DYNAMICS ON BERKOVICH SPACES IN LOW DIMENSIONS

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Abstract. These are expanded lecture notes for the summer school on Berkovich spaces that took place at the Institut de Mathématiques de Jussieu, Paris, during June 28–July 9, 2010. They serve to illustrate some techniques and results from the dynamics on low-dimensional Berkovich spaces and to exhibit the structure of these spaces.

Contents

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1. Introduction

The goal of these notes is twofold. First, I'd like to describe how Berkovich spaces enters naturally in certain instances of discrete dynamical systems. In particular, I will try to show how my own work with Charles Favre [FJ07, FJ11] on valuative dynamics relates to the dynamics of rational maps on the Berkovich projective line as initiated by Juan Rivera-Letelier in his thesis [Riv03a] and subsequently studied by him and others. In order to keep the exposition somewhat, I have chosen three sample problems (Theorems A, B and C below) for which I will present reasonably complete proofs.

The second objective is to show some of the simplest Berkovich spaces "in action". While not necessarily representative of the general situation, they have a structure that is very rich, yet can be described in detail. In particular, they are trees, or cones over trees.

For the purposes of this introduction, the dynamical problems that we shall be interested in all arise from polynomial mappings

$$f : \mathbf{A}^n \to \mathbf{A}^n,$$

where $\mathbf{A}^n$ denotes affine n-space over a *valued field*, that is, a field K complete with respect a norm $|\cdot|$. Studying the dynamics of f means, in rather vague terms, studying the asymptotic behavior of the *iterates* of f:

$$f^m = f \circ f \circ \cdots \circ f$$

(the composition is taken m times) as $m \to \infty$. For example, one may try to identify regular as opposed to chaotic behavior. One is also interested in invariant objects such as fixed points, invariant measures, etc.

When K is the field of complex numbers, polynomial mappings can exhibit very interesting dynamics both in one and higher dimensions. We shall discuss this a little further in §1.1 below. As references we point to [CG93, Mil06] for the one-dimensional case and [Sib99] for higher dimensions.

Here we shall instead focus on the case when the norm on K is *non-Archimedean* in the sense that the strong triangle inequality $|a+b| \le \max\{|a|, |b|\}$ holds. Interesting examples of such fields include the p-adic numbers $\mathbf{Q}_p$, the field of Laurent series $\mathbf{C}((t))$, or any field K equipped with the *trivial* norm.

One motivation for investigating the dynamics of polynomial mappings over non-Archimedean fields is simply to see to what extent the known results over the complex (or real) numbers continue to hold. However, non-Archimedean dynamics sometimes plays a role, even when the original dynamical system is defined over the complex numbers. We shall see some instances of this phenomenon in these notes; other examples are provided by the work of Kiwi [Kiw06], Baker and DeMarco [BdM09], and Ghioca, Tucker and Zieve [GTZ08].

Over the complex numbers, many of the most powerful tools for studying dynamics are either topological or analytical in nature: distortion estimates, potential theory, quasiconformal mappings etc. These methods do not directly carry over to the non-Archimedean setting since K is totally disconnected.

On the other hand, a polynomial mapping f automatically induces a selfmap

$$f : \mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}} \to \mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$$

of the corresponding *Berkovich space* $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$. By definition, $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}} = \mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}(K)$ is the set of multiplicative seminorms on the coordinate ring $R \simeq K[z_1, \dots, z_n]$ of $\mathbf{A}^n$ that extend the given norm on K. It carries a natural topology in which it it locally compact and arcwise connected. It also contains a copy of $\mathbf{A}^n$: a point $x \in \mathbf{A}^n$ is identified with the seminorm $\phi \mapsto |\phi(x)|$. The action of f on $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ is given as follows. A seminorm $|\cdot|$ is mapped by f to the seminorm whose value on a polynomial $\phi \in R$ is given by $|f^*\phi|$.

The idea is now to study the dynamics on $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$. At this level of generality, not very much seems to be known at the time of writing (although the time may be ripe to start looking at this). Instead, the most interesting results have appeared in situations when the structure of the space $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ is better understood, namely in sufficiently low dimensions.

We shall focus on two such situations:

(1) $f : \mathbf{A}^1 \to \mathbf{A}^1$ is a polynomial mapping of the affine line over a general valued field K;
(2) $f : \mathbf{A}^2 \to \mathbf{A}^2$ is a polynomial mapping of the affine plane over a field K equipped with the trivial norm.

In both cases we shall mainly treat the case when K is algebraically closed.

In (1), one makes essential use of the fact that the Berkovich affine line $\mathbf{A}^1_{\mathrm{Berk}}$ is a tree.[1] This tree structure was pointed out already by Berkovich in his original work [Ber90] and is described in great detail in the book [BR10] by Baker and Rumely. It has been exploited by several authors and a very nice picture of the global dynamics on this Berkovich space has taken shape. It is beyond the scope of these notes to give an account of all the results that are known. Instead, we shall focus on one specific problem: equidistribution of preimages of points. This problem, which will be discussed in further detail in §1.1, clearly shows the advantage of working on the Berkovich space as opposed to the "classical" affine line.

As for (2), the Berkovich affine plane $\mathbf{A}^2_{\mathrm{Berk}}$ is already quite a beast, but it is possible to get a handle on its structure. We shall be concerned not with the global dynamics of f, but the local dynamics either at a fixed point $0 = f(0) \in \mathbf{A}^2$, or at infinity. There are natural subspaces of $\mathbf{A}^2_{\mathrm{Berk}}$ consisting of seminorms that "live" at 0 or at infinity, respectively, in a sense that can be made precise. These two spaces are cones over a tree and hence reasonably tractable.

