# A non-Archimedean approach to K-stability and the existence of Kähler-Einstein metrics


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# Plan

1. KE metrics on Fano manifolds and coercivity of functionals.

2. K-stability from several points of view.

3. A variational proof of the YTD conjecture.

4. Extensions and speculations.

Based on joint work with R. Berman, H. Blum, S. Boucksom and T. Hisamoto (in various configurations).

Heavily uses work by many, many other people! References far from complete: apologies!

# Part 1: Kähler-Einstein metrics on Fano manifolds

In this part:

- General remarks on Kähler-Einstein metrics.
- The "calculus" of metrics on line bundles.
- Functionals on the space of metrics.
- Kähler-Einstein metrics and coercivity of Ding and Mabuchi.

# Kähler-Einstein metrics

- $X$ = smooth complex projective variety of dim $n$.
- $K_X$ = canonical bundle (or divisor class).
- $\omega$ = Kähler form on $X$. Also think of as Hermitean metric.
- Say $\omega$ is a *Kähler-Einstein* (KE) metric if

$$\operatorname{Ric}\omega = \lambda\omega$$

  for some $\lambda \in \mathbf{R}$.
- The cases $\lambda < 0$ ($X$ can. polarized) and $\lambda = 0$ ($X$ Calabi-Yau) are understood due to work by Calabi, Aubin and Yau. Namely, there exists a unique Kähler-Einstein metric.
- Remains to consider the case when $\lambda > 0$ and $X$ is *Fano*, i.e. $-K_X$ ample. After scaling, $\lambda = 1$ so we look at

$$\operatorname{Ric}\omega = \omega.$$

- This equation may or may not have a solution!

# KE metrics in Fano case

- Assume $X$ Fano and look at the equation

$$\operatorname{Ric}\omega = \omega, \tag{KE1}$$

  where $\omega \in c_1(X) = c_1(-K_X)$ is a Kähler form on $X$.
- Fix a reference Kähler form $\omega_0 \in c_1(X)$ and write

$$\omega = \omega_0 + dd^c\varphi,$$

  for $\varphi \in C^\infty(X)$, where $dd^c = \frac{i}{\pi}\partial\overline{\partial}$. Then (KE1) becomes

$$(\omega_0 + dd^c\varphi)^n = ce^{-2\varphi}\mu, \tag{KE2}$$

  where $c = c(\varphi) > 0$ is a normalizing constant and $\mu = \mu(\omega_0)$ is a positive volume form on $X$.
- Equation (KE2) is a PDE of *Monge-Ampère* type.
- Both existence and uniqueness are nontrivial.

# Bando-Mabuchi and YTD

- *Uniqueness* governed by the *Bando-Mabuchi theorem*.
- **Thm** [Bando-Mabuchi; Berndtsson] If $X$ is Fano and $\omega, \omega'$ are KE metrics, then there exists $g \in \mathrm{Aut}(X)$ such that $\omega' = g^*\omega$.
- *Existence* is more subtle. Starting with Matsushima '57, people found various *obstructions*.
- Example: $\mathbf{P}^2$ blown up in one pt is Fano but has no KE metric.
- *YTD conjecture*: a KE metric exists iff $X$ is *K-(poly)stable*.
- Discuss K-stability later. In principle algebraic condition on $X$.
- **Thm**[Chen-Donaldson-Sun, Tian] The YTD conjecture is true.

# Methods for constructing KE metrics

- Several approaches to solving the equation

$$(\omega_0 + dd^c\varphi)^n = ce^{-2\varphi}\mu. \qquad \text{(KE2)}$$

  - Cont. method (Chen-Donaldson-Sun, Tian, Székelyhidi).
  - Kähler-Ricci flow (Chen-Sun-Wang).
  - Variational method (Berman-Boucksom-J).
- Will only discuss the variational method in these lectures.
- All methods have versions (easier, but still nontrivial) in the canonically polarized and Calabi-Yau case. In these cases, solutions always exist.
- In the Fano case, solutions do not always exist. The methods must use the K-polystability assumption (explained later).

# Variational approach: basic idea

- Again look at the equation
$$(\omega_0 + dd^c\varphi)^n = ce^{-2\varphi}\mu. \tag{KE2}$$
- Consider the space $\mathcal{H} := \{\varphi \in C^\infty(X) \mid \omega_0 + dd^c\varphi > 0\}$.
- Define a functional $F\colon \mathcal{H} \to \mathbf{R}$ whose critical points, $F'(\varphi) = 0$, are solutions to (KE2).
- Find these critical points as *minima* of $F$ on $\mathcal{H}$.
- Not obvious that a minimizer exists: ignore this issue for now.
- Will use two different functionals: Mabuchi and Ding...
- ...as well as some other functionals.
- Useful to identify elements of $\mathcal{H}$ as *metrics* on $-K_X$.

# Metrics on line bundles

- Equip $\mathbf{C}$ with the usual norm $|a+ib| = \sqrt{a^2+b^2}$ for $a, b \in \mathbf{R}$.
- A *norm* on a $\mathbf{C}$-vector space $V$ is a function $\|\cdot\|\colon V \to \mathbf{R}_+$ s.t.:
  - $\|v\| = 0$ iff $v = 0$;
  - $\|v+w\| \le \|v\| + \|w\|$ for $v, w \in V$;
  - $\|av\| = |a| \cdot \|v\|$ for $a \in \mathbf{C}$ and $v \in V$.
- If $\dim_{\mathbf{C}} V = 1$, any two norms on $V$ are proportional, but there is no canonical norm on $V$ in general.
- If $\pi\colon L \to X$ is a line bundle on a complex manifold $X$, then a *metric* on $L$ is a function $\|\cdot\|\colon L \to \mathbf{R}_+$ whose restriction to $\pi^{-1}(x) \simeq \mathbf{C}$ is a norm for all $x \in X$.
- Use *additive* terminology and identify a metric $\|\cdot\|$ with

$$\phi := -\log\|\cdot\|\colon L^\times \to \mathbf{R},$$

  where $L^\times$ is $L$ with the zero section removed.

# Calculus on metrics on line bundles

- Use additive terminology on line bundles, too: $L_1 + L_2 := L_1 \otimes L_2$.
- $\phi_i$ metric on $L_i$, $a_i \in \mathbf{Z} \implies a_1\phi_1 + a_2\phi_2$ metric on $a_1L_1 + a_2L_2$.
- If $s \in \Gamma(U, L)$ is a local nowhere vanishing section, then $\phi := \log|s|$ is a metric on $L$ over $U$ for which $\phi \circ s \equiv 0$.
- Identify metrics on $\mathcal{O}_X$ with functions on $X$: evaluate at "1".
- Given a reference metric $\phi_0$ on $L$, any other metric on $L$ is of the form $\phi = \phi_0 + \varphi$, where $\varphi$ is a function on $X$.
- Given a metric $\phi$ on $L$, set $dd^c\phi := dd^c(\phi \circ s)$ for any local nonvanishing section $s$ of $L$. Then $dd^c\phi \in c_1(L)$.
- Say $\phi$ is *positive* if $\phi$ smooth and $dd^c\phi$ Kähler, i.e. $dd^c\phi > 0$.
- Any metric $\phi$ on $K_X$ induces *volume form* $e^{2\phi}$ on $X$ and conv'ly.
- In this way, $\operatorname{Ric}\omega = -dd^c\frac{1}{2}\log|\omega^n|$ for any Kähler form $\omega$.
- $X$ Fano, $\phi$ positive metric on $-K_X \implies dd^c\phi$ is a KE metric iff

$$(dd^c\phi)^n = c(\phi)e^{-2\phi}.$$

# Functionals on the space of metrics

- Let $(X, L)$ be a polarized smooth complex projective variety. Identify $X$ and $L$ with their analytifications.
- Redefine $\mathcal{H}$ as the space of *positive metrics* $\phi$ on $L$.
- Can define several *functionals* on $\mathcal{H}$.

