and solvable Lie groups, and further reduce the latter ones to nilpotent Lie groups.

§5. Examples of $\lambda_1(\Gamma\backslash X) = \lambda_1(X)$

After having proved theorems of the end structures of locally symmetric spaces $\Gamma\backslash X$ with $\lambda_1(\Gamma\backslash X) = \lambda_1(X)$ and related spaces in the previous section, it is natural to construct examples satisfying this extremal condition.

One idea is to use the Patterson-Sullivan type theorem on relations between the bottom of the spectrum and the Hausdorff dimension and the critical exponents. See the paper [Le] and the references contained there.

By the results of Elstrodt-Patterson-Sullivan-Corlette for rank one symmetric spaces and of Leuzinger for higher rank spaces (see [Le], p. 920), one way to find examples of $\Gamma\backslash X$ with $\lambda_1(\Gamma\backslash X) = \lambda_1(X)$ is to construct examples of discrete groups Γ with critical exponents $\delta(\Gamma)$ less than or equal to ρ_m, where ρ_m is a positive number defined as follows. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p} \subset \mathfrak{g}$, and $\mathfrak{a}^+$ a positive chamber. Let ρ be the half sum of the positive roots of $\mathfrak{g}$ with respect to $\mathfrak{a}^+$. Then $\rho_m = \sup \rho(H)$, where $H \in \mathfrak{a}^+$, $\|H\| = 1$. When the rank of G is equal to one, then $\rho_m = \|\rho\|$. In the above discussion, the Riemannian metric of X is induced from the Killing form. Then $\lambda_1(X) = \|\rho\|^2$.

If the rank of X (or G) is equal to r, then there exist torsion-free abelian subgroups Γ of A with rank less than or equal to r, where $G = NAK$ is the Iwasawa decomposition of G. As pointed out in [Le], §2.3.1, such groups satisfy $\delta(\Gamma) = 0$ and hence $\lambda_1(\Gamma\backslash X) = \lambda_1(X)$, by the result of Elstrodt-Patterson-Sullivan-Corlette.

Another type of example can be obtained as follows. Let P be a parabolic subgroup of G and let $P = N_P A_P M_P$ be the Langlands decomposition of P. Then M_P is a reductive Lie group and the quotient $X_P = M_P/K \cap M_P$ is a symmetric space of noncompact type. The symmetric space X admits a horospherical decomposition

$$X = N_P \times A_P \times X_P.$$

For every point $n \in N_P$, the product $A_P \times X_P$ is isometrically embedded into X as $\{n\} \times A_P \times X_P$.

For any discrete subgroup $\Gamma_N \subset N_P$, it is known that $\delta(\Gamma) > 0$ by [Le], p. 922. But it is not obvious that $\delta(\Gamma) \leqq \rho_m$. On the other hand, it follows from the observation that the bottom of the spectrum of X is achieved by N_P-invariant functions and the horospherical decomposition of the Laplace operator, it can be shown that $\lambda_1(\Gamma_N\backslash X) = \lambda_1(X)$. (Or we can note that Γ_N is nilpotent, hence amenable, and use Theorem 5.1 below.)

For any discrete subgroup Γ_M of M_P with $\lambda_1(\Gamma_M\backslash X_P) = \lambda_1(X_P)$, for example, Γ_M can be taken as a nilpotent or abelian subgroup. Suppose that Γ_N is a discrete subgroup of N_P normalized by Γ_M. (Such subgroups do exist, for example, induced from arithmetic subgroups of G when P is a parabolic subgroup defined over rational numbers.) Then

$\Gamma_N \Gamma_M$ is also a discrete subgroup of G. Using similar arguments and the fact that the product $A_P \times X_P$ is isometrically embedded into X, it can also be shown as above that $\lambda_1(\Gamma_N \Gamma_M \backslash X) = \lambda_1(X)$.

On the other hand, given a discrete subgroup Γ_N of N_P, there is in general no discrete subgroup of A_P which normalizes Γ_N. So in general we can not combine discrete subgroups of N_P and A_P to get discrete subgroups of G.

Let Γ_M be a discrete subgroup of M_P with $\lambda_1(\Gamma_M \backslash X_P) = \lambda_1(X_P)$, and let Γ_A be a discrete subgroup of A_P. Then $\Gamma_A \Gamma_M$ is a discrete subgroup of G. By similar arguments as above, it can be proved that $\lambda_1(\Gamma_A \Gamma_M \backslash X) = \lambda_1(X)$.

These examples above are all amenable, but some are not elementary in the sense that they fix one common point in the sphere at infinity of X, if the rank of X is greater than 1. The next theorem established the fact that if Γ is amenable, then $\lambda_1(X) = \lambda_1(\Gamma \backslash X)$ for a rather general class of complete manifolds X that are not necessarily a symmetric space.

Theorem 5.1. *Let X be a complete Riemannian manifold whose Ricci curvature is bounded from below by some nonpositive constant. Assume that the volume of geodesic balls of radius 1 satisfies the subexponential decay estimate*

$$V_x(1) \geqq C \exp\bigl(-\varepsilon r(x)\bigr)$$

for any $\varepsilon > 0$, where $p \in X$ is a fixed point and $r(x) = r(p, x)$ is the distance from p to x. If Γ is amenable, then $\lambda_1(\Gamma \backslash X) = \lambda_1(X)$.

