Mini-workshop in complex dynamics

Fields Institute Nov 2008

Lecture 3

Polynomial dynamics at infinity

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(w C. Favre, www.arxiv.org
"Dynamical compactifications of $\mathbb{C}^2$")

Polynomial Mappings

$f: \mathbb{C}^2 \to \mathbb{C}^2$ poly, dominant

Study behavior at ∞ of f^n

Similar to $g: (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ but

- f not proper in general so can't say $f(\infty) = \infty$
- situation less local than $(\mathbb{C}^2, 0)$: "holomorphic objects" defined near ∞ in $\mathbb{C}^2$ extend to all of $\mathbb{C}^2$ (Hartogs)

Strategy: combine methods from

- Lecture 1 (global merom selfmaps)
- Lecture 2 (local dynamics)

Dynamical degrees

$\lambda_2 =$ top. deg of f

$\lambda_1 = \lim_{n\to\infty} (\deg f^n)^{1/n}$

(Assume $\lambda_1 > 1$ for the most part).

Thm A: 2 possibilities:

a) $\deg f^n \sim \lambda_1^n$

b) $\deg f^n \sim n \cdot \lambda_1^n$. In this case, f conjugate to a skew product $(x,y) \mapsto (P(x), Q(x,y))$ (+ cond'n on P, Q)

Thm B: λ_1 quadratic integer: $\lambda_1^2 = A\lambda_1 + B$ $A, B \in \mathbb{Z}$

Thm C: $\deg f^n$ satisfies recursion formula:

$$\deg f^n = \sum_{j=1}^{N} a_j \deg f^{n-j} \qquad a_j \in \mathbb{Z}$$

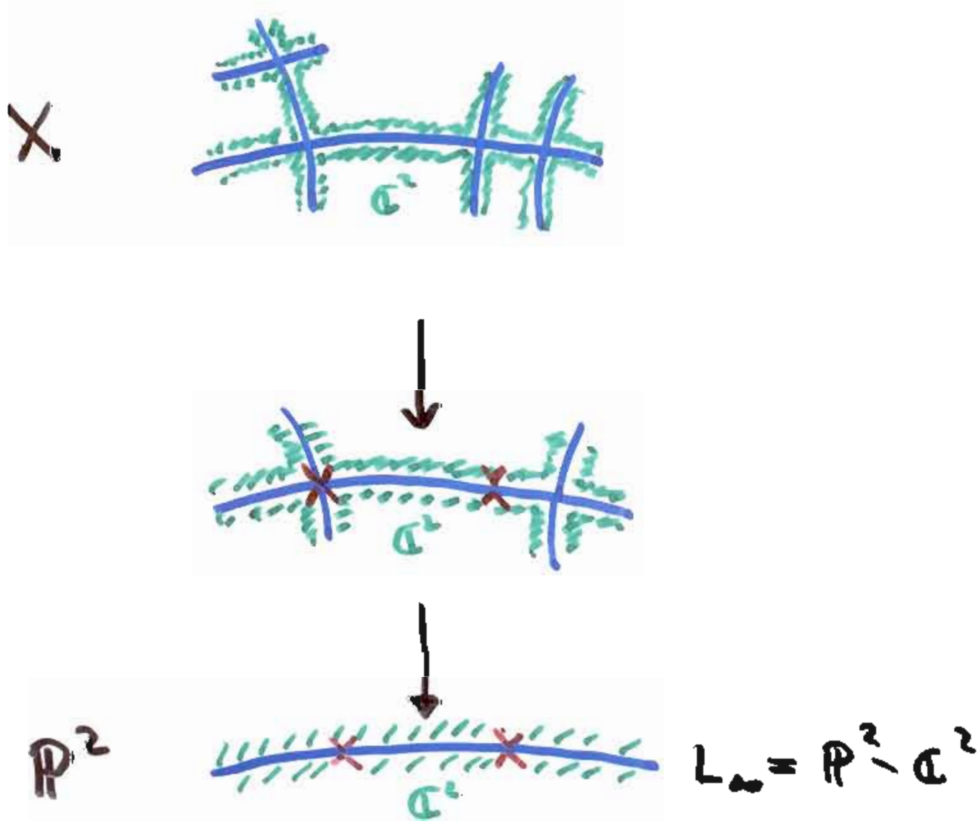
Can approach these result by studying dynamics at ∞.