  The Monge-Ampère energy and related functionals.
  The Mabuchi (or $K$-energy) functional.
  The Ding functional (in the Fano case $L = -K_X$).
- Set $V = (L^n)$. Then $\int_X (dd^c\phi)^n = V$ for all $\phi \in \mathcal{H}$.
- Given metric $\phi \in \mathcal{H}$, set

$$\mathrm{MA}(\phi) := V^{-1}(dd^c\phi)^n.$$

  This is a probability measure on $X$.

# Monge-Ampère energy

- Fix reference metric $\phi_0 \in \mathcal{H}$. Any other metric is of the form $\phi = \phi_0 + \varphi$, where $\varphi$ is a *function* on $X$.
- Define the *Monge-Ampère energy* of $\phi$ as

$$E(\phi) = \frac{1}{n+1} \sum_{j=0}^{n} V^{-1} \int_X \varphi (dd^c \phi)^j \wedge (dd^c \phi_0)^{n-j}$$

  where $V = (L^n)$.
- This is the *antiderivative of the Monge-Ampère* operator:

$$E'(\phi) = \mathrm{MA}(\phi)$$

  i.e. $\frac{d}{dt} E(\phi + tf)\big|_{t=0} = \int_X f\, \mathrm{MA}(\phi)$ for $f \in C^\infty(X)$.
- Also have $E(\phi_0) = 0$ and $E(\phi + c) = E(\phi) + c$ for $c \in \mathbf{R}$.
- Can get rid of reference metric by viewing $E$ as a metric $\langle \phi^{n+1} \rangle$ on the line $\langle L^{n+1} \rangle$ given by the Deligne pairing.

# The functionals $I$, $J$, and $I - J$

- Using the Monge-Ampère energy, can define several functionals that serve as "norms" or exhaustion functions on $\mathcal{H}$.
- The functionals $I$, $J$ and $I - J$ are given by

$$I(\phi) = \int_X \varphi(\mathrm{MA}(\phi_0) - \mathrm{MA}(\phi))$$
$$J(\phi) = \int_X \varphi\, \mathrm{MA}(\phi_0) - E(\phi)$$
$$(I - J)(\phi) = E(\phi) - \int_X \varphi\, \mathrm{MA}(\phi).$$

- They are translation invariant: $I(\phi + c) = I(\phi)$ etc.
- We have $I(\phi) \geq 0$ with equality iff $\phi = \phi_0 + c$. Same for $J$, $I - J$.
- We have the inequality

$$n^{-1} J \leq I - J \leq nJ,$$

so the three functionals are equivalent.

# The Ding functional

- Assume $X$ is Fano and that $L = -K_X$.
- Any metric $\phi \in \mathcal{H}$ then induces a volume form $e^{-2\phi}$ on $X$.
- The Ding functional on $\mathcal{H}$ is defined by

  $$D(\phi) = L(\phi) - E(\phi),$$

  where

  $$L(\phi) = -\frac{1}{2} \log \int_X e^{-2\phi}.$$

- We have

  $$D'(\phi) = e^{-2\phi} / \int_X e^{-2\phi} - \mathrm{MA}(\phi)$$

- Thus the critical points of Ding are Kähler-Einstein metrics!
- More precisely: $D'(\phi) = 0$ iff $\omega = dd^c\phi$ is a KE metric.

# Entropy

- Define a reference prob. measure on $X$ by $\mu_0 = e^{-2\phi_0} / \int_X e^{-2\phi_0}$, where $\phi_0 \in \mathcal{H}$ is the reference metric.
- Define the *entropy* of a probability measure $\mu$ (wrt $\mu_0$) as

$$\mathrm{Ent}(\mu) := \int_X \log \frac{d\mu}{d\mu_0} \mu,$$

  if $\mu \ll \mu_0$, and $\mathrm{Ent}(\mu) = +\infty$ otherwise.
- We have $\mathrm{Ent}(\mu) \geq 0$ with equality iff $\mu = \mu_0$.
- The entropy functional is the Legendre dual of the functional $L$:

$$\mathrm{Ent}(\mu) = \sup\{L(\phi) - \int (\phi - \phi_0)\mu \mid \phi \text{ smooth metric on } L\}$$

$$L(\phi) = \inf\{\mathrm{Ent}(\mu) + \int (\phi - \phi_0)\mu \mid \mu \text{ prob measure on } X\}$$

# The Mabuchi functional

- Define the *Mabuchi functional* on $\mathcal{H}$ by

$$M(\phi) = H(\phi) - (I - J)(\phi),$$

where

$$H(\phi) = \frac{1}{2}\,\mathrm{Ent}(\mathrm{MA}(\phi))$$

- The critical points of the Mabuchi functional

$$M'(\phi) = 0,$$

also give rise to KE metrics, just like for the Ding functional.
- Can define the Mabuchi functional for general polarizations using a different formula. In this case, the critical points define *constant scalar curvature Kähler metrics*.
- The formula above is due to Tian and Chen.

# Coercivity and KE metrics

- For Fano manifolds w/o nontrivial vector fields, the existence of KE metrics can be detected by the Mabuchi and Ding functionals.
- Say $M$ is *coercive* if $\exists \delta, C > 0$ such that

$$M \geq \delta J - C \quad \text{on } \mathcal{H}.$$

- **Thm** [Tian97,. . . , BBEGZ16, DR17] If $\mathrm{Aut}(X)$ finite, TFAE
  (i) $X$ admits a KE metric;
  (ii) $D$ is coercive;
  (iii) $M$ is coercive.
- By [DR17], the theorem is also true if $X$ has nontrivial vector fields if one replaces $J(\phi)$ in the coercivity condition by $J_G(\phi) := \inf\{J(g^*\phi) \mid \phi \in G\}$ where $G = \mathrm{Aut}^0(X)$.
- In these lectures, we shall focus on the case when $G = \{\mathrm{id}\}$.
- Will take Thm for granted, and relate coercivity to K-stability!

# A version of the YTD conjecture

- Goal for rest of lectures is to explain the following result.
- **Thm** [Berman-Boucksom-J] For a Fano manifold $X$ w/o nontrivial vector fields, TFAE
  (i) $X$ admits a KE metric;
  (ii) The Ding functional $D$ is coercive;
  (iii) The Mabuchi functional $M$ is coercive;
  (iv) $X$ is uniformly K-stable;
  (v) $X$ is uniformly Ding-stable;
- Will take the equivalence of (i)–(iii) for granted.
- Need to explain (iv) and (v).
- Will outline proof of (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (iii).

