Mini-workshop in complex dynamics
Fields Institute Nov 2008

Lecture 2

Super-attracting fixed points

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(w. C. Favre, AENS 2007)
"Eigenvaluations"

Local dynamics in $\mathbb{C}^2$

$f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$

Rough classification using $Df(0)$:

- attracting
- repelling
- saddle
- parabolic

.....

We will focus on an extreme case:

Def: f is superattracting if $Df(0)$ is nilpotent ($\Longleftrightarrow Df^2(0) \equiv 0$)

Rem: Assume f dominant ($Jf \not\equiv 0$)
To simplify exposition, sometimes assume f finite (no contracted curves)

Superattracting fixed points

Talk will focus on tools rather than results. Still, what can be said?

$f = f_c + f_{c+1} + f_{c+2} + \dots$

$c = c(f) \geq 1$ order of vanishing

f superattracting $\iff c(f^2) > 1$

$n \longmapsto c(f^n)$ supermultiplicative:

$c(f^{n+m}) \geq c(f^n)\, c(f^m)$

$\Rightarrow c_\infty := c_\infty(f) := \lim\limits_{n\to\infty} c(f^n)^{1/n}$ exists.

Thm A: $\exists\, \delta > 0$ s.t.

$$\delta c_\infty^n \underset{\substack{\uparrow\\ \text{hard}}}{\leq} c(f^n) \underset{\substack{\uparrow\\ \text{obvious}}}{\leq} c_\infty^n$$

Thm B: c_∞ is a quadratic integer:

$$c_\infty^2 = A c_\infty + B, \quad A, B \in \mathbb{Z}$$

Ex: $f(x,y) = (y, xy)$ $\quad c_\infty = \frac{1}{2}(1+\sqrt{5})$

Blowups I

$f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$

$\pi : X \to (\mathbb{C}^2, 0)$ blowup of 0

$E_0 = \pi^{-1}(0)$ exc. div.

$$\begin{tikzcd} X \arrow[r, dashed, "\tilde{f}"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"'] \\ (\mathbb{C}^2,0) \arrow[r, "f"] & (\mathbb{C}^2,0) \end{tikzcd}$$

0 parabolic fixed point $(Df(0) = \mathrm{id})$

$\Rightarrow \tilde{f}$ good tool for understanding dynamics

0 superattracting fixed point $(Df^2(0) = 0)$

$\Rightarrow \begin{cases} \tilde{f} \text{ usually not holo} \\ \tilde{f}(E_0) = \text{point on } E_0. \end{cases}$

Blowups II

$\pi : X_\pi \longrightarrow (\mathbb{C}^2, 0)$ composition of blowups

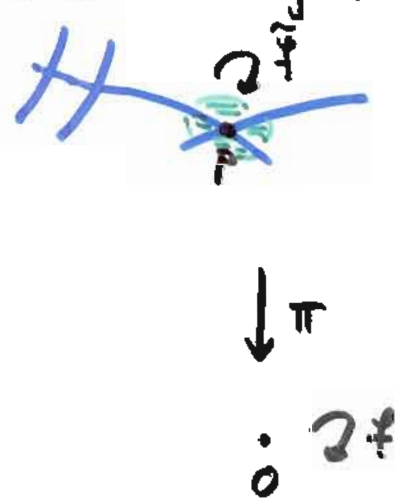
$$\begin{tikzcd} X_\pi \arrow[r, dashed, "\tilde{f}"] \arrow[d] & X_\pi \arrow[d] \\ (\mathbb{C}^2,0) \arrow[r, "f"] & (\mathbb{C}^2,0) \end{tikzcd}$$

Can we make $\tilde{f}$ "nice" by clever choice of π?

<u>Thm C</u>: Can choose $\pi : X_\pi \to (\mathbb{C}^2, 0)$ and a point $p \in \pi^{-1}(0)$ s.t. :

- $\tilde{f}$ holomorphic at p
- $\tilde{f}(p) = p$
- $\tilde{f} : (X_\pi, p) \circlearrowleft$ <u>rigid germ</u>

(Critical set of $\tilde{f}$ contained in a totally invariant set with snc sings)



<u>Rem</u>: Can find local normal forms for $\tilde{f} : (X_\pi, p) \circlearrowleft$

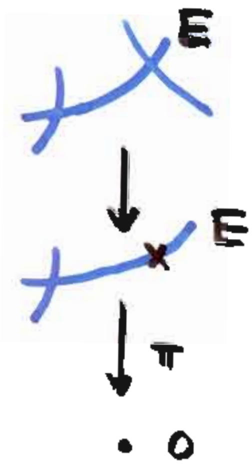
E.g. $\tilde{f}$ <u>monomial</u> map

Blowups III

How do we pick $\pi : X_\pi \longrightarrow (\mathbb{C}^2, 0)$?