While it is of general interest to study the dynamics in (2) for a general field K, there are surprising applications to *complex* dynamics when using $K = \mathbf{C}$ equipped with the trivial norm. We shall discuss this in §1.2 and §1.3 below.

1.1. **Polynomial dynamics in one variable.** Our first situation is that of a polynomial mapping

$$f : \mathbf{A}^1 \to \mathbf{A}^1$$

[1] For a precise definition of what we mean by "tree", see §2.

of degree $d > 1$ over a complete valued field K, that we here shall furthermore assume to be algebraically closed and, for simplicity, of characteristic zero.

When K is equal to the (archimedean) field $\mathbf{C}$, there is a beautiful theory describing the polynomial dynamics. The foundation of this theory was built in the 1920's by Fatou and Julia, who realized that Montel's theorem could be used to divide the phase space $\mathbf{A}^1 = \mathbf{A}^1(\mathbf{C})$ into a region where the dynamics is tame (the Fatou set) and a region where it is chaotic (the Julia set). In the 1980's and beyond, the theory was very significantly advanced, in part because of computer technology allowing people to visualize Julia sets as fractal objects, but more importantly because of the introduction of new tools, in particular quasiconformal mappings. For further information on this we refer the reader to the books [CG93, Mil06].

In between, however, a remarkable result by Hans Brolin [Bro65] appeared in the 1960's. His result seems to have gone largely unnoticed at the time, but has been of great importance for more recent developments, especially in higher dimensions. Brolin used potential theoretic methods to study the asymptotic distribution of preimages of points. To state his result, let us introduce some terminology. Given a polynomial mapping f as above, one can consider the *filled Julia set* of f, consisting of all points $x \in \mathbf{A}^1$ whose orbit is bounded. This is a compact set. Let ρ_f be *harmonic measure* on the filled Julia set, in the sense of potential theory. Now, given a point $x \in \mathbf{A}^1$ we can look at the distribution of preimages of x under f^n. There are d^n preimages of x, counted with multiplicity, and we write $f^{n*}\delta_x = \sum_{f^n y = x} \delta_y$, where the sum is taken over these preimages. Thus $d^{-n} f^{n*}\delta_x$ is a probability measure on $\mathbf{A}^1$. Brolin's theorem now states

Theorem. *For all points $x \in \mathbf{A}^1$, with at most one exception, we have*

$$\lim_{n\to\infty} d^{-n} f^{n*}\delta_x \to \rho_f.$$

Furthermore, a point $x \in \mathbf{A}^1$ is exceptional iff there exists a global coordinate z on $\mathbf{A}^1$ vanishing at x such that f is given by the polynomial $z \mapsto z^d$. In this case, $d^{-n} f^{n*}\delta_x = \delta_x$ for all n.

A version of this theorem for selfmaps of $\mathbf{P}^1$ was later proved independently by Lyubich [Lyu83] and by Freire-Lopez-Mañé [FLM83]. There have also been far-reaching generalizations of Brolin's theorem to higher-dimensional complex dynamics. However, we shall stick to the one-dimensional polynomial case in this introduction.

It is now natural to ask what happens when we replace $\mathbf{C}$ by a *non-Archimedean* valued field K. We still assume that K is algebraically closed and, as above, that it is of characteristic zero. An important example is $K = \mathbf{C}_p$, the completed algebraic closure of the p-adic numbers $\mathbf{Q}_p$. However, while most of the early work focused on $\mathbf{C}_p$, and certain deep results that are true for this field do not hold for general K, we shall not assume $K = \mathbf{C}_p$ in what follows.

Early on, through work of Silverman, Benedetto, Hsia and others [Ben00, Ben01a, Ben02b, Hsi00, MS95, Riv03a] it became clear that there were some significant differences to the archimedean case. For example, with the most direct translations of the definitions from the complex numbers, it may well happen that the Julia set of a polynomial over a non-Archimedean field K is empty. This is in clear distinction with the complex case. Moreover, the topological structure of K is vastly different from that of $\mathbf{C}$. Indeed, K is totally disconnected and usually not even locally compact. The lack of compactness is inherited by the space of probability measures on K: there is a priori no reason for the sequence of probability measures on K to admit a convergent subsequence. This makes it unlikely that a naïve generalization of Brolin's theorem should hold.

Juan Rivera-Letelier was the first one to realize that Berkovich spaces could be effectively used to study the dynamics of rational functions over non-Archimedean fields. As we have seen above, $\mathbf{A}^1$ embeds naturally into $\mathbf{A}^1_{\mathrm{Berk}}$ and the map f extends to a map

$$f : \mathbf{A}^1_{\mathrm{Berk}} \to \mathbf{A}^1_{\mathrm{Berk}} .$$

Now $\mathbf{A}^1_{\mathrm{Berk}}$ has good topological properties. It is locally compact[2] and contractible. This is true for the Berkovich affine space $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ of any dimension. However, the structure of the Berkovich affine $\mathbf{A}^1_{\mathrm{Berk}}$ can be understood in much greater detail, and this is quite helpful when analyzing the dynamics. Specifically, $\mathbf{A}^1_{\mathrm{Berk}}$ has a structure of a tree and the induced map $f : \mathbf{A}^1_{\mathrm{Berk}} \to \mathbf{A}^1_{\mathrm{Berk}}$ preserves the tree structure, in a suitable sense.