# Part 2: K-stability from several points of view

In this part ($X$ Fano):

- K-stability via test configurations for $X$.
- K-stability via anticanonical $\mathbf{Q}$-divisors on $X$.
- K-stability via (divisorial) valuations on $X$.
- K-stability via valuations on the cone $\mathbf{C}(X)$.
- Berkovich analytifications.
- Test configurations as non-Archimedean metrics.
- K-stability via functionals on non-Archimedean metrics.
- K-stability and Ding stability.

# K-stability

- The notion of K-stability was introduced by Tian and Donaldson to understand obstructions for KE metrics.
- It is inspired by and related to stability in the sense of GIT.
- Its is algebraic in the sense that it works over any algebraically closed field of characteristic zero.
- Here, will explain K-stability from 5 points of view:
  (1) Test configurations for $(X, -K_X)$.
  (2) Singularities of special divisors in $|-mK_X|$, $m \gg 0$.
  (3) Divisorial valuations on $X$.
  (4) Valuations on the cone $\mathbf{C}(X)$ of $X$.
  (5) Non-Archimedean metrics on $-K_X$.
- Will be sloppy with Cartier divisors, $\mathbf{Q}$-Cartier divisors, line bundles,...

# Test configurations

- Let $(X, L)$ be a polarized variety. A *test-configuration* for $(X, L)$ is essentially a 1-parameter degeneration of $(X, L)$. Consists of:
  (1) a flat scheme $\mathcal{X} \to \mathbf{P}^1$ and a $\mathbf{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$;
  (2) a $\mathbf{C}^*$-action on $(\mathcal{X}, \mathcal{L})$ lifting the action on $\mathbf{P}^1$;
  (3) a $\mathbf{C}^*$-equivariant isomorphism

$$(\mathcal{X} \setminus \mathcal{X}_0, \mathcal{L}|_{\mathcal{X} \setminus \mathcal{X}_0}) \stackrel{\sim}{\to} (X \times (\mathbf{P}^1 \setminus \{0\}), p_1^* L).$$

**** PICTURE ****

- The test configuration is *normal* if $\mathcal{X}$ is normal. It is *ample/semiample/nef* if $\mathcal{L}$ is *relatively* ample/semiample/nef.

# More on test configurations

- Any $\mathbf{C}^*$-action on $X$ induces a *product* test configuration $(X \times \mathbf{P}^1, p_1^* L)$, with the diagonal $\mathbf{C}^*$-action on $X \times \mathbf{P}^1$.
- As a special case, if the $\mathbf{C}^*$-action on $X$ is trivial, we get the *trivial* test configuration for $(X, L)$.
- If $(\mathcal{X}, \mathcal{L})$ is a test configuration for $(X, L)$, so is its *normalization*.
- A test configuration is *almost trivial* if its normalization is trivial.
- Suppose $rL$ is very ample, where $r \in \mathbf{N}^*$ and consider $X \hookrightarrow \mathbf{P}V$, where $V = H^0(X, rL)$. Then any 1-PS

$$\mathbf{C}^* \to \mathrm{PGL}(V)$$

  gives rise to an ample (not necessarily normal) test configuration. Every ample test configuration is obtained in this way.
- Will not use this characterization of ample test configurations.

# Donaldson-Futaki invariant

- Consider *ample* test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, L)$.
- Set $N_m = h^0(X, mL)$. Note that $h^0(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0}) = N_m$ for $m \gg 0$.
- Have induced $\mathbf{C}^*$-action on $H^0(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0})$ and its determinant line $\det H^0(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0})$. Let $w_m$ be the weight of the latter action.
- By (suitable versions of) Riemann-Roch, have expansion

$$\frac{w_m}{mN_m} = F_0 + m^{-1}F_1 + m^{-2}F_2 + \dots$$

  where $F_i \in \mathbf{Q}$. The *Donaldson-Futaki* invariant of $(\mathcal{X}, \mathcal{L})$ is

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = -2F_1$$

- Can check that the Donaldson-Futaki invariant does not change when replacing a tc by its normalization.
- Odaka, Wang: intersection theoretic formula for $\mathrm{DF}(\mathcal{X}, \mathcal{L})$.

# K-stability via test configurations

- Say that the polarized pair $(X, L)$ is
  (i) *K-semistable* if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for every ample tc $(\mathcal{X}, \mathcal{L})$.
  (ii) *K-stable* if it is K-semistable and $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$ iff $(\mathcal{X}, \mathcal{L})$ almost trivial (i.e. its normalization is trivial).
  (iii) *K-polystable* if it is K-semistable and $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$ iff $(\mathcal{X}, \mathcal{L})$ almost product (i.e. its normalization is a product).
- $X$ Fano is K-semistable (etc) iff $(X, -K_X)$ is K-semistable.
- Can also consider the canonically polarized case $L = K_X$ and the Calabi-Yau case $K_X = 0$. In these cases $(X, L)$ is always K-stable.
- Can also allow singularities. Odaka proved:
  (i) a can. polarized normal variety is K-stable iff it is lc;
  (ii) a polarized normal Calabi-Yau variety is K-stable iff it is klt.
- Can also allow non-normal varieties and pairs, but will stick to the smooth case here.

# Uniform K-stability

- K-stability means $\mathrm{DF}(\mathcal{X},\mathcal{L}) > 0$ for any ample tc $(\mathcal{X},\mathcal{L})$ for $(X, L)$, except if $(\mathcal{X},\mathcal{L})$ is almost trivial.
- Can make this inequality quantitative.
- Introduce *norm* $\|(\mathcal{X},\mathcal{L})\|$ of a test configuration. See later.
- Key property: $\|(\mathcal{X},\mathcal{L})\| = 0$ iff $(\mathcal{X},\mathcal{L})$ almost trivial.
- *Uniform K-stability* then means $\exists\delta > 0$ such that

$$\mathrm{DF}(\mathcal{X},\mathcal{L}) \geq \delta\|(\mathcal{X},\mathcal{L})\|$$

  for every ample test configuration $(\mathcal{X},\mathcal{L})$ for $(X, L)$.
- Uniform K-stability implies that $\mathrm{Aut}(X, L)$ finite.
- Notion of *uniform K-polystability* being developed, won't discuss.

# Other approaches to K-stability

- The above definition of K-stability is the "traditional" one.
- Next, characterize K-stability of Fano mfld $X$ in four other ways:
  (1) Singularities of certain anticanonical $\mathbf{Q}$-divisors on $X$.
  (2) Invariants of divisorial valuations on $X$.
  (3) Invariants of valuations on the cone $\mathbf{C}(X)$.
  (4) Non-Archimedean metrics on $-K_X$.
- Will focus on the last one.

# K-stability via anticanonical $\mathbf{Q}$-divisors

- Define the *alpha-invariant* of a Fano manifold $X$ by

$$\alpha(X) := \inf\{\mathrm{lct}(D) \mid D \text{ effective } \mathbf{Q}\text{-divisor}, D \sim_{\mathbf{Q}} -K_X\},$$

  an algebraic version of the invariant introduced by Tian.
- Fujita and Odaka introduced a new *delta-invariant*.
- Given $m \gg 0$, say $D$ is of *m-basis type* if there exists a basis $s_1, \dots, s_{N_m}$ for $\Gamma(X, -mK_X)$ such that $D = \frac{1}{mN_m}\sum_{j=1}^{N_m} \mathrm{div}(s_j)$, and set $\delta_m(X) := \inf\{\mathrm{lct}(D) \mid D \text{ of } m\text{-basis type}\}$.
- **Thm** [Blum-J] The limit $\delta(X) := \lim_m \delta_m(X)$ exists, and
  - (i) $X$ is K-semistable iff $\delta(X) \geq 1$.
  - (ii) $X$ is uniformly K-stable iff $\delta(X) > 1$.
- The result was conjectured by Fujita and Odaka.
- Proof uses result by Fujita and C. Li (and Li-Xu), see later.