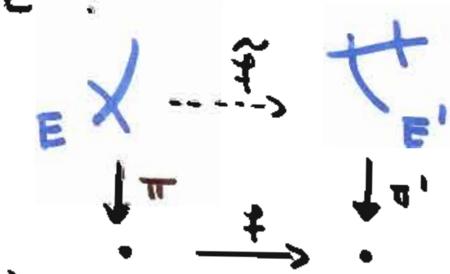
Idea: look at all possible π at the same time!

Def: An exceptional prime (divisor) is an irreducible component $E \subseteq \pi^{-1}(0)$ for some (composition of) blowup(s) $\pi : X_\pi \longrightarrow (\mathbb{C}^2, 0)$.



Convention: identify E with its strict transform under further blowups.

Fact (Zariski): f maps exceptional primes to exc. primes: given E $\exists E'$ s.t. $\tilde{f}E = E'$.



How to study induced dynamics on exc. primes?

1. Riemann-Zariski (Lecture 1)
2. Valuations (Lecture 2)
3. Combo (Lecture 3)

Divisorial valuations

$R = \mathcal{O}_0$ ring of holo germs at $0 \in \mathbb{C}^2$
K fraction field of R

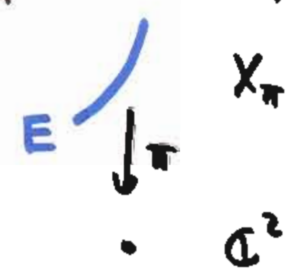
Idea: identify exc. prime $E \subseteq \pi^{-1}(0)$
with divisorial valuation ord_E

$\phi \in R$ (or K) $\Rightarrow$

$\mathrm{ord}_E(\phi)$ = order of vanishing of ϕ along E

(Convention: identify fcn field of X_π with K)

$\mathrm{ord}_E : \begin{cases} R \to \mathbb{N} \cup \{\infty\} \\ K \to \mathbb{Z} \cup \{\infty\} \end{cases}$ satisfies



- $\mathrm{ord}_E(\phi\psi) = \mathrm{ord}_E(\phi) + \mathrm{ord}_E(\psi)$
- $\mathrm{ord}_E(\phi+\psi) \geq \min\{\mathrm{ord}_E(\phi), \mathrm{ord}_E(\psi)\}$.
- $\mathrm{ord}_E(0) = \infty$
- $\mathrm{ord}_E(c) = 0 \quad c \in \mathbb{C}^*$
- $\phi \in \mathfrak{m}$ i.e. $\phi(0) = 0 \Rightarrow \mathrm{ord}_E(\phi) > 0$.

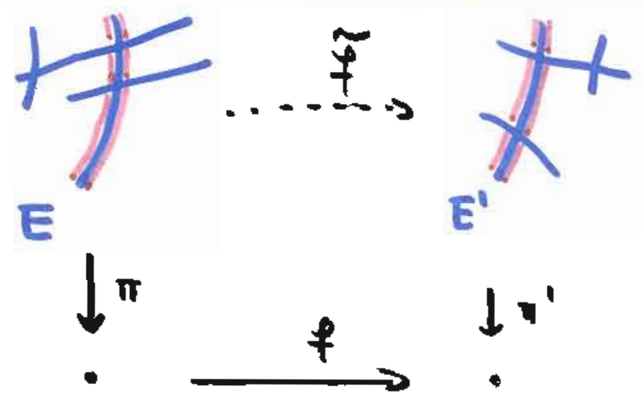
Ex: ord_0 (blow up 0 once)

Divisorial dynamics

Given f, E define new valuation $f_* \mathrm{ord}_E$:

$(f_* \mathrm{ord}_E)(\phi) := \mathrm{ord}_E(f^* \phi) = \mathrm{ord}_E(\phi \circ f)$

Fact (Zariski) $\Longrightarrow$ $\boxed{f_* \mathrm{ord}_E = k \cdot \mathrm{ord}_{E'}}$



$k =$ coeff of E in $\tilde{f}^* E'$

Set $\hat{\mathcal{V}}_{\mathrm{div}} = \{ c \cdot \mathrm{ord}_E \mid c > 0,\ E \text{ exc. prime} \}$

Get induced dynamics $f_* : \hat{\mathcal{V}}_{\mathrm{div}} \to \hat{\mathcal{V}}_{\mathrm{div}}$.

