Dynamics of monomial mappings

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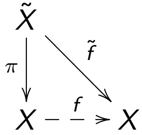
Plan

- Stability: definition and motivation.
- Old and new results.
- Toric varieties: translation.
- Ideas of proof.

Pullback by meromorphic maps

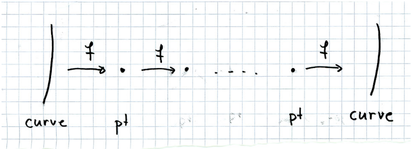
- Consider $f : X \dashrightarrow X$ meromorphic, X complex manifold.
- In dynamics, look for invariant objects, e.g. T positive closed $(1,1)$-current with $f^*T = \lambda T$, $\lambda > 0$.
- What is meant by f^* when f meromorphic?
- Resolve singularities of f:



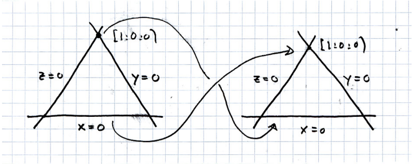
- Set $f^*T := \pi_* \tilde{f}^* T$.
- Also set $f^*\alpha := \pi_* \tilde{f}^* \alpha$, $\alpha \in H^2(X; \mathbf{R})$.
- Problem: won't have $f^{n*} = f^{*n}$ in general!

Stability

- **Def**: f is 1-*stable* if $f^{n*} = f^{*n}$ on $H^2(X; \mathbf{R})$.
- **Prop** [Fornæss-Sibony]: f is not 1-stable $\equiv f^n(H) \subset \mathrm{Indet}(f)$ for some $n \geq 1$ and some hypersurface H.
- Picture in dimension two!



- **Ex**: $f : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$ given by $f[x : y : z] = [yz : zx : xy]$. Then $f^* = 2\,\mathrm{id}$ but $f^2 = 2\,\mathrm{id}$ so $f^{2*} \neq f^{*2}$.



Stabilization: definition and know results

- **Question**: given $f : X \dashrightarrow X$, $\exists$? modification $\pi : X' \to X$ such that the induced map $f' : X' \dashrightarrow X'$ is 1-stable?
- **Thm** [Diller-Favre]: *YES*, when X is a complex surface and $f : X \dashrightarrow X$ is bimeromorphic.
- **Thm** [Favre]: Complete characterization for $f : X \dashrightarrow X$ monomial and $X = \overline{(\mathbf{C}^*)^2}$ a toric surface. Write

$$f(z_1, z_2) = (z_1^{a_{11}} z_2^{a_{21}}, z_1^{a_{12}} z_2^{a_{22}}),$$

  and μ_1, μ_2 eigenvalues of $A = (a_{ij}) \in M_2(\mathbf{Z})$. Then TFAE:
  (a) f not stabilizable, even on toric surface X' with quotient sings;
  (b) $\overline{\mu_1} = \mu_2$ and μ_1/μ_2 is not a root of unity.
- **Thm** [Favre-J]: *YES* for $f : \mathbf{C}^2 \to \mathbf{C}^2$ polynomial (inducing $f : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$). Here X' has quotient singularities but can often be chosen smooth.

Stabilization: new results

- **Thm** [J-Wulcan]: Suppose $X = \overline{(\mathbf{C}^*)^m}$ is a toric variety and $f(z) = z^A$ a monomial map, where A has real eigenvalues $\mu_1 > \mu_2 > \cdots > \mu_m$. Then there exists toric variety X' with quotient singularities and modification $X' \to X$ such that the induced map $f' : X' \dashrightarrow X'$ is 1-stable.
- Can pick X' smooth after replacing f by iterate f^N.
- Examples in dimension two where X' cannot be smooth unless we replace f by f^2.
- Conjecturally, *dynamical degrees* $\lambda_1, \dots, \lambda_m$ are given by

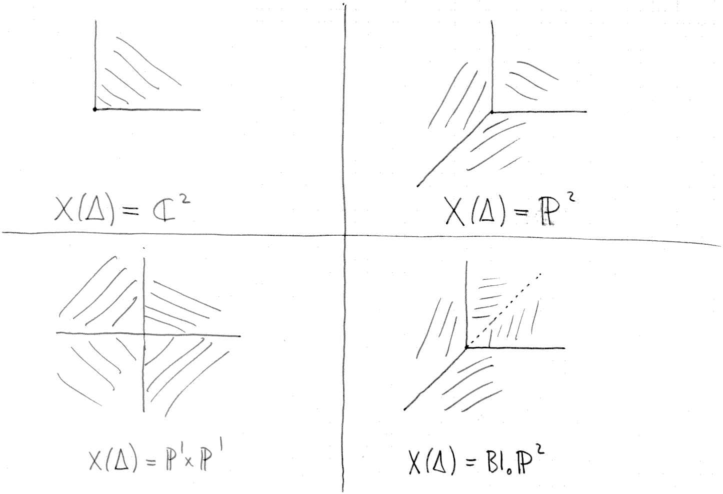
  $$\lambda_j = |\mu_1 \dots \mu_j|.$$

  The condition $\mu_1 > \mu_2 > \cdots > \mu_m > 0$ is true (for f^2) iff $j \mapsto \log \lambda_j$ is strictly concave.
- Similar results obtained independently by Jan-Li Lin.