Introducing the Berkovich space $\mathbf{A}^1_{\mathrm{Berk}}$ is critical for even formulating many of the known results in non-Archimedean dynamics. This in particular applies to the non-Archimedean version of Brolin's theorem:

Theorem A. *Let $f : \mathbf{A}^1 \to \mathbf{A}^1$ be a polynomial map of degree $d > 1$ over an algebraically closed field of characteristic zero. Then there exists a probability measure $\rho = \rho_f$ on $\mathbf{A}^1_{\mathrm{Berk}}$ such that for all points $x \in \mathbf{A}^1$, with at most one exception, we have*

$$\lim_{n\to\infty} d^{-n} f^{n*}\delta_x \to \rho.$$

Furthermore, a point $x \in \mathbf{A}^1$ is exceptional iff there exists a global coordinate z on $\mathbf{A}^1$ vanishing at x such that f is given by the polynomial $z \mapsto z^d$. In this case, $d^{-n} f^{n*}\delta_x = \delta_x$ for all n.

In fact, we could have started with any point $x \in \mathbf{A}^1_{\mathrm{Berk}}$ assuming we are careful with the definition of $f^{n*}\delta_x$. Notice that when $x \in \mathbf{A}^1$, the probability measures $d^{-n} f^{n*}\delta_x$ are all supported on $\mathbf{A}^1 \subseteq \mathbf{A}^1_{\mathrm{Berk}}$, but the limit measure may very well give no mass to $\mathbf{A}^1$. It turns out that if we define the Julia set J_f of f as the support of the measure ρ_f, then J_f shares many properties of the Julia set of complex polynomials. This explains why we may not see a Julia set when studying the dynamics on $\mathbf{A}^1$ itself.

Theorem A is due to Favre and Rivera-Letelier [FR10]. The proof is parallel to Brolin's original proof in that it uses *potential theory*. Namely, one can define a Laplace operator Δ on $\mathbf{A}^1_{\mathrm{Berk}}$ and to every probability measure ρ on $\mathbf{A}^1_{\mathrm{Berk}}$ associate a subharmonic function $\varphi = \varphi_\rho$ such that $\Delta\varphi = \rho - \rho_0$, where ρ_0 is a fixed reference measure (typically a Dirac mass at a point of $\mathbf{A}^1_{\mathrm{Berk}} \setminus \mathbf{A}^1$). The function φ is unique up to an additive constant. One can then translate convergence of the measures in Theorem A to the more tractable statement about convergence of potentials.

[2] Its one-point compactification is the Berkovich projective line $\mathbf{P}^1_{\mathrm{Berk}} = \mathbf{A}^1_{\mathrm{Berk}} \cup \{\infty\}$.

The Laplace operator itself can be very concretely interpreted in terms of the tree structure on $\mathbf{A}^1_{\mathrm{Berk}}$. All of this will be explained in §§2–5.

The story does not end with Theorem A. For instance, Favre and Rivera-Letelier analyze the ergodic properties of f with respect to the measure ρ_f. Okuyama [Oku11b] has given a quantitative strengthening of the equidistribution result in Theorem A. The measure ρ_f also describes the distribution of periodic points, see [FR10, Théorème B] as well as [Oku11a].

As already mentioned, there is also a very interesting Fatou-Julia theory. We shall discuss this a little further in §4 but the discussion will be brief due to limited space. The reader will find many more details in the book [BR10]. We also recommend the recent survey by Benedetto [Ben10].

1.2. **Local plane polynomial dynamics.** The second and third situations that we will study both deal with polynomial mappings

$$f : \mathbf{A}^2 \to \mathbf{A}^2$$

over a valued field K. In fact, they originally arose from considerations in *complex* dynamics and give examples where non-Archimedean methods can be used to study Archimedean problems.

Thus we start out by assuming that $K = \mathbf{C}$. Polynomial mappings of $\mathbf{C}^2$ can have quite varied and very interesting dynamics; see the survey by Sibony [Sib99] for some of this. Here we will primarily consider local dynamics, so we first consider a *fixed point* $0 = f(0) \in \mathbf{A}^2$. For a detailed general discussion of local dynamics in this setting we refer to Abate's survey [Aba10].

The behavior of f at the fixed point is largely governed by the tangent map $df(0)$ and in particular on the eigenvalues λ_1, λ_2 of the latter. For example, if $|\lambda_1|, |\lambda_2| < 1$, then we have an *attracting* fixed point: there exists a small neighborhood $U \ni 0$ such that $f(\overline{U}) \subseteq U$ and $f^n \to 0$ on U. Further, when there are no *resonances* between the eigenvalues λ_1, λ_2, the dynamics can in fact be *linearized*: there exists a local biholomorphism $\phi : (\mathbf{A}^2, 0) \to (\mathbf{A}^2, 0)$ such that $f \circ \phi = \phi \circ \Lambda$, where $\Lambda(z_1, z_2) = (\lambda_1 z_1, \lambda_2 z_2)$. This in particular gives very precise information on the rate at which typical orbits converge to the origin: for a "typical" point $x \approx 0$ we have $\|f^n(x)\| \sim \max_{i=1,2} |\lambda_i|^n \|x\|$ as $n \to \infty$.