Problem : $\hat{\mathcal{V}}_{\mathrm{div}}$ not "nice"

Solution : Make it nice! $\hat{\mathcal{V}}_{\mathrm{div}} \subseteq \hat{\mathcal{V}}$

Analogy : $\mathbb{Q} \subseteq \mathbb{R}$

Valuations

$R = \mathcal{O}_0$ germs at $0 \in \mathbb{C}^2$

$\mathfrak{m}$ = max ideal $\phi \in \mathfrak{m} \Leftrightarrow \phi(0) = 0$.

Def: A valuation on R is a function

$\nu : R \longrightarrow [0, +\infty]$ s.t

- $\nu(\phi\psi) = \nu(\phi) + \nu(\psi)$
- $\nu(\phi+\psi) \geq \min\{\nu(\phi), \nu(\psi)\}$
- $\nu(0) = +\infty$, $\nu(c) = 0$, $c \in \mathbb{C}^*$

Def: A valuation ν is:

- centered at 0 if $\nu(\mathfrak{m}) := \min_{\phi \in \mathfrak{m}} \nu(\phi) > 0$
- normalized if $\nu(\mathfrak{m}) = 1$.

Def: $\hat{\mathcal{V}} := \{$val's centered at $0\}$

$\mathcal{V} := \{$normalized val's$\}$.

Rem: $\nu \in \hat{\mathcal{V}} \Leftrightarrow e^{-\nu}$ non-archimedean seminorm

$\hat{\mathcal{V}} \approx$ Berkovich space.

Valuative dynamics

$f : (\mathbb{C}^2, 0) \circlearrowleft$

[Assume f finite for simplicity ; need slight modification if f dominant].

Get induced map $f_* : \hat{\mathcal{V}} \longrightarrow \hat{\mathcal{V}}$.

$\nu \in \mathcal{V}$ normalized $\not\Rightarrow f_* \nu$ normalized.

$c(f, \nu) := \nu(f^* \mathfrak{m}) \quad (< \infty$ since f finite$)$

$f_\bullet \nu := c(f, \nu)^{-1} \cdot f_* \nu$

$\rightsquigarrow$ Induced map $f_\bullet : \mathcal{V} \longrightarrow \mathcal{V}$

<u>Rem</u>: $(f^n)_* = (f_*)^n$ and $(f^n)_\bullet = (f_\bullet)^n$

$$c(f^n, \nu) = \prod_{j=0}^{n-1} c(f, f_\bullet^j \nu)$$ <u>multiplicative cocycle</u>

In particular:

$$c(f^n) = c(f^n, \mathrm{ord}_0) = \prod_{j=0}^{n-1} c(f, f_\bullet^j \mathrm{ord}_0)$$

Executive summary of approach

1) $\mathcal{V}$ has structure of $\mathbb{R}$-tree

2) $f_\bullet : \mathcal{V} \to \mathcal{V}$ preserves this structure

3) $\exists$ fixed point (eigenvaluation) : $f_\bullet \nu_* = \nu_*$

4) Several cases ... in one of them, ν_* "interior" pt of $\mathcal{V}$, attracting :

a) ν_* interior pt $\Rightarrow$ (Skoda-Izumi)

$$\delta \nu_* \le \mathrm{ord}_0 \le \nu_*$$

Also : $f_* \nu_* = \lambda \nu_*$ $\lambda > 0$ $\Rightarrow$

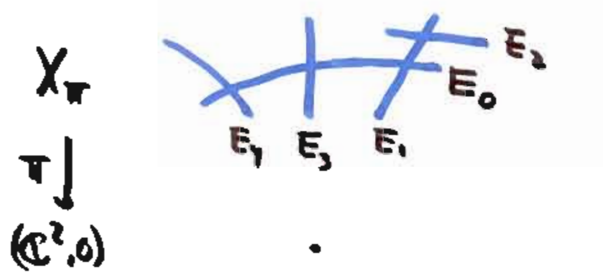
$$\delta \cdot \lambda^n \le c(f^n) \le \lambda^n \Rightarrow \text{Thm A}$$