Toric varieties

- Toric varieties are algebraic varieties that can be understood/analyzed combinatorially.
- Combinatorial data:
  - *lattice* $N \simeq \mathbf{Z}^m$; write $N_{\mathbf{R}} := M \otimes_{\mathbf{Z}} \mathbf{R} \simeq \mathbf{R}^m$;
  - *fan* Δ; this is finite collection of (rational, strongly convex) cones $\sigma = \sum_{i=1}^k \mathbf{R}_+ v_i \subset N_{\mathbf{R}}$, $v_i \in N$, satisfying:
    - a face of a cone in Δ is in Δ;
    - distinct cones in Δ have disjoint relative interiors;
- To each $\sigma \in \Delta$ can associate affine variety U_σ.
- Glue together to get toric variety $X(\Delta)$.
- Torus $U_{\{0\}} \simeq (\mathbf{C}^*)^m$ is dense in $X(\Delta)$ and acts on it.
- Translations:
  - $X(\Delta)$ compact iff Δ complete, i.e. cones in Δ cover all of $N_{\mathbf{R}}$;
  - $X(\Delta)$ has quotient sings if Δ simplicial;
  - $X(\Delta)$ smooth iff Δ regular;

Examples of toric varieties



Stability criterion

- More translations:
  - Refinement $\Delta \to \Delta'$ (subdivide cones) $\implies$ modification $X(\Delta') \to X(\Delta)$;
  - $\mathbf{Z}$-linear map $\phi : N \to N \implies$ monomial map $f : X(\Delta) \dashrightarrow X(\Delta)$;
- f is holomorphic iff m-dimensional cones in Δ are mapped into m-dimensional cones in Δ.
- **Prop**: Assume Δ simplicial and complete, Suppose there exists $\mathcal{S} \subset \Delta$ such that:
  - $\phi(\sigma) \subset \sigma$ for all $\sigma \in \mathcal{S}$;
  - if $\rho \in \Delta$ is a 1-dimensional cone and $n \geq 1$, then $\phi^n(\rho) \in \Delta$ or $\phi^n(\rho) \subset \sigma$ for some $\sigma \in \mathcal{S}$.

  Then $f : X(\Delta) \dashrightarrow X(\Delta)$ is 1-stable.
- Proof involves $H^2(X(\Delta); \mathbf{Z}) \simeq \mathrm{Pic}(X(\Delta))$ and representation of Cartier divisors as piecewise linear functions on $N_{\mathbf{R}}$.
- From now on, suffices to work with combinatorial data!

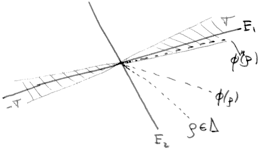
Stability in dimension two

- Suppose $N \cong \mathbf{Z}^2$.
- $\phi : N \to N$ $\mathbf{Z}$-linear with eigenvalues μ_1, μ_2.
- Recall criterion for stability: if there exists $\mathcal{S} \subset \Delta$ such that:
  - $\phi(\sigma) \subset \sigma$ for all $\sigma \in \mathcal{S}$;
  - if $\rho \in \Delta$ is a 1-dimensional cone and $n \geq 1$, then $\phi^n(\rho) \in \Delta$ or $\phi^n(\rho) \subset \sigma$ for some $\sigma \in \mathcal{S}$.

  Then can stabilize f on model with quotient singularities.
- *Cannot* achieve this if $\mu_2 = \overline{\mu_1}$ and μ_2/μ_1 not a root of unity: (irrational rotation).
- *Can* achieve this if $|\mu_1| > |\mu_2| > 0$ (pictures on next slide).
- Can also achieve this in all other cases, so get Favre's result.

Stabilization in dimension two: pictures

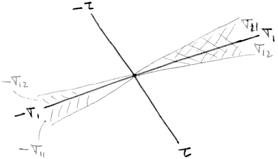
$\mu_1 > \mu_2 > 0$
$\mu_1, \mu_2 \in \mathbb{R} \setminus \mathbb{Q}$



$S = \{\sigma, -\sigma\}$

[Pick σ small, invariant,
incorporate σ,
incorporate images of rays
$\phi^k(\rho) \notin \pm\sigma \quad \rho \in \Delta$]

$\mu_1 > \mu_2 > 0$
$\mu_1, \mu_2 \in \mathbb{R} \setminus \mathbb{Q}$



$S = \{\sigma_1, -\sigma_1, \pm\sigma_{11}, \pm\sigma_{12}, \pm\tau\}$

Higher dimensions

- Difficulties in higher dimensions:
  - book-keeping problems: many possibilities (picture);
  - subdividing cones is more complicated;
  - toy problem: how to subdivide a simplex σ_0 containing another simplex σ_1 so that σ_1 is one of the new simplies?