# K-stability via divisorial valuations

- Next, characterize K-stability of a Fano manifold as a condition on divisorial valuations.
- Let $X$ be a smooth variety. A *prime divisor over* $X$ is a prime divisor $E \subset Y$, where $Y$ is a normal variety together with a birational morphism $\pi\colon Y \to X$.
- Such a divisor induces a *divisorial valuation*

$$\mathrm{ord}_E\colon \kappa(X) \to \mathbf{Z}$$

- Two prime divisors $E \subset Y$ and $E' \subset Y'$ over $X$ induce the same divisorial valuation iff the canonical birational map $Y \dashrightarrow Y'$ sends $E$ onto $E'$.

# Invariants of divisorial valuations

- Consider divisorial valuation $\mathrm{ord}_E$, associated to prime divisor $E \subset Y$ over $X$, with birational morphism $\pi \colon Y \to X$.
- The *log discrepancy* of $\mathrm{ord}_E$ is

$$A_X(\mathrm{ord}_E) := 1 + \mathrm{ord}_E(K_{Y/X}),$$

  where $K_{Y/X}$ is the relative canonical divisor.
- Fix an ample line bundle $L$ on $X$. Set $V = \mathrm{Vol}(L) = (L^n)$.
- Define

$$T(\mathrm{ord}_E) := \sup\{t > 0 \mid \pi^*L - tE \text{ pseudoeffective}\}$$

  and

$$S(\mathrm{ord}_E) := V^{-1} \int_0^{T(\mathrm{ord}_E)} \mathrm{Vol}(\pi^*L - tE)\, dt.$$

- These invariants depend only on $\mathrm{ord}_E$ (and not on $Y$).

# K-stability of Fano manifolds via divisorial valuations

- Can characterize K-stability of a Fano mfld using div. valuations.
- **Thm** [Fujita, C. Li] If $X$ is a Fano manifold, then
  (i) $X$ is K-semistable iff $A_X(\mathrm{ord}_E) \geq S(\mathrm{ord}_E)$ for all $E$.
  (ii) $X$ is uniformly K-stable iff $\inf_E \frac{A_X(\mathrm{ord}_E)}{S(\mathrm{ord}_E)} > 1$.
- The invariants of $X$ defined via anticanonical $\mathbf{Q}$-divisors satisfy:

$$\alpha(X) = \inf_E \frac{A_X(\mathrm{ord}_E)}{T(\mathrm{ord}_E)} \quad \text{and} \quad \delta(X) = \inf_E \frac{A_X(\mathrm{ord}_E)}{S(\mathrm{ord}_E)}$$

- This is easy for $\alpha(X)$ and was proved for $\delta(X)$ by Blum-J using Okounkov bodies.
- The theorem by Fujita and Li thus implies the above characterization of K-stability via anticanonical $\mathbf{Q}$-divisors.
- The proof of the theorem uses MMP as in Li-Xu as well as the notion of Ding-stability. More on this below.

# K-stability via valuations on the cone $\mathbf{C}(X)$

- Consider cone $Y = \mathbf{C}(X)$ of a Fano mfld $X$. This has a klt singularity at 0 and comes with $\mathbf{C}^*$-action.
- Consider valuation $v$ on $\mathbf{C}[Y]$ centered at 0.
- C. Li introduced the *normalized volume* of $v$:

$$\widehat{\mathrm{vol}}(v) := A_X(v)^{n+1}\,\mathrm{vol}(v),$$

where $A_X(v)$ is the log discrepancy and $\mathrm{vol}(v)$ is the volume of $v$.

$$\mathrm{vol}(v) = \limsup_{m\to\infty} \frac{\dim_{\mathbf{C}}(\mathcal{O}_{Y,0}/\{v \geq m\}}{m^{n+1}/(n+1)!}$$

- **Thm** [Li, Li-Liu] $X$ is K-semistable iff $\mathrm{vol}(v)$ is minimized for the divisorial valuation $\mathrm{ord}_0$ obtained by blowing up $0 \in Y$.
- Will not discuss this further here but the idea of working on the cone is similar in spirit to working with test configurations.
- Recent work on general klt sings: Li-Liu, Li-Xu, Liu-Xu, Blum,...

# K-stability via non-Archimedean geometry

- Next, interpret K-stability via non-Archimedean (NA) geometry.
- NA geometry is analogue of complex geometry when replacing the complex numbers by elements of a NA field.
- NA field: field $k$ with NA multiplicative norm $|\cdot|\colon k \to \mathbf{R}_+$, i.e. $|a+b| \le \max\{|a|,|b|\}$, $|a| = 0$ iff $a = 0$, $|ab| = |a| \cdot |b|$.
- Key examples:
  - (i) $k = \mathbf{C}$ with the *trivial norm*: $|a| = 1$ for $a \neq 0$.
  - (ii) $k = \mathbf{C}((t))$ with $|f| = r^{\mathrm{ord}_0(f)}$ for some $r \in (0,1)$.
- Other examples: $k = \mathbf{Q}_p$ and $k = \mathbf{F}_p((t))$. Won't be used here.
- Will use the approach to NA geometry due to V. Berkovich.
- Don't need general Berkovich spaces but only analytifications of complex algebraic varieties. Further, mainly treat them as topological spaces, ignoring structure sheaf.

# Berkovich analytifications

- Fix NA field $k$ (e.g. $k = \mathbf{C}$ with trivial norm).
- Given scheme $X/k$ of f.t., define Berkovich analytification $X^{\mathrm{an}}$.
- If $X = \operatorname{Spec} A$ is affine, with $A =$ f.g. $k$-algebra, then

$$X^{\mathrm{an}} = \{\text{multiplicative seminorms on } A \text{ extending norm on } k\}$$

  with weakest top. s.t. $X^{\mathrm{an}} \ni |\cdot| \to |f| \in \mathbf{R}_+$ cont. $\forall f \in A$.
- In general, get $X^{\mathrm{an}}$ by gluing $U^{\mathrm{an}}$ for open affine subsets $U \subset X$.
- As a set, $X^{\mathrm{an}}$ is the set of pairs $(\xi, |\cdot|)$, where $\xi \in X$ is a scheme pt, and $|\cdot|$ mult. *norm* on $k(\xi)$ extending norm on $k$.
- Thm: $X^{\mathrm{an}}$ is locally compact and locally arcwise connected.
- Morphism $f\colon Y \to X$ induces continous map $f^{\mathrm{an}}\colon Y^{\mathrm{an}} \to X^{\mathrm{an}}$.
- Have various GAGA results, such as:
  - (i) $X^{\mathrm{an}}$ compact (Hausdorff) iff $X$ proper.
  - (ii) $X^{\mathrm{an}}$ arcwise connected iff $X$ connected.
- Have embedding $X(k) \hookrightarrow X^{\mathrm{an}}$. Not surjective if $\dim X > 0$.