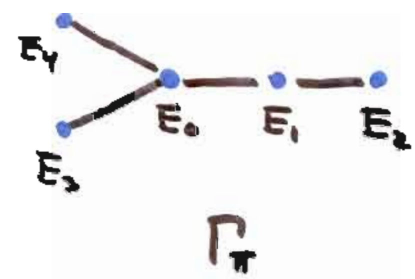
b) ν_* attracting $\rightsquigarrow$
construct X_π, p from suitable basin of attraction.

Dual graphs I

$\pi : X_\pi \to (\mathbb{C}^2, 0)$ (composition of) blowup(s)

$\Gamma_\pi :=$ dual graph of exceptional divisor





Structures on Γ_π :

- Partial ordering $\leq$, root $= E_0$
- Metric :

$$b_E := \mathrm{ord}_E(\mathfrak{m})$$

$$\mathrm{dist}(E,F) := \frac{1}{b_E b_F} \quad (E,F \text{ adjacent})$$

Compatibility : $\pi' \geq \pi \implies \Gamma_\pi \hookrightarrow \Gamma_{\pi'}$
order-preserving isometry

Dual graphs II

Can embed Γ_π inside $\mathcal{V}$:

Vertices:

E exc prime $\longmapsto$ normalized div. val'n

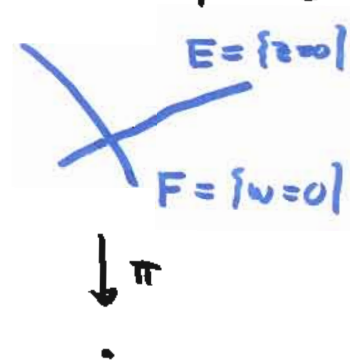
$\nu_E := b_E^{-1} \operatorname{ord}_E$

Edges:

$[E,F]$ edge in Γ_π

Parametrization : $s b_E + t b_F = 1 \quad s,t \geq 0$

Associate to (s,t) val'n $\nu_{s,t}$ which is monomial in (z,w):



$$\nu_{s,t}\left(\sum a_{ij} z^i w^j\right) = \min_{a_{ij} \neq 0} \{si + tj\}$$

Get embedding $\Gamma_\pi \overset{i_\pi}{\hookrightarrow} \mathcal{V}$

(Compatibility: $\Gamma_\pi \hookrightarrow \Gamma_{\pi'} \hookrightarrow \mathcal{V}$)

Def: $\mathcal{V}_{qm} := \varinjlim_\pi \Gamma_\pi$ quasimonomial val'ns

(union)

Dual graphs III

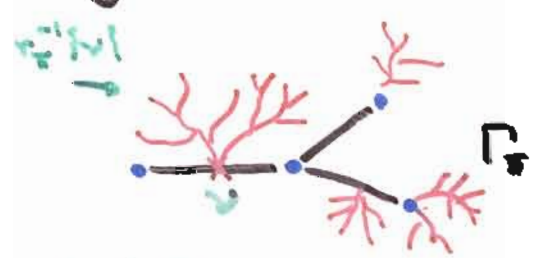
Given blowup $\pi : X_\pi \longrightarrow (\mathbb{C}^2, 0)$
and valuation $\nu \in \mathcal{V}$ can define
center of ν on X_π:

a) $\nu = \nu_E \Rightarrow$ center $= E$
b) otherwise $\exists!\, p \in X_\pi$ s.t. $\nu > 0$ on $\mathfrak{m}_p$
center $= p$

Refined construction $\rightsquigarrow$ retraction $r_\pi : \mathcal{V} \to \Gamma_\pi$

$$\boxed{\text{Thm}: \ \mathcal{V} := \varprojlim_{\pi} \Gamma_\pi}$$

Cor: $\mathcal{V}$ is an $\mathbb{R}$-tree
(partial ordering + metric)