On the other hand, in the *superattracting* case, when $\lambda_1 = \lambda_2 = 0$, the action of f on the tangent space $T_0\mathbf{C}^2$ does not provide much information about the dynamics. Let us still try to understand at what rate orbits tend to the fixed point. To this end, let

$$f = f_c + f_{c+1} + \cdots + f_d$$

be the expansion of f in homogeneous components: $f_j(\lambda z) = \lambda^j f_j(z)$ and where $f_c \not\equiv 0$. Thus $c = c(f) \geq 1$ and the number $c(f)$ in fact does not depend on the choice of coordinates. Note that for a typical point $x \approx 0$ we will have

$$\|f(x)\| \sim \|x\|^{c(f)}.$$

Therefore, one expects that the speed at which the orbit of a typical point x tends to the origin is governed by the growth of $c(f^n)$ as $n \to \infty$. This can in fact be made precise, see [FJ07], but here we shall only study the sequence $(c(f^n))_n$.

Note that this sequence is supermultiplicative: $c(f^{n+m}) \geq c(f^n)c(f^m)$. This easily implies that the limit

$$c_\infty(f) := \lim_{n\to\infty} c(f^n)^{1/n}$$

exists. Clearly $c_\infty(f^n) = c_\infty(f)^n$ for $n \geq 1$.

Example 1.1. If $f(z_1, z_2) = (z_2, z_1 z_2)$, then $c(f^n)$ is the $(n+2)$th Fibonacci number and $c_\infty(f) = \frac{1}{2}(\sqrt{5}+1)$ is the golden mean.

Our aim is to give a proof of the following result, originally proved in [FJ07].

Theorem B. *The number $c_\infty = c_\infty(f)$ is a quadratic integer: there exists $a, b \in \mathbf{Z}$ such that $c_\infty^2 = ac_\infty + b$. Moreover, there exists a constant $\delta > 0$ such that*

$$\delta c_\infty^n \leq c(f^n) \leq c_\infty^n$$

for all $n \geq 1$.

Note that the right-hand inequality $c(f^n) \leq c_\infty^n$ is an immediate consequence of supermultiplicativity. It is the left-hand inequality that is nontrivial.

To prove Theorem B we study the induced dynamics

$$f : \mathbf{A}^2_{\mathrm{Berk}} \to \mathbf{A}^2_{\mathrm{Berk}}$$

of f on the Berkovich affine plane $\mathbf{A}^2_{\mathrm{Berk}}$. Now, if we consider $K = \mathbf{C}$ with its standard Archimedean norm, then it is a consequence of the Gelfand-Mazur theorem that $\mathbf{A}^2_{\mathrm{Berk}} \simeq \mathbf{A}^2$, so this may not seem like a particularly fruitful approach. If we instead, however, consider $K = \mathbf{C}$ equipped with the *trivial* norm, then the associated Berkovich affine plane $\mathbf{A}^2_{\mathrm{Berk}}$ is a totally different creature and the induced dynamics is very interesting.

By definition, the elements of $\mathbf{A}^2_{\mathrm{Berk}}$ are multiplicative seminorms on the coordinate ring of $\mathbf{A}^2$, that is, the polynomial ring $R \simeq K[z_1, z_2]$ in two variables over K. It turns out to be convenient to instead view these elements "additively" as *semivaluations* $v : R \to \mathbf{R} \cup \{+\infty\}$ such that $v|_{K^*} \equiv 0$. The corresponding seminorm is $|\cdot| = e^{-v}$.

Since we are interested in the local dynamics of f near a (closed) fixed point $0 \in \mathbf{A}^2$, we shall study the dynamics of f on a corresponding subspace of $\mathbf{A}^2_{\mathrm{Berk}}$, namely the set $\hat{\mathcal{V}}_0$ of semivaluations v such that $v(\phi) > 0$ whenever ϕ vanishes at 0. In valuative terminology, these are the semivaluations $v \in \mathbf{A}^2_{\mathrm{Berk}} \setminus \mathbf{A}^2$ whose *center* on $\mathbf{A}^2$ is the point 0. It is clear that $f(\hat{\mathcal{V}}_0) \subseteq \hat{\mathcal{V}}_0$.

Note that $\hat{\mathcal{V}}_0$ has the structure of a cone: if $v \in \hat{\mathcal{V}}_0$, then $tv \in \hat{\mathcal{V}}_0$ for $0 < t \leq \infty$. The apex of this cone is the image of the point $0 \in \mathbf{A}^2$ under the embedding $\mathbf{A}^2 \hookrightarrow \mathbf{A}^2_{\mathrm{Berk}}$. The base of the cone can be identified with the subset $\mathcal{V}_0 \subseteq \hat{\mathcal{V}}_0$ consisting of semivaluations that are normalized by the condition $v(\mathfrak{m}_0) = \min_{p\in\mathfrak{m}_0} v(\phi) = +1$, where $\mathfrak{m}_0 \subseteq R$ denotes the maximal ideal of 0. This space $\mathcal{V}_0$ is compact and has a structure of an $\mathbf{R}$-tree. We call it the *valuative tree* at the point 0. Its structure is investigated in detail in [FJ04] and will be examined in §7.[3]

[3]In [FJ04, FJ07], the valuative tree is denoted by $\mathcal{V}$. We write $\mathcal{V}_0$ here in order to emphasize the choice of point $0 \in \mathbf{A}^2$.