# The Berkovich affine line

- The Berkovich affine line $\mathbf{A}_k^{1,\mathrm{an}}$ looks like a tree.

**** PICTURES (triv and nontriv norm) ****

- It has a *skeleton* consisting of norms on $k[T]$ of the form

$$|\sum_i a_i T^i| = \max_i |a_i| r^i,$$

for some fixed $r \in \mathbf{R}_+$.

# Metrics on line bundles

- A line bundle $L \to X$ analytifies to $L^{\mathrm{an}} \to X^{\mathrm{an}}$
- Fiber over $x \in X^{\mathrm{an}}$ is isomorphic to $\mathbf{A}^{1,\mathrm{an}}_{\mathcal{H}(x)}$, where $\mathcal{H}(x)$ is the *complete residue field* of $x$, a NA extension of $k$.
- A *metric* on $L^{\mathrm{an}}$ is a function

  $$\|\cdot\| \colon L^{\mathrm{an}} \to \mathbf{R}_+$$

  whose restriction to any fiber $L^{\mathrm{an}}_x \simeq \mathbf{A}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ is of the form

  $$|\cdot| \to c(x)|T|$$

  for some $c(x) \in \mathbf{R}^*_+$.
- Completely analogous to the Archimedean situation!
- Will again work *additively*, replacing $\|\cdot\|$ by $\phi := -\log\|\cdot\|$.

# Calculus on metrics on line bundles

- Much of the “calculus” in the Archimedean case carries over.
- $\phi_i$ metric on $L_i$, $a_i \in \mathbf{Z} \implies a_1\phi_1 + a_2\phi_2$ metric on $a_1L_1 + a_2L_2$.
- If $s \in \Gamma(U, L)$ is a nowhere vanishing local section, then $\phi := \log |s|$ is a metric on $L$ over $U$ for which $\phi \circ s \equiv 0$.
- Can and will identify metrics on $\mathcal{O}_X^{\mathrm{an}}$ with functions on $X^{\mathrm{an}}$.
- Given a reference metric $\phi_0$ on $L^{\mathrm{an}}$, any other metric on $L^{\mathrm{an}}$ is of the form $\phi = \phi_0 + \varphi$, where $\varphi$ is a function on $X^{\mathrm{an}}$.
- Chambert-Loir and Ducros have given meaning to $dd^c\phi$ for suitably nice metrics $\phi$ on $L^{\mathrm{an}}$. Will not define this here, but will later use Monge-Ampère measure $(dd^c\phi)^n$.
- Metrics on $K_X$ play special role: more on this later.

# Fubini-Study metrics

- Now focus on case $k = \mathbf{C}$ with trivial norm and $X$ projective.
- Instead of "metric on $L^{\mathrm{an}}$" will say "NA metric on $L$".
- Given glob sections $s_j$ of $mL$ w/o common zeros, and $\lambda_j \in \mathbf{Z}$,

$$\phi := m^{-1} \max_{1 \le j \le l} (\log |s_j| + \lambda_j)$$

  is a continuous NA metric on $L$, called *Fubini-Study* (FS) metric.
- If $\lambda_j = 0 \ \forall j$, get can. NA metric on $L$, the *trivial metric* $\phi_{\mathrm{triv}}$.
- A *DFS metric* is a difference of FS metrics. A *DFS function* is a DFS metric on $\mathcal{O}_X$, viewed as fcn on $X^{\mathrm{an}}$. Notation: DFS($X$).
- Fact: DFS($X$) is dense in $C^0(X^{\mathrm{an}})$.
- DFS metrics are NA analogues of smooth metrics.
- FS metrics are NA analogues of positive smooth metrics. For this reason, also write $\mathcal{H}^{\mathrm{NA}} := \mathrm{FS}(L)$.

# Test configurations and divisorial valuations

- Let $X$ be a smooth (or normal) projective variety.
- A *valuation on* $X$ is a valuation $v$ on the function field $k(X)$ that is trivial on the ground field $\mathbf{C}$.
- Say $v$ is a *divisorial valuation* on $X$ if $v = c\,\mathrm{ord}_F$, where $c \in \mathbf{Q}_+^*$ and $F$ is a prime divisor over $X$.
- Every valuation $v$ on $X$ defines a point $x = \exp(-v)$ in $X^{\mathrm{an}}$.
- **Fact**: the set $X^{\mathrm{div}} \subset X^{\mathrm{an}}$ consisting of points corresponding to divisorial valuations is dense in $X^{\mathrm{an}}$.
- Consider normal test configuration $\mathcal{X}$ for $X$ and an irreducible component $E \subset \mathcal{X}_0$ of the central fiber.
- Then $\mathrm{ord}_E$ is a $\mathbf{C}^*$-invariant divisorial valuation on $\mathcal{X}$.
- **Fact**: The restriction of $\mathrm{ord}_E$ to $\mathbf{C}(X) \subset \mathbf{C}(\mathcal{X})$ is a valuation on $X$ that is either divisorial or the trivial valuation on $X$. Further, every such valuation on $X$ arises in this way.

# Test configurations and NA metrics

- Consider $X/\mathbf{C}$ sm proj and $L \to X$ line bundle (not nec. ample).
- Any tc $(\mathcal{X}, \mathcal{L})$ for $(X, L)$ induces a NA metric $\phi_{\mathcal{L}}$ on $L$.
- Won't give complete definition, but if $s \in H^0(X, L)$, let $\bar{s}$ be the canonical $\mathbf{C}^*$-equivariant extension of $s$ to a rational section of $\mathcal{L}$. Then $\bar{s} \in H^0(\mathcal{X}, \mathcal{L})$ iff $\phi_{\mathcal{L}} \circ s \geq 0$.
- The metric $\phi_{\mathcal{L}}$ does not change under pullbacks:

$$\begin{cases} \rho\colon \mathcal{X}' \to \mathcal{X} \text{ birat'l, } \mathbf{C}^*\text{-equivariant} \\ \mathcal{L}' = \rho^* \mathcal{L} \end{cases} \quad \Longrightarrow \quad \phi_{\mathcal{L}'} = \phi_{\mathcal{L}}.$$

- In general, $\phi_{\mathcal{L}} = \phi_{\mathcal{L}'}$ iff $(\mathcal{X}, \mathcal{L})$, $(\mathcal{X}', \mathcal{L}')$ admit common pullback.
- **Fact:** Let $\phi$ be a NA metric on $L$
  - (i) $\phi \in \mathrm{DFS}(L)$ iff $\phi = \phi_{\mathcal{L}}$ for some $\mathcal{L}$;
  - (ii) $\phi \in \mathrm{FS}(L)$ iff $\phi = \phi_{\mathcal{L}}$ for some semiample (or ample) $\mathcal{L}$.
- Thus DFS (resp. FS) metrics on $L$ can be though of as equivalence classes of tc's (resp. semiample tc's) for $(X, L)$.

