Pluripotential theory in a non-archimedean setting

Mattias Jonsson (University of Michigan)

NUS, Jan 6, 2011

Plan

- Archimedean pluripotential theory.
- Non-archimedean pluripotential theory.
- Joint work with S. Boucksom and C. Favre.

Pluripotential theory on Kähler manifolds

- Pluripotential theory = study of plurisubharmonic (psh) fcns.
- (X, ω) compact Kähler manifold. Assume $\int_X \omega^n = 1$.
- Say $\varphi : X \to [-\infty, \infty[$ *ω-psh* if $dd^c\varphi + \omega \geq 0$.
- Example: $X = \mathbf{P}^n$, $\omega =$ Fubini-Study and

$$\varphi = \frac{1}{2m} \log \sum_{k=1}^{N} \frac{|f_k|^2}{\| \cdot \|^{2m}},$$

  where $f_1, \dots, f_N$ homo polys on $\mathbf{C}^{n+1}$ with $\bigcap_k f_k^{-1}(0) = \{0\}$.
- **Compactness property**: the function

$$\mathrm{PSH}(X, \omega) \ni \varphi \mapsto \sup_X \varphi \in \mathbf{R}$$

  is continuous and proper in the L^1-topology.

Geometric Monge-Ampère equation

- Monge-Ampère equation: given probability measure μ on X, find ω-psh function φ on X such that

$$\mathrm{MA}(\varphi) := (\omega + dd^c\varphi)^n = \mu.$$

- **Thm** by Calabi/Yau: uniqueness/existence when $\mu > 0$ *smooth*.
- **Thm** by Guedj-Zeriahi/Dinew: existence/uniqueness when μ *non-pluripolar* measure (no mass on $\{u = -\infty\}$, u psh).
- Example: $X = \mathbf{P}^1$, $\omega =$ Fubini-Study.

$$\varphi(z) = \int \log \frac{|z - w|}{\sqrt{1 + |w|^2}} d\mu(w).$$

- Non-linear problem in dim > 1!

Capacity and extremal functions

- To study MA eqn, useful to develop capacity theory.
- For any subset $E \subseteq X$, define *extremal function*

$$u_E = \sup\{\varphi \in \mathrm{PSH}(X,\omega) \mid \varphi \leq 0, \varphi|_E \leq -1\}.$$

- For $E \subseteq X$ Borel, define the *capacity*

$$\mathrm{Cap}(E) = \sup\{\int_E \mathrm{MA}(\varphi) \mid \varphi \in \mathrm{PSH}(X,\omega), -1 \leq \varphi \leq 0\}.$$

- **Thm**: for $K \subseteq X$ compact, $\mathrm{Cap}(K) = \int_K \mathrm{MA}(u_K^*)$ and $\mathrm{supp}\,\mathrm{MA}(u_K^*) \subseteq K$.
- **Thm**: for any $E \subseteq X$, TFAE:
  - E is pluripolar, i.e. $E \subseteq \{u = -\infty\}$, u psh;
  - $\forall \varepsilon\ \exists G \supseteq E$ open with $\mathrm{Cap}(G) < \epsilon$;
  - $u_E^* \equiv 0$;
  - E is "negligable".

Key tools

- **Regularization**: can approximate any $\varphi \in \mathrm{PSH}(X,\omega)$ by a *decreasing sequence* $\varphi_j \searrow \varphi$, with $\varphi_j \in \mathrm{PSH}(X,\omega) \cap C^\infty(X)$.
- **Comparison principle**:

$$\int_{\varphi<\psi} \mathrm{MA}(\psi) \leq \int_{\varphi<\psi} \mathrm{MA}(\varphi)$$

  for $\varphi, \psi \in \mathrm{PSH}(X,\omega)$ sufficiently regular.
- **"Balayage"**: Can solve $\mathrm{MA} = 0$ with Dirichlet boundary condition. Implies $\operatorname{supp} \mathrm{MA}(u_K^*) \subset K$.
- **Countability**: X has countable basis for topology. Used to prove e.g. Choquet's Lemma.