Now $\mathcal{V}_0$ is in general not invariant by f. Instead f induces a selfmap

$$f_\bullet : \mathcal{V}_0 \to \mathcal{V}_0$$

and a "multiplier" function $c(f, \cdot) : \mathcal{V}_0 \to \mathbf{R}_+$ such that

$$f(v) = c(f, v) f_\bullet v$$

for $v \in \mathcal{V}_0$. The number $c(f)$ above is exactly equal to $c(f, \mathrm{ord}_0)$, where $\mathrm{ord}_0 \in \mathcal{V}_0$ denotes the order of vanishing at $0 \in \mathbf{A}^2$. Moreover, we have

$$c(f^n) = c(f^n, \mathrm{ord}_0) = \prod_{i=0}^{n-1} c(f, v_i), \quad \text{where } v_i = f_\bullet^i \mathrm{ord}_0;$$

this equation will allow us to understand the behavior of the sequence $c(f^n)$ through the dynamics of $f_\bullet$ on $\mathcal{V}_0$.

The proof of Theorem B given in these notes is simpler than the one in [FJ07]. Here is the main idea. Suppose that there exists a valuation $v \in \mathcal{V}_0$ such that $f_\bullet v = v$, so that $f(v) = cv$, where $c = c(f, v) > 0$. Then $c(f^n, v) = c^n$ for $n \geq 1$. Suppose that v satisfies an Izumi-type bound:

$$v(\phi) \leq C \,\mathrm{ord}_0(\phi) \quad \text{for all polynomials } \phi, \tag{1.1}$$

where $C > 0$ is a constant independent of ϕ. This is true for many, but not all semivaluations $v \in \mathcal{V}_0$. The reverse inequality $v \geq \mathrm{ord}_0$ holds for all $v \in \mathcal{V}_0$ by construction. Then we have

$$C^{-1} c^n = C^{-1} c(f^n, v) \leq c(f^n) \leq c(f^n, v) \leq c^n.$$

This shows that $c_\infty(f) = c$ and that the bounds in Theorem B hold with $\delta = C^{-1}$. To see that c_∞ is a quadratic integer, we look at the value group Γ_v of v. The equality $f(v) = cv$ implies that $c\Gamma_v \subseteq \Gamma_v$. If we are lucky, then $\Gamma \simeq \mathbf{Z}^d$, where $d \in \{1, 2\}$, which implies that $c_\infty = c$ is an algebraic integer of degree one or two.

The two desired properties of v hold when the eigenvaluation v is *quasimonomial* valuation. In general, there may not exist a quasimonomial eigenvaluation, so the argument is in fact a little more involved. We refer to §8 for more details.

1.3. **Plane polynomial dynamics at infinity.** Again consider a polynomial mapping

$$f : \mathbf{A}^2 \to \mathbf{A}^2$$

over the field $K = \mathbf{C}$ of complex numbers. In the previous subsection, we discussed the dynamics of f at a (superattracting) fixed point in $\mathbf{A}^2$. Now we shall consider the dynamics at infinity and, specifically, the rate at which orbits tend to infinity. Fix an embedding $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$. It is then reasonable to argue that the rate at which "typical" orbits tend to infinity is governed by the *degree growth sequence* $(\deg f^n)_{n\geq 1}$. Precise assertions to this end can be found in [FJ07, FJ11]. Here we shall content ourselves with the study on the degree growth sequence.

In contrast to the local case, this sequence is *sub*multiplicative: $\deg f^{n+m} \leq \deg f^n \deg f^m$, but again the limit

$$d_\infty(f) := \lim_{n\to\infty} (\deg f^n)^{1/n}$$

exists. Apart from some inequalities being reversed, the situation is very similar to the local case, so one may hope for a direct analogue of Theorem B above. However, the skew product example $f(z_1, z_2) = (z_1^2, z_1 z_2^2)$ shows that we may have $\deg f^n \sim n d_\infty^n$. What does hold true in general is

Theorem C. *The number $d_\infty = d_\infty(f)$ is a quadratic integer: there exist $a, b \in \mathbf{Z}$ such that $d_\infty^2 = a d_\infty + b$. Moreover, we are in exactly one of the following two cases:*

(a) *there exists $C > 0$ such that $d_\infty^n \le \deg f^n \le C d_\infty^n$ for all n;*

(b) $\deg f^n \sim n d_\infty^n$ *as $n \to \infty$.*

Moreover, case (b) occurs iff f, after conjugation by a suitable polynomial automorphism of $\mathbf{C}^2$, is a skew product of the form

$$f(z_1, z_2) = (\phi(z_1), \psi(z_1) z_2^{d_\infty} + O_{z_1}(z_2^{d_\infty - 1})),$$

where $\deg \phi = d_\infty$ *and* $\deg \psi > 0$.

As in the local case, we approach this theorem by considering the induced dynamics

$$f : \mathbf{A}^2_{\mathrm{Berk}} \to \mathbf{A}^2_{\mathrm{Berk}},$$

where we consider $K = \mathbf{C}$ equipped with the trivial norm. Since we are interested in the dynamics of f at infinity, we restrict our attention to the space $\hat{\mathcal{V}}_\infty$ consisting of semivaluations $v : R \to \mathbf{R} \cup \{+\infty\}$ whose center is at infinity, that is, for which $v(\phi) < 0$ for some polynomial ϕ. This space has the structure of a pointed[4] cone. To understand its base, note that our choice of embedding $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$ determines the space $\mathcal{L}$ of affine functions on $\mathbf{A}^2$ (the polynomials of degree at most one). Define

$$\mathcal{V}_\infty := \{v \in \mathbf{A}^2_{\mathrm{Berk}} \mid \min_{L \in \mathcal{L}} v(L) = -1\}.$$

We call $\mathcal{V}_\infty$ the *valuative tree at infinity*.[5] This subspace at first glance looks very similar to the valuative tree $\mathcal{V}_0$ at a point but there are some important differences. Notably, for a semivaluation $v \in \mathcal{V}_0$ we have $v(\phi) \ge 0$ for all polynomials ϕ. In contrast, while a semivaluations in $\mathcal{V}_\infty$ must take some negative values, it can take positive values on certain polynomials.