Compactifications

Fix embedding $\mathbb{C}^2 \hookrightarrow \mathbb{P}^2$
($\Rightarrow$ can talk about affine fcns on $\mathbb{C}^2$)

Def: An admissible compactification $X \supseteq \mathbb{C}^2$ is obtained from $\mathbb{P}^2$ by finitely many blowups at ∞

Def: Primes of X = irr. comp's of $X \setminus \mathbb{C}^2$
(e.g. $L_\infty = \mathbb{P}^2 \setminus \mathbb{C}^2$)



Dynamical compactifications

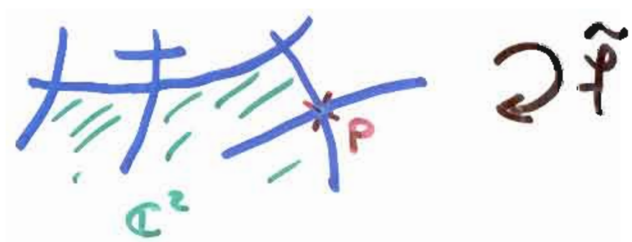
Thm D: Assume $\lambda_2 < \lambda_1$ "low top deg."

Then there exists:

- An admissible comp'n $X \supset \mathbb{C}^2$
- A point $p \in X \setminus \mathbb{C}^2$
- An integer $N \geq 1$

s.t:

- the lift $\tilde{f} : X \dashrightarrow X$ is holo at p, $\tilde{f}(p) = p$
- the germ $\tilde{f} : (X,p) \to (X,p)$ is superattn and admits "simple" normal form
- $\tilde{f}^N E = \{p\}$ for all primes E of X (with at most one exception)



Can deduce Thms B, C (when $\lambda_2 < \lambda_1$) from this.

Can also deduce that Green fcn

$$G := \lim_{n \to \infty} \lambda_1^{-n} \log^+ \| f^n \|$$

is well behaved

Focus on proving Thm A, D

The Riemann-Zariski approach

$\mathfrak{X} := \varprojlim X$ X admissible comp'n

Riemann-Zariski space at ∞ (don't touch $\mathbb{C}^2$)

$W(\mathfrak{X}) := \varprojlim H^{1,1}_{\mathbb{R}}(X)$ Weil classes

$C(\mathfrak{X}) := \varinjlim H^{1,1}_{\mathbb{R}}(X)$ Cartier ——

$L^2(\mathfrak{X})$

$\mathrm{Nef}(\mathfrak{X})$

f_*, f^* act on W, C, L^2, Nef

[Use f holo on $\mathbb{C}^2$ and $f(\mathbb{C}^2) \subset \mathbb{C}^2$]

Thm: Assume $\lambda_1^2 > \lambda_2$. Then:

a) $\exists\ \theta_*, \theta^* \in \mathrm{Nef}(\mathfrak{X})$, unique up to scaling,
s.t. $f_* \theta_* = \lambda_1 \theta_*$, $f^* \theta^* = \lambda_1 \theta^*$

b) $\alpha \in L^2(\mathfrak{X}) \Rightarrow$

$$\lambda_1^{-n} f_*^n \alpha \longrightarrow \frac{(\theta^* \cdot \alpha)}{(\theta^* \cdot \theta_*)} \theta_* \quad \text{(fast)}$$

(+ same for f^*)

Will interpret θ_*

Valuations

$R = \mathbb{C}[x,y]$ coordinate ring of $\mathbb{C}^2$

$\hat{\mathcal{V}}_0 = \{$valuations $\nu : R \to (-\infty, +\infty]$ centered at ∞ : $\nu(P) < 0$ for some polynomial $P\}$.

$\mathcal{V}_0 = \{$normalized valuations in $\hat{\mathcal{V}}_0\}$

Normalization: $\min\{\nu(L) \mid L \text{ affine}\} = -1$
$\iff \nu(L) = -1$ for generic L.

Prime $E \subset X \rightsquigarrow$ divisorial valuations

$\mathrm{ord}_E \in \hat{\mathcal{V}}_0$

$\nu_E \in \mathcal{V}_0$

$\nu_E = b_E^{-1} \mathrm{ord}_E$

$b_E = -\mathrm{ord}_E(L)$

Ex: $E = L_\infty = \mathbb{P}^2 \setminus \mathbb{C}^2 \Rightarrow$
$\nu_E = \mathrm{ord}_E = -\deg$ $\quad (b_E = 1)$

Dual graphs

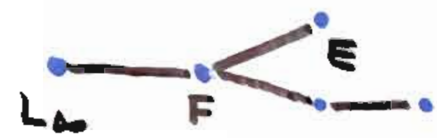
X admissible comp'n of $\mathbb{C}^2$

Γ_X dual graph:

- partial ordering (root $= L_\infty$)
- metric

X

$\mathrm{dist}(E,F) = \dfrac{1}{b_E b_F}$



Γ_X

$X' \geq X$ (can get from X to X' by blowing up)

$\Rightarrow \Gamma_X \hookrightarrow \Gamma_{X'}$ order-preserving isometry

Also have:

- Embedding: $\Gamma_X \hookrightarrow \mathcal{V}_0$
- Retraction: $\mathcal{V}_0 \twoheadrightarrow \Gamma_X$