Beyond Archimedes

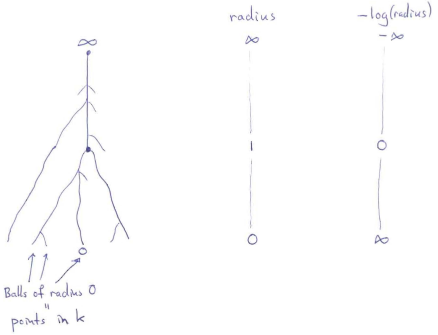
- Try to extend previous analysis to non-archimedean setting.
- Natural for problems of arithmetic nature.
- Relevant work by several people: Baker-Rumely, Boucksom-Favre-J, Chambert-Loir, Chinburg-Rumely, Favre-Rivera-Letelier, Gubler, Kontsevich-Tschinkel, Liu, Thuillier, Yuan-Zhang, ...
- **Definition**. Archimedean=not non-Archimedean.
- **Factoid**: Archimedes died 2222 years ago.

Non-archimedean analysis

- Assume k field equipped with *non-archimedean* norm: $|a+b| \leq \max\{|a|, |b|\}$. Also assume k *complete*.
- Example: $\mathbf{C}((t))$ (Laurent series) $0 < |t| < 1$, $|c| = 1$.
- Example: any field k with *trivial* norm: $|a| = 1$, $a \neq 0$.
- Can try to develop analytic geometry as over $\mathbf{C}$.
- Problem 1: k totally disconnected.
- Problem 2: in general, k not locally compact.
- Various ways to deal with this.
- Approach: replace k (or k^n) by suitable *Berkovich space* (space of valuations).

The Berkovich affine space

- **Def**: $\mathbf{A}^n_{\text{Berk}}$ is the set of multiplicative seminorms on $k[z_1, \dots, z_n]$ extending the given norm on k.
- When $k = \mathbf{C}$, $\mathbf{A}^n_{\text{Berk}} = \mathbf{C}^n$ by Gelfand-Mazur.
- Points of $\mathbf{A}^1_{\text{Berk}}$ can be viewed as *balls* in k. Thus $\mathbf{A}^1_{\text{Berk}}$ admits *tree structure*.



- For $n > 1$, $\mathbf{A}^n_{\text{Berk}}$ harder to visualize!

The valuation space $\mathcal{V}$

- Try to adapt previous results to Berkovich spaces. Look at special case: $k = \mathbf{C}$ *with trivial valuation*.
- Define subsets $\mathcal{V} \subset \widehat{\mathcal{V}} \subset \mathbf{A}^n_{\mathrm{Berk}}$ by

$$\widehat{\mathcal{V}} := \{\text{seminorms on } \mathbf{C}[z_1, \dots, z_n] \mid \max_i |z_i| < 1\}.$$

$$\mathcal{V} := \{\text{seminorms on } \mathbf{C}[z_1, \dots, z_n] \mid \max_i |z_i| = e^{-1}\}.$$

- Useful for studying singularities at a point in $\mathbf{C}^n$.
- $\widehat{\mathcal{V}} \simeq$ cone over $\mathcal{V}$.
- $\mathcal{V}$ contractible compact Hausdorff space. No countable basis.
- $\mathcal{V} \simeq$ limit of simplicial complexes. $\mathbf{R}$-tree when $n = 2$.
- Define class $\mathrm{PSH}(\mathcal{V})$ and "do" pluripotential theory on it.
- Functions in $\mathrm{PSH}(\mathcal{V})$ extend to $\widehat{\mathcal{V}}$ by homogeneity.

Plurisubharmonic functions

- View elements of $\mathcal{V}$ as *semivaluations*

$$\nu : \mathbf{C}[z_1, \dots, z_n] \to [0, +\infty]$$

using $|\cdot| = e^{-\nu}$.

- Define psh fcn as *decreasing limit* of fcns of the form

$$c \max_{1 \le j \le N} \log |f_j|,$$

where $c > 0$, $f_j \in \mathbf{C}[z_1, \dots, z_n]$ and $\bigcap_j \{f_j = 0\} = \{0\}$.

- Here $\log |f|(\nu) := -\nu(f)$.
- Can define analogy of L^1-topology (but not metrizable).
- **Thm**: the function

$$\mathrm{PSH}(\mathcal{V}) \ni \varphi \mapsto \sup_{\mathcal{V}} \varphi \in \mathbf{R}_-$$

is continuous and proper.