Proof. Note that when $\lambda_1(X) = 0$, then obviously $\lambda_1(\Gamma \backslash X) = 0$ by monotonicity property.

Assuming that $\lambda_1(X) > 0$, for $\varepsilon > 0$ sufficiently small, there exists a minimal positive Green's function $G(x, y)$ to the operator $\Delta + \bigl(\lambda_1(X) - \varepsilon\bigr)$ on X. Moreover, by the estimate in [LW4] (see Corollary 2.2), we have

$$\int_{B_p(R+1) \backslash B_p(R-1)} G^2(p, y)\, dy \leqq C e^{-2\sqrt{\varepsilon} R} \tag{5.1}$$

for all $R > 0$. Now let $\phi(x) \geqq 0$ be a smooth function with compact support on X and $\phi(p) = 1$. Let us define the function

$$u(x) = \int_X G(x, y) \phi(y)\, dy.$$

Obviously, $u(x) \geqq 0$ and

$$\bigl(\Delta + \bigl(\lambda_1(X) - \varepsilon\bigr)\bigr) u(x) = -\phi(x) \leqq 0,$$

hence

$$\Delta u(x) \leqq -\bigl(\lambda_1(X) - \varepsilon\bigr) u(x). \tag{5.2}$$

On the other hand, the decay estimate (5.1) implies that for any point $x \in X$ with $r(p,x) = R$, we have

$$\int_{B_x(1)} u^2(y)\,dy \leqq C_1 \int_{B_p(R+1)\setminus B_p(R-1)} G^2(p,y)\,dy \tag{5.3}$$
$$\leqq C_2 e^{-2\sqrt{\varepsilon}R}.$$

Since

$$\big(\Delta + \big(\lambda_1(X) - \varepsilon\big)\big)u(x) = 0$$

away from the support of ϕ, the standard mean value inequality [LT1] implies

$$u^2(x) \leqq C_4 V_x^{-1}(1) \int_{B_x(1)} u^2(y)\,dy.$$

Combining with (5.3), we conclude that

$$u(x) \leqq C_5 V_x^{-1}(1)\, \exp(-\sqrt{\varepsilon}R).$$

In particular, the assumption on the volume decay on X implies that u must be bounded.

The assumption that Γ is amenable implies there exists an invariant mean μ on the space $L^\infty(\Gamma)$. Moreover, according to [G], Theorem 3.6.1, μ can be obtained as

$$\mu(f) = \lim_{j\to\infty} \frac{1}{|U_j|} \sum_{g\in U_j} f(g)$$

for a net of finite subsets $\{U_j\}$ in Γ. We use this invariant mean μ and the function u on X to define a function v on $\Gamma\backslash X$ by

$$v(z) = \mu\big(f_z(g)\big),$$

where we view $\Gamma\backslash X$ as a fundamental domain in X and $f_z(g) = u\big(g(z)\big)$ for $g \in \Gamma$. Since u is bounded, $f_z \in L^\infty(\Gamma)$. By the fact that μ is Γ invariant, v is well-defined on $\Gamma\backslash X$ and $v \geqq 0$. Also, using the explicit form of the invariant mean μ and (5.2), it is straightforward to check that the function v satisfies

$$\Delta v(z) \leqq -\big(\lambda_1(X) - \varepsilon\big)v(z). \tag{5.4}$$

By [LW4], Proposition 1.1, we conclude that

$$\lambda_1(\Gamma\backslash X) \geqq \lambda_1(X) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this implies that

$$\lambda_1(\Gamma\backslash X) \geqq \lambda_1(X)$$

and

$$\lambda_1(\Gamma\backslash X) = \lambda_1(X)$$

as the reverse inequality always holds. □

Corollary 5.2. *Let X be a symmetric space of noncompact type, and $\Gamma\backslash X$ a noncompact quotient by a discrete, amenable, group of isometries of X. Then $\lambda_1(X) = \lambda_1(\Gamma\backslash X)$. In particular, $\Gamma\backslash X$ either has one end, which is necessarily of infinite volume, or it must have two ends, one of infinite volume and another of finite volume, and is diffeomorphic to a product $\mathbb{R} \times N$, where N is compact.*

We would like to remark that Theorem 5.1 is related to the results of R. Brooks [Br2], where he also considers the same question with the assumption that $\Gamma\backslash X$ is topologically finite. His proof used the isoperimetric inequality, while ours is mostly analytical. More importantly, the assumption on Brooks result is on the base space $\Gamma\backslash X$, compare to the rather mild assumption of Theorem 5.1 is on the covering X itself. In particular, even for the case of $X = \mathbb{H}^3$, where the assumption of Theorem 5.1 is automatic, but the assumption that $\Gamma\backslash X$ is topologically finite is a nontrivial one. Brooks also pointed out that the converse statement to Theorem 5.1 does not hold in general.

Remark 5.3. In [BeK], quotients by amenable discrete isometric groups of simply connected Riemannian manifolds with negatively pinched sectional curvature are studied. It follows from their result that such a quotient has either one end, or is diffeomorphic to a product $\mathbb{R} \times N$, where N is a compact nilmanifold. Theorem 5.1 is a generalization of this result to an important class of nonpositively curved manifolds.

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