Thm: $\mathcal{V}_0 \cong \varprojlim \Gamma_X$

Cor: $\mathcal{V}_0$ is an $\mathbb{R}$-tree

Valuative dynamics

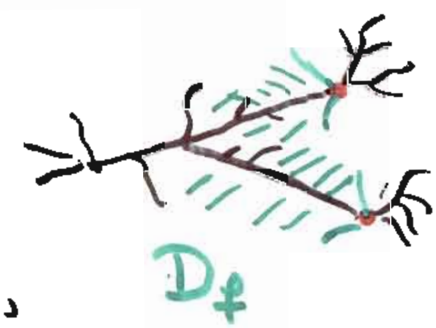
$f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ poly, dominant

Want to define: $\begin{cases} f_* : \hat{\mathcal{V}}_0 \to \hat{\mathcal{V}}_0 \\ f_\bullet : \mathcal{V}_0 \to \mathcal{V}_0 \end{cases}$

as in local case.

Problem: $f_* \nu$ may not be centered at ∞.

Get subtree $D_f \subseteq \mathcal{V}_0$ on which $f_\bullet$ is defined.



$f_\bullet : D_f \longrightarrow \mathcal{V}_0$

respects tree structure,
has dense image.

Problem: D_f depends on f.

Is $D_f \cap D_{f^2} \cap \ldots \cap D_{f^n} \cap \ldots \neq \emptyset$?

Answer: YES!

The subtree $\mathcal{V}_1$: I

The $\mathbb{R}$-tree $\mathcal{V}_0$ admits 2 natural parametrizations:

$$\alpha : \mathcal{V}_0 \longrightarrow [-\infty, 1] \qquad \text{"skewness"}$$
$$A : \mathcal{V}_0 \longrightarrow [-2, \infty] \qquad \text{"thinness"}$$

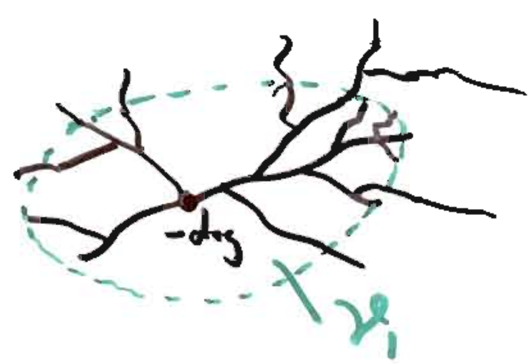
1) $\alpha(\nu) = 1 - \mathrm{dist}(\nu, \mathrm{root})$

2) Thinness A defined as "log-discrepancy"

$$A(\mathrm{ord}_E) = 1 + \mathrm{ord}_E(\omega) \qquad \omega = dx \wedge dy \text{ on } \mathbb{C}^2$$
$$A(\nu_E) = b_E^{-1} A(\mathrm{ord}_E)$$

<u>Def</u>: $\nu \in \mathcal{V}_1$ iff $\begin{cases} \alpha(\nu) \geq 0 \\ A(\nu) \leq 0 \end{cases}$

"ν close enough to the root of $\mathcal{V}_0$"



The subtree $\mathcal{V}_1$: II

Valuations in $\mathcal{V}_1$ have good properties

Thm: If $\alpha(\nu) > 0$ (e.g. ν in "interior" of $\mathcal{V}_1$) $\Rightarrow$

1) $\nu(P) < 0$ for every nonconst pol P
2) $\nu(P) \leq -\alpha(\nu) \cdot \deg P$

Thm: If $\alpha(\nu) = 0$, $A(\nu) < 0$ and ν is divisorial ($\Rightarrow \nu$ endpt in $\mathcal{V}_1$) then ν is a rational pencil valuation:

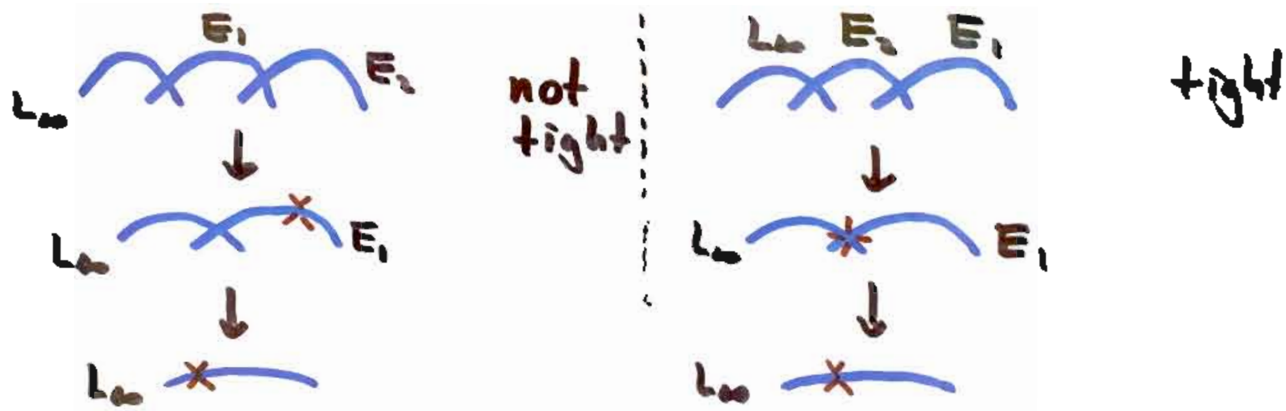
$$\nu(Q) = \text{const} \cdot \text{ord}_\infty (Q|_{P=\text{const}})$$