# Monge-Ampère measures

- For $1 \leq j \leq n$, let $\phi_i$ be a NA metric on a line bundles $L_j$ represented by a test configuration $(\mathcal{X}_j, \mathcal{L}_j)$ for $(X, L_j)$.
- The *mixed Monge-Ampère measure* $dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n$ is a finite signed atomic measure on $X^{\mathrm{div}} \subset X^{\mathrm{an}}$ defined as follows.
- May assume $\mathcal{X}_j = \mathcal{X}$ independent of $j$. Write the central fiber as

$$\mathcal{X}_0 = \sum_{i \in I} b_i E_i,$$

and let $x_i \in X^{\mathrm{div}}$ be the point corresponding to $E_i$. Then

$$dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n = \sum_{i \in I} b_i(\mathcal{L}_1|_{E_i} \cdot \ldots \cdot \mathcal{L}_n|_{E_i})\delta_{x_i}.$$

- If the $L_i$ are ample and the $\phi_i$ positive, then the mixed Monge-Ampère measure is a positive measure.
- If $L$ is ample and $\phi \in \mathcal{H}^{\mathrm{NA}}$, then we write

$$\mathrm{MA}(\phi) := V^{-1}(dd^c\phi)^n = V^{-1}dd^c\phi \wedge \cdots \wedge dd^c\phi,$$

where $V = (L^n) = \mathrm{Vol}(L)$.

# Energy functionals

- Define the energy functionals $E$, $I$, $J$ in exactly the same way as before, using $\phi_{\mathrm{triv}}$ as reference metric:

$$E(\phi) = \frac{1}{n+1}\sum_{j=0}^{n} V^{-1}\int_X \varphi (dd^c\phi)^j \wedge (dd^c\phi_{\mathrm{triv}})^{n-j}$$

$$I(\phi) = \int_X \varphi(\mathrm{MA}(\phi_{\mathrm{triv}}) - \mathrm{MA}(\phi))$$

$$J(\phi) = \int_X \varphi\,\mathrm{MA}(\phi_{\mathrm{triv}}) - E(\phi).$$

  where $\varphi = \phi - \phi_{\mathrm{triv}}$ is a function on $X^{\mathrm{an}}$.
- If $\phi = \phi_{\mathcal{L}}$, can also write these using intersection nos on $\mathcal{X}$, e.g.

$$E(\phi_{\mathcal{L}}) = \frac{1}{n+1}V^{-1}(\mathcal{L}^{n+1}).$$

- The functionals $I$, $J$ and $I - J$ are $\geq 0$ and comparable. We have $J(\phi) \geq 0$ with equality iff $\phi = \phi_{\mathrm{triv}}$. Same for $I$, $I - J$.
- Use any one of them as "norm" on $\mathcal{H}^{\mathrm{NA}}$, e.g. $\|(\mathcal{X}, \mathcal{L})\| := J(\phi_{\mathcal{L}})$.

# The Ding functional

- Now assume $X$ Fano and $L = -K_X$.
- Extend log discrepancy $A = A_X$ as lsc function on $X^{\mathrm{an}}$.
- The *Ding functional*

$$D \colon \mathcal{H}^{\mathrm{NA}} \to \mathbf{R}$$

  is again defined as

$$D(\phi) = L(\phi) - E(\phi),$$

  where the functional $L$ is defined by

$$L(\phi) = \inf_{X^{\mathrm{an}}} (A + \phi - \phi_{\mathrm{triv}}).$$

- Say $X$ is *Ding semistable* if $D \geq 0$ on $\mathcal{H}^{\mathrm{NA}}$.
- Say $X$ is *uniformly Ding stable* if $\exists \delta > 0$ s.t. $D \geq \delta J$ on $\mathcal{H}^{\mathrm{NA}}$.

# Entropy and the Mabuchi functional

- The *entropy* of a probability measure $\mu$ on $X^{\mathrm{an}}$ is defined in terms of the log discrepancy:

$$\mathrm{Ent}(\mu) := \int_{X^{\mathrm{an}}} A(x)\, d\mu(x).$$

- The *Mabuchi functional* $M\colon \mathcal{H}^{\mathrm{NA}} \to \mathbf{R}$ is now defined as

$$M(\phi) = \mathrm{Ent}(\mathrm{MA}(\phi)) - (I - J)(\phi).$$

- The Mabuchi functional can be defined for more general polarizations. The formula above is the NA Chen-Tian formula.
- The Mabuchi functional is closely related to the DF invariant.
- If $\phi = \phi_{\mathcal{L}} \in \mathcal{H}^{\mathrm{NA}}$ is defined by a normal ample tc $(\mathcal{X}, \mathcal{L})$, then

$$\begin{aligned} \mathrm{DF}(\mathcal{X}, \mathcal{L}) &= M(\phi_{\mathcal{L}}) + V^{-1}((\mathcal{X}_0 - \mathcal{X}_{0,\mathrm{red}}) \cdot \mathcal{L}^n) \\ &\geq M(\phi_{\mathcal{L}}) \end{aligned}$$

  with equality iff $\mathcal{X}_0$ is reduced.
- The Mabuchi functional is "better" than the DF invariant.

# K-stability and Ding stability

- **Thm** [Boucksom-Hisamoto-J] For $X$ Fano and $\delta \geq 0$ we have:

$$\mathrm{DF} \geq \delta J \text{ on } \mathcal{H}^{\mathrm{NA}} \quad \Leftrightarrow \quad M \geq \delta J \text{ on } \mathcal{H}^{\mathrm{NA}}. \qquad \text{(uKs)}$$

- Setting $\delta = 0$ shows $X$ K-semistable iff $M \geq 0$ on $\mathcal{H}^{\mathrm{NA}}$.
- If ineqs in (uKs) hold for some $\delta > 0$, say $X$ *uniformly K-stable*.
- Notion makes sense for general $L$. Also introduced by Dervan.
- **Thm** [Berman-Boucksom-J, Fujita] For $X$ Fano:
  (i) $X$ K-semistable iff $X$ Ding-semistable
  (ii) $X$ uniformly K-stable iff $X$ uniformly Ding stable.
  In fact, can use the same $\delta \in (0, 1]$ in (ii).
- The proof uses MMP as pioneered by Li-Xu. Sketch:
  - Always have $M \geq D$ so "if" is clear.
  - For "only if" run MMP starting with tc. Check that $D - \delta J$ decreasing and end up with tc where $D = M$.
- Alternative proof in preparation uses *Legendre duality* between the functionals $L$ and Ent (cf. Berman in the Arch. case.)

# Part 3: A variational proof of the YTD conjecture

In this part:

- Partial summary of Parts 1 and 2.
- Remarks on non-Archimedean degenerations.
- Proof of necessity of K-stability.
- Proof of sufficiency ot K-stability: simplified picture.
- More details: singular metrics of finite energy.

# Partial summary of Lectures 1 and 2

- Goal: variational proof of YTD conjecture in special case.
- Let $X$ be a complex Fano manifold with $\mathrm{Aut}(X)$ finite.
- Black box: $\exists$ KE metric on $X \Leftrightarrow$ Mabuchi or Ding *coercive*:

$$M \geq \delta J - C \quad \Leftrightarrow \quad D \geq \delta J - C' \quad \text{on } \mathcal{H},$$

  where $\mathcal{H} = \{\text{positive metrics on } -K_X\}$ and $\delta, C, C' > 0$.
- Study K-stability using Berkovich geometry wrt trivial norm on $\mathbf{C}$.
- Saw: $X$ *uniformly K-stable* iff $X$ *uniformly Ding stable*

$$M \geq \delta J \quad \Leftrightarrow \quad D \geq \delta J \quad \text{on } \mathcal{H}^{\mathrm{NA}},$$

  where $\mathcal{H}^{\mathrm{NA}} = \{\text{Fubini Study metrics on } -K_X\}$ and $\delta > 0$.
- Now want to show that all four conditions above are equivalent.
- Idea: the metrics in $\mathcal{H}^{\mathrm{NA}}$ arise as *degenerations* of metrics in $\mathcal{H}$.