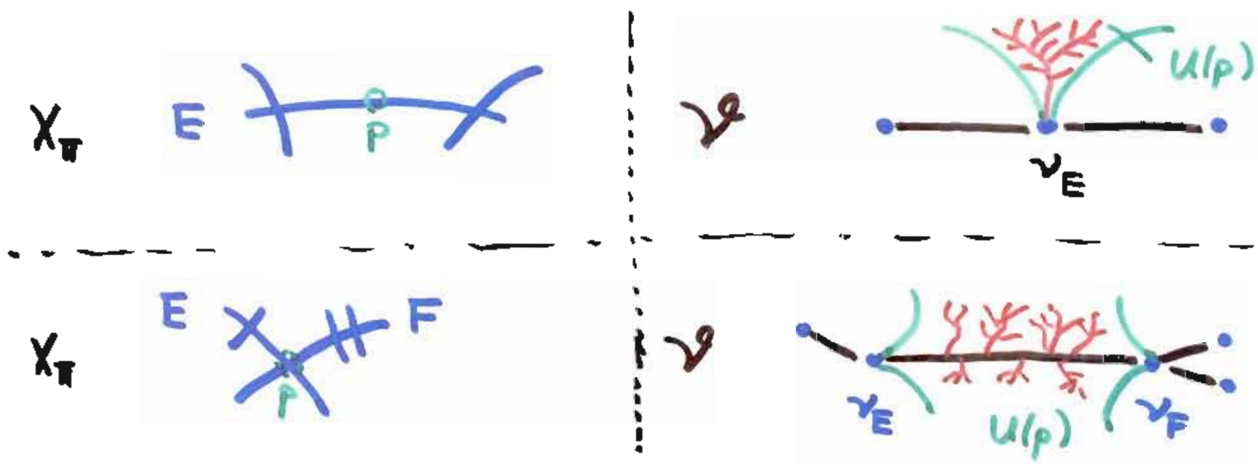
Rem: $\mathcal{V}_{qm}$ is "interior" of $\mathcal{V}$.

Rem: 2 types of non-quasimonomial val's:
- curve valuations
- infinitely singular valuations

Geometric partitions of $\mathcal{V}$

Given blowup $\pi : X_\pi \longrightarrow (\mathbb{C}^2, 0)$
and a point $p \in \pi^{-1}(0)$ "infinitely near pt"
get open subset $U(p) \subseteq \mathcal{V}$

$$U(p) = \{ \nu \mid \text{center of } \nu \text{ on } X_\pi \text{ is } p \}$$



This gives partition of $\mathcal{V}$ into:

- vertices ν_E of Γ_π
- open subsets $U(p)$ $\quad p \in \pi^{-1}(0)$

Rem: replacing π by $\pi' \geq \pi$
$\Rightarrow$ refining partition

Recognizing indeterminacy points

$f : (\mathbb{C}^2, 0) \circlearrowleft$

induced map $f_\bullet : \mathcal{V} \longrightarrow \mathcal{V}$

π, π' blowups of $(\mathbb{C}^2, 0)$

$$\begin{tikzcd} X_\pi \arrow[r, dashed, "\tilde{f}"] \arrow[d, "\pi"'] & X_{\pi'} \arrow[d, "\pi'"] \\ (\mathbb{C}^2, 0) \arrow[r, "f"] & (\mathbb{C}^2, 0) \end{tikzcd}$$

$p \in \pi^{-1}(0) \quad , \quad p' \in (\pi')^{-1}(0)$

Then : $\left\{ \begin{array}{l} \tilde{f} \text{ holo at } p \\ \tilde{f}(p) = p' \end{array} \right. \iff f_\bullet U(p) \subset U(p')$

Preservation of tree structure

① γ (ordered) segment in $\mathcal{V}$
s.t. $c(f,\cdot)$ constant on γ

$\Longrightarrow$

$f_\bullet$ maps γ homeom onto segment γ'

$\nu \le \nu' \iff \nu(\phi) \le \nu'(\phi)\ \forall \phi$

"Pf": $\begin{cases} f_* \text{ order-preserving} \\ c(f,\cdot) \text{ constant on } \gamma \end{cases} \Rightarrow f_\bullet$ o-p on γ

② $\exists$ finite subtree $T_f \subset \mathcal{V}$ s.t.:

a) $c(f,\cdot)$ locally constant outside T_f

b) T_f can be decomposed into finitely many segments on each of which $f_\bullet$ is a homeomorphism.