Tight compactifications

$X \supset \mathbb{C}^2$ admissible compactification

Def: X is **tight** if $\forall$ prime E of X, the valuation ν_E lies in $\mathcal{V}_1$

[Restriction on blowups from $\mathbb{P}^2$ to X].



Tight compactifications $X \supseteq \mathbb{C}^2$ have nice geometric properties:

- $\mathrm{Nef}(X)$ is a <u>simplicial</u> cone
- Every nef line bundle on X is generated by global sections

Dynamics on $\mathcal{V}_1$ I

$f: \mathbb{C}^2 \to \mathbb{C}^2$

Thm: $f_\bullet$ well defined on $\mathcal{V}_1$ ($D_f \supset \mathcal{V}_1$) and $f_\bullet \mathcal{V}_1 \subset \mathcal{V}_1$

Cor: $\exists$ eigenvaluation $\nu_* \in \mathcal{V}_1$:

$$f_\bullet \nu_* = \nu_* \qquad f_* \nu_* = \lambda \nu_*$$

Pf of Thm A [Behavior of $\deg f^n$].

(i) ν_* rat'l pencil valuation $\Rightarrow f \sim$ skew product

(ii) $\alpha(\nu_*) > 0 \Rightarrow \lambda_1^n \le \deg f^n \le D\lambda_1^n$ where $D = \alpha(\nu_*)^{-1}$.

[(iii) one more case ...]

Rem: $\lambda_2 < \lambda_1 \Rightarrow \nu_*$ not divisorial

Rem: $A(\nu_*) = 0 < \alpha(\nu_*) \Rightarrow$ f counterexample to JC !

Dynamics of $\mathcal{V}_1$ II

Thm: Assume $\lambda_1^2 > \lambda_2$. Then:

$f_\bullet^n \nu \longrightarrow \nu_*$ as $n \to \infty$

for all $\nu \in \mathcal{V}_1$ with at most one exception ν^*, for which $f_\bullet \nu^* = \nu^*$.

Pf: Associate Weil class $Z_\nu \in W(\mathfrak{X})$ to any valuation $\nu \in \mathcal{V}_0$.

$\nu \in \mathcal{V}_1 \Rightarrow Z_\nu$ nef

$f_* Z_\nu = Z_{f_* \nu}$

$\Rightarrow \ldots \Rightarrow$

$f_\bullet^n \nu \longrightarrow \nu_*$ unless

$(Z_\nu \cdot \theta^*) = 0 \Rightarrow Z_\nu = \theta^*$ ... $\square$

Construction of Z_ν when $\nu = \mathrm{ord}_E$

E prime of X. $Z_{\mathrm{ord}_E} = Z_E$ defined by:

$$(Z_E \cdot F) = \begin{cases} 1 & F = E \\ 0 & F \neq E. \end{cases}$$

[General ν by homog. + approx.]

Proof of Thm D

Thm D: Assume $\lambda_2 < \lambda_1$. Then $\exists X, p, N$ ...

X 

$\circlearrowleft \tilde{f}$

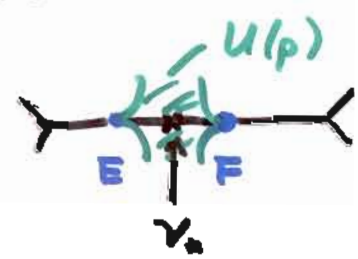
$\tilde{f} : (X,p) \circlearrowleft$ superattr. fixed pt germ

$\tilde{f}^N E = p \quad \forall E$.

"Pf": ν_* cannot be divisorial $(\lambda_2 < \lambda_1)$

- Assume ν_* irrational quasimonomial
- Successively blow up center of ν_* many times
- Get tight compactification X, p = center of ν_* on X

X 

$\nu_\bullet$ 

- ν_* locally attracting $\Rightarrow$ $f_\bullet U(p) \subset U(p)$ $\Rightarrow$ $\tilde{f}$ holo at p etc
- $f_\bullet^n \nu_E \longrightarrow \nu_* \quad \forall E \Rightarrow \tilde{f}^N E = \{p\}$.