- Proof uses multiplier ideals to construct $\varphi_j \searrow \varphi$.

Monge-Ampère operator on $\mathcal{V}$

- First define $\mathrm{MA}(\varphi)$ for $\varphi = \max_j \log |f_j|$.
- Suffices to define $\int_{\mathcal{V}} \psi \, \mathrm{MA}(\varphi)$ for $\psi = \max_i \log |g_i|$.
- Do this as a local intersection number. Analytically:

$$\int_{\mathcal{V}} \psi \, \mathrm{MA}(\varphi) = -((dd^c\varphi)^{n-1} \wedge dd^c\psi)\{0\},$$

  where in the RHS we view φ and ψ as functions on $\mathbf{C}^n$!
- For general $\varphi \in \mathrm{PSH}(\mathcal{V})$, define $\mathrm{MA}(\varphi) = \lim_j \mathrm{MA}(\varphi_j)$, where $\varphi_j \searrow \varphi$.
- **Thm**. Unique solution to $\mathrm{MA}(\varphi) = \mu$ for any non-pluripolar measure μ.
- Existence proof uses variational approach (Alexandrov; Berman-Boucksom-Guedj-Zeriahi).
- Need to develop capacity theory along the way.

Capacity and extremal functions

- For any subset $E \subseteq \mathcal{V}$, define *extremal function*

$$u_E = \sup\{\varphi \in \mathrm{PSH}(\mathcal{V}) \mid \varphi|_E \leq -1\}.$$

- For $E \subseteq \mathcal{V}$ Borel, define the *capacity*

$$\mathrm{Cap}(E) = \sup\{\int_E \mathrm{MA}(\varphi) \mid \varphi \in \mathrm{PSH}(\mathcal{V}), \varphi \geq -1\}.$$

- **Thm**: for $K \subseteq \mathcal{V}$ compact, $\mathrm{Cap}(K) = \int_K \mathrm{MA}(u_K^*)$ and $\operatorname{supp} \mathrm{MA}(u_K^*) \subseteq K$.
- **Thm**: for any $E \subseteq \mathcal{V}$, TFAE:
  - E is pluripolar, i.e. $E \subseteq \{u = -\infty\}$, u psh;
  - $\forall \varepsilon \; \exists G \supseteq E$ open with $\mathrm{Cap}(G) < \epsilon$;
  - $u_E^* \equiv 0$;
  - E is "negligable".
- Results exactly parallel to the archimedean situation!

Key tools I

- Many proofs are the same as in the archimedean situation, but a few basic ingredients are different.
- **Countability**: $\mathcal{V}$ has no countable basis for topology but $\mathbf{C}[z_1, \dots, z_n]$ is noetherian. Used to prove Choquet's Lemma.
- **Regularization**: can approximate any $\varphi \in \mathrm{PSH}(\mathcal{V})$ by a *decreasing sequence* $\varphi_j \searrow \varphi$, with $\varphi_j \in \mathrm{PSH}(\mathcal{V})$ of nice form.
- This was built into the definition of $\mathrm{PSH}(\mathcal{V})$...but then properness of $\varphi \to \sup \varphi$ was hard to establish!
- In $\mathbf{C}^n$, can regularize using convolution. On $\mathcal{V}$, we use multiplier ideals.

Key tools II

- **Comparison principle**:

$$\int_{\varphi<\psi} \mathrm{MA}(\psi) \le \int_{\varphi<\psi} \mathrm{MA}(\varphi)$$

  for $\varphi, \psi \in \mathrm{PSH}(\mathcal{V})$ not too singular.
- As in $\mathbf{C}^n$ prove this by reduction to the case φ, ψ "smooth".
- In $\mathbf{C}^n$, the smooth case uses Stokes' theorem. On $\mathcal{V}$, uses positivity of certain intersection nos.
- **"Balayage"**: Don't know how to solve Dirichlet problem locally. Replacement is the *orthogonality property* of asymptotic Zariski decompositions in the sense of Boucksom-Demailly-Păun-Peternell.

Future work

- Would like to do pluripotential theory on other Berkovich spaces.
- Example: work on all of $\mathbf{A}^n_{\mathrm{Berk}}$, not just localized at one point.
- Example: analytification of projective varieties over $\mathbf{C}((t))$.