# Non-Archimedean degenerations

- Variational proof of YTD conjecture involves considering NA metrics as degenerations of Archimedean metrics.
- The idea of NA objects as limits of degenerations has appeared several times in the literature, for example:
  (i) Bergman '71: the logarithmic limit set of an alg variety.
  (ii) Morgan-Shalen '84: compactifications of affine algebraic varieties using valuations.
  (iii) Mikhalkin '01, Rullgård '01, J '16: degenerations of amoebas to tropical varieties.
  (iv) Kiwi '06, '15; DeMarco-Faber '14, 16; Favre '16: degeneration of complex dynamics to NA dynamics.
  (v) Boucksom-J '17: Degenerations of Calabi-Yau varieties.
- A trivial but relevant remark is that if $|\cdot|$ is the Euclidean norm on $\mathbf{C}$, then $|\cdot|^\rho$ converges pointwise to the trivial norm as $\rho \to 0$.

# Necessity of K-stability for existence of KE metrics

- Suppose $X$ has KE metric, so Ding and Mabuchi are coercive:

$$M \geq D \geq \delta J - C \quad \text{on } \mathcal{H}.$$

- Consider any ample test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, -K_X)$.
- Pick a smooth $S^1$-invariant semipositive metric $\Phi$ for $\mathcal{L}$ near $\mathcal{X}_0$.
- Using $\mathbf{C}^*$-action, $\Phi$ induces a ray $(\phi^t)_0^\infty$ of metrics on $L$.
- **Thm**: if $F$ is any of the functionals $E$, $I$, $J$, $M$ or $D$, then

$$\lim_{t\to\infty} t^{-1} F(\phi^t) = F(\phi_{\mathcal{L}}),$$

  where $\phi_{\mathcal{L}} \in \mathcal{H}^{\mathrm{NA}}$ is the NA metric defined by $(\mathcal{X}, \mathcal{L})$.
- Thus $D, M \geq \delta J$ on $\mathcal{H}^{\mathrm{NA}}$ i.e. $X$ is unif. Ding and K-stable.
- The case $F = D$ due to Berman.
- The cases $F = E, I, J, M$ proved in [BHJ2] using Deligne pairings. Earlier versions exist in the literature.

# Sufficiency of K-stability: outline

- Now suppose $X$ uniformly Ding (and hence K-)stable:

$$M \geq D \geq \delta J$$

  on $\mathcal{H}^{\mathrm{NA}}$ for some $\delta \in (0,1)$.
- Want to prove Mabuchi coercive: for any $\delta' \in (0,\delta)$

$$M \geq \delta' J - C$$

  on $\mathcal{H}$ for some $C = C_{\delta'} > 0$.
- Argue by contradiction: suppose there exists $(\phi_j)_1^\infty$ such that

$$M(\phi_j) \leq \delta' J(\phi_j) - j.$$

- Idea is to "extract" from $(\phi_j)_1^\infty$ a ray $(\phi^t)_0^\infty$ in $\mathcal{H}$ and a NA metric $\phi \in \mathcal{H}^{\mathrm{NA}}$ such that $D(\phi^t) \leq \delta' J(\phi^t)$ and

$$\lim t^{-1} F(\phi^t) = F(\phi)$$

  where $F = D, J$. This contradicts $D \geq \delta J$ on $\mathcal{H}^{\mathrm{NA}}$.

# Difficulties and remedies

- Unclear how to "extract" (subgeodesic) ray from sequence in $\mathcal{H}$.
- Even if this can be done, no reason why this ray is associated to a test configuration, i.e. to a metric in $\mathcal{H}^{\mathrm{NA}}$.
- Both problems can be overcome by working with more general Archimedean and NA metrics on $-K_X$.
- On the Archimedean side, we use the space $\mathcal{E} \supset \mathcal{H}$ of (singular) *metrics of finite energy*. Extensively studied by Darvas, following work of Cegrell, Guedj-Zeriahi, BBEGZ, ...
- The space $\mathcal{E}$ has a Finsler metric with certain compactness properties, allowing us to construct generalized geodesic rays.
- On the NA side, there is a corresponding space $\mathcal{E}^{\mathrm{NA}} \supset \mathcal{H}^{\mathrm{NA}}$. A geodesic ray in $\mathcal{E}$ induces a metric in $\mathcal{E}^{\mathrm{NA}}$.
- Instead of metrics in $\mathcal{E}^{\mathrm{NA}}$, can work with *sequences of test configurations*, so don't need to formally introduce $\mathcal{E}^{\mathrm{NA}}$.

# Metrics of finite energy

- Consider general smooth polarized variety $(X, L)$.
- There exists a natural space $\mathrm{PSH}(L)$ of (singular) metrics on $L$. Can define it as the set of *decreasing limits* of metrics in $\mathcal{H}$.
- Extend the Monge-Ampère energy functional $E$ to $\mathrm{PSH}(L)$ via

$$E(\phi) = \inf\{E(\psi) \mid \psi \geq \phi, \psi \in \mathcal{H}\},$$

  and define $\mathcal{E} = \mathcal{E}(L)$ as the space of metrics of finite energy:

$$\mathcal{E} = \{\phi \in \mathrm{PSH}(L) \mid E(\phi) > -\infty\}.$$

- Can define Monge Ampère measures and extend the functionals $I$, $J$, $M$ to the space $\mathcal{E}$. Same for $D = \mathrm{Ding}$ if $L = -K_X$.
- **Thm** [GZ]. The Monge-Ampère operator induces a bijection

$$\mathrm{MA}\colon \mathcal{E}/\mathbf{R} \xrightarrow{\sim} \mathcal{M},$$

  with $\mathcal{M}$ the space of probability measures $\mu$ on $X$ of finite energy,

$$E^*(\mu) := \sup\{E(\phi) - \int (\phi - \phi_0)\, \mathrm{MA}(\phi) \mid \phi \in \mathcal{H}\} < +\infty.$$

  Further, we have $E^*(\mathrm{MA}(\phi)) = (I - J)(\phi)$.

# Non-Archimedean metrics of finite energy

- Consider smooth polarized complex variety $(X, L)$.
- Can define the space $\mathcal{E}^{\mathrm{NA}}$ of NA (singular) metrics of finite energy exactly as in the Archimedean case.
- Same for the space $\mathcal{M}^{\mathrm{NA}}$ of Radon probability measures $\mu$ on $X^{\mathrm{an}}$ of finite energy, $E^*(\mu) < +\infty$.
- Can define Monge Ampère measures and extend the functionals $I$, $J$, $M$ to the space $\mathcal{E}^{\mathrm{NA}}$. Same for $D =$ Ding if $L = -K_X$.
- **Thm** [Boucksom-Favre-J, Boucksom-J]. We have a bijection

$$\mathrm{MA}\colon \mathcal{E}^{\mathrm{NA}}/\mathbf{R} \overset{\sim}{\to} \mathcal{M}^{\mathrm{NA}},$$

  and the supremum in $E^*(\mathrm{MA}(\phi))$ is attained at $\phi$, i.e.

$$E^*(\mathrm{MA}(\phi)) = (I - J)(\phi)$$

  for any $\phi \in \mathcal{E}^{\mathrm{NA}}$.
- Can use this result to prove that uniform K-stability is equivalent to uniform Ding-stability.