"Pf": Use monomialization $\exists \pi, \pi'$

$$\begin{tikzcd} X_\pi \arrow[r, "\tilde{f}"] \arrow[d] & X_{\pi'} \arrow[d] \\ (\mathbb{C}^2, 0) \arrow[r, "f"] & (\mathbb{C}^2, 0) \end{tikzcd}$$

$\tilde{f}$ holo + locally monomial

Existence of eigenvaluation

<u>Thm</u>: $\exists \nu_* \in \mathcal{V}$ s.t. $f_\bullet \nu_* = \nu_*$

Moreover, either:

1) ν_* quasimonomial (not endpoint in $\mathcal{V}$)

2) ν_* not qm, locally attracting

<u>Pf</u>: Arboreal $\square$

Growth of $c(f^n)$

Assume ν_* quasimonomial

$f_\bullet \nu_* = \nu_* \implies f_* \nu_* = \lambda \nu_* \quad \lambda > 0$

$\implies c(f^n; \nu_*) = \nu_*(f^{n*} m) = \lambda^n$

Izumi-Skoda inequality (ν_* qm)

$\exists \delta > 0$ s.t $\delta \nu_* \leq \mathrm{ord}_0 \leq \nu_*$

automatic

$$\implies \begin{cases} c(f^n) = \mathrm{ord}_0(f^{n*} m) \geq \delta \cdot \lambda^n \\ c(f^n) \leq \lambda^n \end{cases}$$

$\implies c_\infty = \lambda$ and $\delta c_\infty^n \leq c(f^n) \leq c_\infty^n$

i.e Thm A

(modify argument when ν_* not qm)

Normal forms I

Assume ν_* irrational quasimonomial
(nondivisorial)

$f_\bullet$ must map I into J for tree reasons.
Assume $f_\bullet$ order-preserving at ν_*
(replace f by f^2 if not)

<u>Claim</u>: $f_\bullet I \subset I$ for any sufficiently small segment $I \ni \nu_*$.

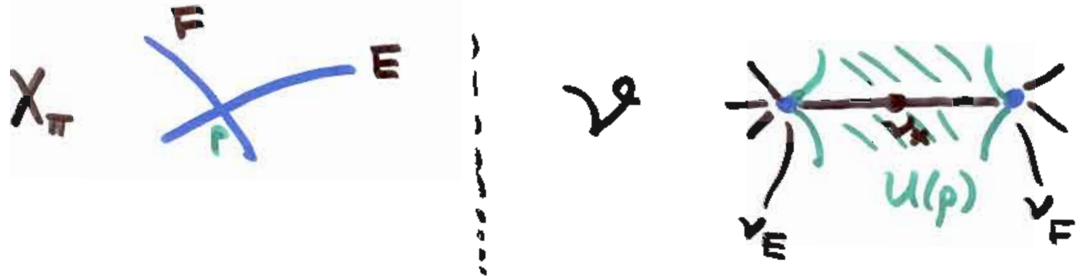
"Pf" In parametrization of I, $f_\bullet$ given by $t \longmapsto \frac{at+b}{ct+d}$ $a,b,c,d \in \mathbb{N}$ $\square$

<u>Claim</u>: For I small enough, $f_\bullet U \subset U$

<u>Pf</u>: Preservation of tree str.

Normal forms II

Can choose $\mathcal{U}$ of "geometric" form $\mathcal{U} = \mathcal{U}(p)$



$$\Rightarrow \begin{cases} \tilde{f} : X_\pi \dashrightarrow X_\pi \text{ holo at } p \\ \tilde{f}(p) = p \end{cases}$$

Can make sure $\begin{cases} C_{\tilde{f}} \subset E \cup F \\ \tilde{f}^{-1}(E \cup F) \subset E \cup F \end{cases}$ locally

$\Rightarrow \tilde{f} : (X_\pi, p) \circlearrowleft$ rigid

Can make $\tilde{f}(z,w) = (z^a w^b, z^c w^d)$

$\Rightarrow c_\infty =$ spectral radius of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\Rightarrow$ Thm B and Thm C !