Assuming for simplicity that f is *proper*, we obtain a dynamical system $f : \hat{\mathcal{V}}_\infty \to \hat{\mathcal{V}}_\infty$, which we can split into an induced map $f_\bullet : \mathcal{V}_\infty \to \mathcal{V}_\infty$ and a multiplier $d(f, \cdot) : \mathcal{V}_\infty \to \mathbf{R}_+$ such that $f(v) = d(f, v) f_\bullet v$.

The basic idea in the proof of Theorem C is again to look for an eigenvaluation, that is, a semivaluation $v \in \mathcal{V}_\infty$ such that $f_\bullet v = v$. However, even if we can find a "nice" (say, quasimonomial) eigenvaluation, the proof in the local case does not automatically go through. The reason is that Izumi's inequality (1.1) may fail.

The remedy to this problem is to use an invariant subtree $\mathcal{V}'_\infty \subseteq \mathcal{V}_\infty$ where the Izumi bound almost always holds. In fact, the valuations $v \in \mathcal{V}'_\infty$ for which Izumi's inequality does not hold are of a very special form, and the case when we end up with a fixed point of that type corresponds exactly to the degree growth $\deg f^n \sim n d_\infty^n$.

[4]The apex of the cone does not define an element in $\mathbf{A}^2_{\mathrm{Berk}}$.

[5]In [FJ07, FJ11], the valuative tree at infinity is denoted by $\mathcal{V}_0$, but the notation $\mathcal{V}_\infty$ seems more natural.

In these notes, $\mathcal{V}'_\infty$ is called the *tight tree at infinity.* I expect it to have applications beyond the situation here.

1.4. **Philosophy and scope.** When writing these notes I was faced with the question of how much material to present, and at what level of detail to present it. Since I decided to have Theorems A, B and C as goals for the presentation, I felt it was necessary to provide enough background for the reader to go through the proofs, without too many black boxes. As it turns out, there is quite a lot of background to cover, so these notes ended up rather expansive!

All the main results that I present here can be found in the literature, However, we draw on many different sources that use different notation and terminology. In order to make the presentation coherent, I have tried to make it self-contained. Many complete proofs are included, others are sketched in reasonable detail.

While the point of these notes is to illustrate the usefulness of Berkovich spaces, we only occasionally draw on the general theory as presented in [Ber90, Ber93]. As a general rule, Berkovich spaces obtained by analytification of an algebraic variety are much simpler than the ones constructed by gluing affinoid spaces. Only at a couple of places in §3 and §4 do we rely on (somewhat) nontrivial facts from the general theory. On the other hand, these facts, mainly involving the local rings at a point on the Berkovich space, are very useful. We try to exploit them systematically. It is likely that in order to treat higher-dimensional questions, one has to avoid simple topological arguments based on the tree structure and instead use algebraic arguments involving the structure sheaf of the space in question.

At the same time, the tree structure of the spaces in question is of crucial importance. They can be viewed as the analogue of the conformal structure on Riemann surfaces. For this reason I have included a self-contained presentation of potential theory and dynamics on trees, at least to the extent that is needed for the later applications in these notes.

I have made an attempt to provide a unified point of view of dynamics on low-dimensional Berkovich spaces. One can of course try to go further and study dynamics on higher-dimensional Berkovich spaces over a field (with either trivial or nontrivial valuation). After all, there has been significant progress in higher dimensional complex dynamics over the last few years. For example, it is reasonable to hope for a version of the Briend-Duval equidistribution theorem [BD01].

Many interesting topics are not touched upon at all in these notes. For instance, we say very little about the dynamics on structure of the Fatou set of a rational map and we likewise do not study the ramification locus. Important contributions to these and other issues have been made by Matt Baker, Robert Benedetto, Laura DeMarco, Xander Faber, Charles Favre, Liang-Chung Hsia, Jan Kiwi, Yûsuke Okuyama, Clayton Petsche, Juan Rivera-Letelier, Robert Rumely Lucien Szpiro, Michael Tepper, Eugenio Trucco and others. For the relevant results we refer to the original papers [BdM09, Bak06, Bak09, BH05, BR06, Ben98, Ben00, Ben01a, Ben01b, Ben02a, Ben05a, Ben05b, Ben06, Fab09, Fab11a, Fab11b, Fab11c, FKT11, FR04, FR06, FR10, Hsi00, Kiw06, Kiw11, Oku11a, Oku11b, PST09, Riv03a, Riv03b, Riv04, Riv05, Tru09]. Alternatively, many of these results can be found in the book [BR10] by Baker and Rumely or the lecture notes [Ben10] by Benedetto.

Finally, we say nothing about arithmetic aspects such as the equidistribution of points of small height [BR10, CL06, FR06, Yua08, Gub08, Fab09, YZ09a, YZ09b]. For an introduction to arithmetic dynamics, see [Sil07] and [Sil10].