## Sufficiency of K-stability: some further details

- To reach contradiction, assume $0 < \delta' < \delta < 1$,

$$D \geq \delta J \quad \text{on } \mathcal{H}^{\mathrm{NA}}.$$

and there exists a sequence $(\phi_j)_1^\infty$ in $\mathcal{H}$ such that

$$M(\phi_j) \leq \delta' J(\phi_j) - j.$$

- Chen: join $\phi_0$ and $\phi_j$ by geodesic segment $\phi_j^t$, $0 \leq t \leq T_j$ in $\mathcal{E}$.
- Convexity of $M$ [Berman-Berndtsson] $\implies$ $M(\phi_j^t) \leq \delta' t$.
- Thus, for fixed $T$, the metrics $\phi_j^t$, $t \leq T$, lie in *compact* subset of $\mathcal{E}$. Use Ascoli to construct geodesic ray $(\phi^t)_0^\infty$ in $\mathcal{E}$ such that

$$D(\phi^t) \leq M(\phi^t) \leq \delta' t$$

# Sufficiency of K-stability: cont.

- The constructed geodesic ray $(\phi^t)_0^\infty$ in $\mathcal{E}$ satisfies

$$E(\phi^t) = -t, \quad \sup(\phi^t - \phi_0) = 0 \quad \text{and} \quad D(\phi^t) \le \delta' t \text{ for all } t.$$

- Will use ray to construct NA metric $\psi \in \mathcal{E}^{\mathrm{NA}}$. How?
- Ray induces $S^1$-invariant psh metric $\Phi$ on the lb $p_1^* L$ on $X \times \mathbf{D}^*$. Extends to the central fiber, i.e. psh metric on $p_1^* L$ on $X \times \mathbf{D}$.
- For $m \ge 1$, let $\mu_m \colon \mathcal{X}_m \to X \times \mathbf{P}^1$ be the normalized blowup along the *multiplier ideal* $\mathcal{J}(m\Phi)$.
- Set $\mathcal{L}_m := \mu_m^* p_1^* L - \frac{1}{m+m_0} E_m$, where $E_m =$ exc div and $m_0 \gg 0$.
- Then $(\mathcal{X}_m, \mathcal{L}_m)$ is a tc for $X$, defining metric $\psi_m \in \mathcal{H}^{\mathrm{NA}}$.
- By subadditivity of multiplier ideals, the sequence $\psi_m$ in $\mathcal{H}^{\mathrm{NA}}$ is essentially decreasing, and has a psh limit $\psi$.
- Can now show $\psi \in \mathcal{E}^{\mathrm{NA}}$ and $D(\psi) \le \delta' J(\psi)$, a contradiction.

# Sufficiency of K-stability: cont.

- Pick $\Psi_m$ = smooth $S^1$-inv. psh metric on $\mathcal{L}_m$ near $\mathcal{X}_0$. It defines:
  (i) subgeodesic ray $(\psi_m^t)_t$ in $\mathcal{H}$.
  (ii) an $S^1$-invariant psh metric $\Phi_m$ on $p_1^*L$ above $X \times \mathbf{D}^*$.
- Know $\lim_t t^{-1} L(\psi_m^t) = L(\psi_m)$ and $\lim_t t^{-1} E(\psi_m^t) = E(\psi_m)$.
- Demailly's reg. thm $\implies$ $\Phi_m$ less singular than $\Phi$. Thus

$$E(\psi_m) = \lim t^{-1} E(\psi_m^t) \geq \lim_t t^{-1} E(\psi^t) = -1,$$

  and hence $E(\psi) = \lim_m E(\psi_m) \geq -1$. In particular, $\psi \in \mathcal{E}^{\mathrm{NA}}$.
- **Thm**: $\lim_t L(\phi^t) = L(\psi)$. This follows from the fact [BFJ08] that *integrability can be detected using valuations*.
- Since $C \geq D(\phi^t) - \delta' J(\phi^t) = L(\phi^t) - (1-\delta')t$, it follows that $L(\psi) \leq 1 - \delta'$; hence $D(\psi) - \delta' J(\psi) = L(\psi) - (1-\delta')E(\psi) \leq 0$.
- This completes the proof of sufficiency!
- Again: can formulate everything in terms of sequences of test configurations if one wants to avoid NA language.

# Part 4: Extensions, comments, speculations

- K-stability and uniform K-stability.
- The case of vector fields, and singular Fanos.
- The case of cscK metrics.
- Twisted Kähler-Einstein metrics.

# K-stability and uniform K-stability

- Suppose $X$ Fano with $\mathrm{Aut}(X)$ finite.
- By [CDS, Tian] and [BBJ], $X$ K-stable iff $X$ unif K-stable.
- Is there a direct algebraic (or NA) proof, not using KE metrics?

# The case of vector fields, and singular Fanos

- Should be able to modify proof to allow for vector fields.
- Darvas-Rubinstein: coercivity of modified versions of Ding and Mabuchi equivalent to existence of KE metrics
- On NA side, need to develop analogous notions of uniform K-polystability and uniform Ding-polystability.
- This is work in progress.
- Should also be able to treat the case $X = \mathbf{Q}$-Fano (or log Fano) but there are some analytic difficulties.

# The case of cscK metrics

- More general YTD conjecture: the existence of a cscK metric $\omega \in c_1(L)$ is equivalent to (some kind of) K-stability for $(X, L)$.
- If $\mathrm{Aut}(X, L)$ finite, uniform K-stability may be the right condition.
- Berman-Darvas-Lu: cscK metric implies stability. Other direction completely open in general.
- Unfortunately, our proof uses $X$ Fano at several places...

# Twisted KE metrics

- Method *does* generalize to existence of *twisted* KE metrics.
- Consider any smooth complex polarized variety $(X, L)$.
- **Question**: for what $\delta \geq 0$ is the following true: for *any* $\alpha \in c_1(-K_X - \delta L)$ there exists $\omega \in c_1(L)$ Kähler with

$$\operatorname{Ric} \omega = \delta\omega + \alpha?$$

- Yau's theorem (for integral classes) exactly says that $\delta = 0$ works!
- **Thm** [BBJ]: The supremum of such $\delta$ is the *delta-invariant* introduced by Fujita-Odaka (for $L = -K_X$):

$$\delta(L) = \lim_{m \to \infty} \inf\{\operatorname{lct}(D) \mid 0 \leq D \sim_{\mathbf{Q}} L \text{ of } m\text{-basis type}\},$$

i.e.

$$D = (mN_m)^{-1} \sum_1^{N_m} \operatorname{div}(s_j) \quad \text{for basis } s_1, \ldots, s_{N_m} \text{ of } H^0(X, mL).$$

where the existence of the limit is proved by Blum-J.