Non-archimedean pluripotential theory

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Plan

- Pluripotential theory.
- Non-archimedean potential theory (dim 1).
- Non-archimedean pluripotential theory (dim > 1).

Pluripotential theory

- Pluripotential theory = study of plurisubharmonic (psh) fcns.
- A smooth function φ on $\mathbf{C}^n$ is psh if the complex Hessian

$$H(\varphi) := \left(\frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}\right)_{i,j}$$

  is positive semidefinite.
- Example: $\varphi = c \log \sum_{k=1}^{N} |f_k|^2$, $c > 0$, $f_1, \dots, f_N$ holomorphic.
- General psh fcn: *decreasing limit* of psh fcns of this form.
- Monge-Ampère operator: $MA(\varphi) := \mathrm{const} \cdot \det(H(\varphi))$

Geometric Monge-Ampère equation

- (X,ω) compact Kähler manifold. Assume $\int_X \omega^n = 1$.
- Monge-Ampère equation: given probability measure μ on X, find function φ on X such that

$$(\omega + dd^c\varphi)^n = \mu; \qquad (\mathrm{MA}_\mu)$$

  Locally this means $\mathrm{MA}(\psi + \varphi) = \mu$.
- Yau's Theorem (solution to Calabi conjecture): can solve (MA_μ) when $\mu > 0$ *smooth*. Solution (almost) unique.
- Work by Guedj-Zeriahi and Dinew: unique solution to (MA_μ) for μ *non-pluripolar* measure (no mass on $\{u = -\infty\}$, u psh).
- Applications to complex geometry and dynamics.
- Example: $X = \mathbf{P}^1$, $\omega =$ Fubini-Study.

$$\varphi(z) = \int \log \frac{|z - w|}{\sqrt{1 + |w|^2}} d\mu(w).$$

- Non-linear problem in dim > 1!

Beyond Archimedes

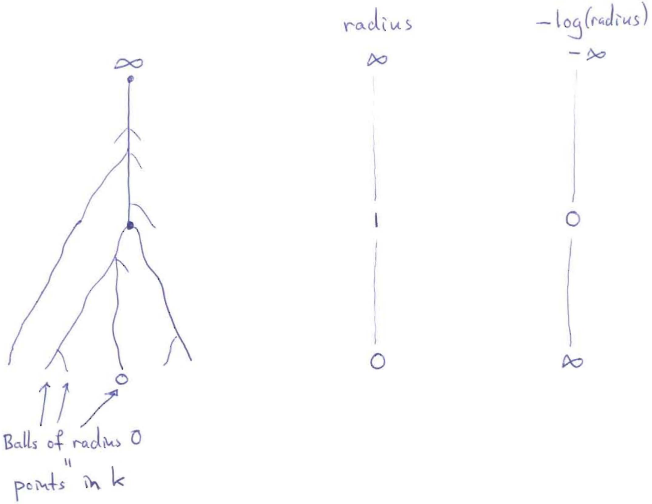
- Try to extend previous analysis to non-archimedean setting.
- Natural for problems of arithmetic nature.
- Relevant work by several people: Baker-Rumely, Boucksom-Favre-J, Chambert-Loir, Chinburg-Rumely, Favre-Rivera-Letelier, Kontsevich-Tschinkel, Liu, Thuillier, Yuan-Zhang, ...
- **Definition**. Archimedean=not non-Archimedean.
- **Factoid**: Archimedes died 2222 years ago.

Non-archimedean analysis

- Assume k field equipped with *non-archimedean* norm: $|a+b| \leq \max\{|a|, |b|\}$. Also assume k *complete*.
- Example: $\mathbf{Q}_p$ (p-adic nos). $0 < |p| < 1$, $|q| = 1$.
- Example: $\mathbf{C}((t))$ (Laurent series) $0 < |t| < 1$, $|c| = 1$.
- Can try to develop analytic geometry as over $\mathbf{C}$.
- Problem 1: k totally disconnected.
- Problem 2: in general, k not locally compact.
- Various ways to deal with this.
- Approach: replace k (or k^n) by suitable *Berkovich space* (space of valuations).

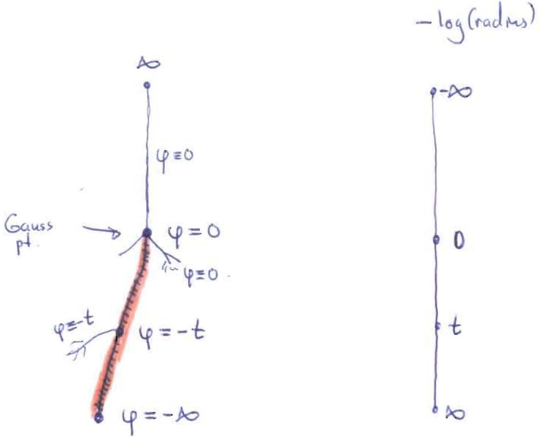
The Berkovich line

- Points of $\mathbf{A}^1_{\text{Berk}}$ can be viewed as *balls* in k.
- Thus $\mathbf{A}^1_{\text{Berk}}$ admits *tree structure*.



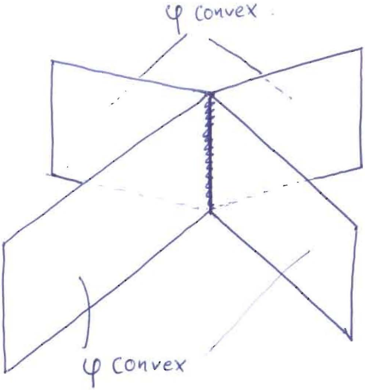
Solving Monge-Ampère in dimension 1

- Can define Monge-Ampère operator using tree structure.
- Extends Laplacian on $\mathbf{R}$ and discrete Laplacian on graphs.
- Can solve $\mathrm{MA}(\varphi) = \mu$ when $\mu(\mathbf{A}^1_{\mathrm{Berk}}) = 0$.
- Example: $\mathrm{MA}(\varphi) = \delta_a - \delta_{\mathrm{Gauss}}$, $a \in k$.



Berkovich spaces in higher dimensions

- In general $\mathbf{A}^n_{\mathrm{Berk}}$ is "limit" of simplicial complexes of dim $\leq n$.
- A psh function φ will be convex on each simplex $\implies$ can consider *real* Monge-Ampère.
- What happens at branch points?



Our result

- $\mathcal{V} := \{\text{normalized valuations on } \mathcal{O}_{\mathbf{C}^n,0}\}$.
- Compact Hausdorff space. Limit of simplicial complexes.
- Define psh fcn as *decreasing limit* of fcns of the form

$$c \max_{1 \leq j \leq N} \log |f_j|,$$

where $f_j \in \mathcal{O}_{\mathbf{C}^n,0}$ and $\bigcap_j \{f_j = 0\} = \{0\}$.
- Here $\log |f|(\nu) := -\nu(f)$.
- For φ psh, define positive measure $\mathrm{MA}(\varphi)$ on $\mathcal{V}$ using intersection theory (algebraic geometry).
- **Thm** (Boucksom-Favre-J). Unique solution to $\mathrm{MA}(\varphi) = \mu$ for any non-pluripolar measure μ.
- Existence proof uses variational approach (Alexandrov; Berman-Boucksom-Guedj-Zeriahi).
- Develop pluripotential/capacity theory along the way.