1.5. **Comparison to other surveys.** Beyond research articles such as the ones mentioned above, there are several useful sources that contain a systematic treatment of material related to the topics discussed in these notes.

First, there is a significant overlap between these notes and the material in the *Thèse d'Habilitation* [Fav05] of Charles Favre. The latter thesis, which is strongly recommended reading, explains the usage of tree structures in dynamics and complex analysis. It treats Theorems A-C as well as some of my joint work with him on the singularities of plurisubharmonic functions [FJ05a, FJ05b]. However, the presentation here has a different flavor and contains more details.

The book by [BR10] by Baker and Rumely treats potential theory and dynamics on the Berkovich projective line in great detail. The main results in §§3–5 are contained in this book, but the presentation in these notes is at times a little different. We also treat the case when the ground field has positive characteristic and discuss the case when it is not algebraically closed and/or trivially valued. On the other hand, [BR10] contains a great deal of material not covered here. For instance, it contains results on the structure of the Fatou and Julia sets of rational maps and it gives a much more systematic treatment of potential theory on the Berkovich line.

The lecture notes [Ben10] by Benedetto are also recommended reading. Just as [BR10], they treat the dynamics on the Fatou and Julia sets in detail. It also contains results in "classical" non-Archimedean analysis and dynamics, not involving Berkovich spaces.

The Ph.D. thesis by Amaury Thuillier [Thu05] gives a general treatment of potential theory on Berkovich curves. It is written in a less elementary way than the treatment in, say, [BR10] but on the other hand is more amenable to generalizations to higher dimensions. Potential theory on curves is also treated in [Bak08].

The valuative tree in §7 is discussed in detail in the monograph [FJ04]. However, the exposition here is self-contained and leads more directly to the dynamical applications that we have in mind.

As already mentioned, we do not discuss arithmetic dynamics in these notes. For information on this fascinating subject we again refer to the book and lecture notes by Silverman [Sil07, Sil10].

1.6. **Structure.** The material is divided into three parts. In the first part, §2, we discuss trees since the spaces on which we do dynamics are either trees or cones over trees. The second part, §§3–5, is devoted to the Berkovich affine and projective lines and dynamics on them. Finally, in §§6–10 we study polynomial dynamics on the Berkovich affine plane over a trivially valued field.

We now describe the contents of each chapter in more detail. Each chapter ends with a section called "Notes and further references" containing further comments.

In §2 we gather some general definitions and facts about trees. Since we shall work on several spaces with a tree structure, I felt it made sense to collect the material in a separate section. See also [Fav05]. First we define what we mean by a tree, with or without a metric. Then we define a Laplace operator on a general metric tree, viewing the latter as a pro-finite tree. In our presentation, the Laplace operator is defined on the class of quasisubharmonic functions and takes values in the space of signed measures with total mass zero and whose negative part is a finite atomic measure. Finally we study maps between trees. It turns out that simply assuming that such a map is finite, open and surjective gives quite strong properties. We also prove a fixed point theorem for selfmaps of trees.

The structure of the Berkovich affine and projective lines is outlined in §3. This material is described in much more detail in [BR10]. One small way in which our presentation stands out is that we try to avoid coordinates as far as possible. We also point out some features of the local rings that turn out to be useful for analyzing the mapping properties and we make some comments about the case when the ground field is not algebraically closed and/or trivially valued.

In §4 we start considering rational maps. Since we work in arbitrary characteristic, we include a brief discussion of separable and purely inseparable maps. Then we describe how polynomial and rational maps extend to maps on the Berkovich affine and projective line, respectively. This is of course only a very special case of the analytification functor in the general theory of Berkovich spaces, but it is useful to see in detail how to do this. Again our approach differs slightly from the ones in the literature that I am aware of, in that it is coordinate free. Having extended a rational map to the Berkovich projective line, we look at the important notion of the local degree at a point.[6] We adopt an algebraic definition of the local degree and show that it can be interpreted as a local expansion factor in the hyperbolic metric. While this important result is well known, we give an algebraic proof that I believe is new. We also show that the local degree is the same as the multiplicity defined by Baker and Rumely, using the Laplacian (as was already known.) See [Fab11a, Fab11b] for more on the local degree and the ramification locus, defined as the subset where the local degree is at least two. Finally, we discuss the case when the ground field is not algebraically closed and/or is trivially valued.

We arrive at the dynamics on the Berkovich projective line in §5. Here we do not really try to survey the known results. While we do discuss fixed points and the Fatou and Julia sets, the exposition is very brief and the reader is encouraged to consult the book [BR10] by Baker and Rumely or the notes [Ben10] by Benedetto for much more information. Instead we focus on Theorem A in the introduction, the equidistribution theorem by Favre and Rivera-Letelier. We give a complete proof which differs in the details from the one in [FR10]. We also give some consequences of the equidistribution theorem. For example, we prove Rivera-Letelier's dichotomy that the Julia set is either a single point or else a perfect set. Finally, we discuss the case when the ground field is not algebraically closed and/or is trivially valued.

At this point, our attention turns to the Berkovich affine plane over a trivially valued field. Here it seems more natural to change from the multiplicative terminology of seminorms to the additive notion of semivaluations. We start in §6 by introducing the home and the center of a valuation. This allows us to stratify the Berkovich affine space. This stratification is very explicit in dimension one, and possible (but

[6] In [BR10], the local degree is called *multiplicity*.

nontrivial) to visualize in dimension two. We also introduce the important notion of a quasimonomial valuation and discuss the Izumi-Tougeron inequality.

In §7 we come to the valuative tree at a closed point 0. It is the same object as in the monograph [FJ04] but here it is defined as a subset of the Berkovich affine plane. We give a brief, but self-contained description of its main properties with a presentation that is influenced by my joint work with Boucksom and Favre [BFJ08b, BFJ11a, BFJ11b] in hiugher dimensions. As before, our treatment is coordinate-free. A key result is that the valuative tree at 0 is homeomorphic to the inverse limit of the dual graphs over all birational morphisms above 0. Each dual graph has a natural metric, so the valuative tree is a pro-finite metric tree, and hence a metric tree in the sense of §2. In some sense, the cone over the valuative tree is an even more natural object. We define a Laplace operator on the valuative tree that takes this fact into account. The subharmonic functions turn out to be closely related to ideals in the ring of polynomials that are primary to the maximal ideal at 0. In general, the geometry of blowups of the point 0 can be well understood and we exploit this systematically.

Theorem B is proved in §8. We give a proof that is slightly different and shorter than the original one in [FJ07]. In particular, we have a significantly simpler argument for the fact that the number c_∞ is a quadratic integer. The new argument makes more systematic use of the value groups of valuations.

Next we move from a closed point in $\mathbf{A}^2$ to infinity. The valuative tree at infinity was first defined in [FJ07] and in §9 we review its main properties. Just as in the local case, the presentation is supposed to be self-contained and also more geometric than in [FJ07]. There is a dictionary between the situation at a point and at infinity. For example, a birational morphism above the closed point $0 \in \mathbf{A}^2$ corresponds to a compactification of $\mathbf{A}^2$ and indeed, the valuative tree at infinity is homeomorphic to the inverse limit of the dual graphs of all (admissible) compactifications. Unfortunately, the dictionary is not perfect, and there are many subtleties when working at infinity. For example, a polynomial in two variables tautologically defines a function on both the valuative tree at a point and at infinity. At a point, this function is always negative but at infinity, it takes on both positive and negative values. Alternatively, the subtelties can be said to stem from the fact that the geometry of compactifications of $\mathbf{A}^2$ can be much more complicated than that of blowups of a closed point.

To remedy some shortcomings of the valuative tree at infinity, we introduce a subtree, the tight tree at infinity. It is an inverse limit of dual graphs over a certain class of tight compactifications of $\mathbf{A}^2$. These have much better properties than general compactifications and should have applications to other problems. In particular, the nef cone of a tight compactification is always simplicial, whereas the nef cone in general can be quite complicated.

Finally, in §10 we come to polynomial dynamics at infinity, in particular the proof of Theorem C. We follow the strategy of the proof of Theorem B closely, but we make sure to only use tight compactifications. This causes some additional complications, but we do provide a self-contained proof, that is simpler than the one in [FJ07].

1.7. **Novelties.** While most of the material here is known, certain proofs and ways of presenting the results are new.

The definitions of a general tree in §2.1 and metric tree in §2.2 are new, although equivalent to the ones in [FJ04]. The class of quasisubharmonic functions on a general tree also seems new, as are the results in §2.5.6 on their singularities. The results on tree maps in §2.6 are new in this setting: they can be found in e.g. [BR10] for rational maps on the Berkovich projective line.

Our description of the Berkovich affine and projective lines is new, but only in the way that we insist on defining things in a coordinate free way whenever possible. The same applies to the extension of a polynomial or rational map from $\mathbf{A}^1$ or $\mathbf{P}^1$ to $\mathbf{A}^1_{\mathrm{Berk}}$ or $\mathbf{P}^1_{\mathrm{Berk}}$, respectively.

While Theorem 4.7, expressing the local degree as a dilatation factor in the hyperbolic metric, is due to Rivera-Letelier, the proof here is directly based on the definition of the local degree and seems to be new. The remarks in §4.11 on the non-algebraic case also seem to be new.

The structure of the Berkovich affine plane over a trivially valued field, described in §6.7 was no doubt known to experts but not described in the literature. In particular, the valuative tree at a closed point and at infinity were never explicitly identified as subsets of the Berkovich affine plane.

Our exposition of the valuative tree differs from the treatment in the book [FJ04] and instead draws on the analysis of the higher dimensional situation in [BFJ08b].

The proof of Theorem B in §8 is new and somewhat simpler than the one in [FJ07]. In particular, the fact that c_∞ is a quadratic integer is proved using value groups, whereas in [FJ07] this was done via rigidification. The same applies to Theorem C in §10.

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2. Tree structures

We shall do dynamics on certain low-dimensional Berkovich spaces, or subsets thereof. In all cases, the space/subset has the structure of a tree. Here we digress to discuss exactly what we mean by this. We also present a general version of potential theory on trees. The definitions that follow are slightly different from, but equivalent to the ones in [FJ04, BR10, Fav05], to which we refer for details. The idea is that any two points in a tree should be joined by a unique interval. This interval should look like a real line interval but may or may not be equipped with a distance function.

2.1. **Trees.** We start by defining a general notion of a tree. All our trees will be modeled on the real line (as opposed to a general ordered group Λ).[7] In order to avoid technicalities, we shall also only consider trees that are complete in the sense that they contain all their endpoints.

Definition 2.1. An *interval structure* on a set I is a partial order $\leq$ on I under which I becomes isomorphic (as a partially ordered set) to the real interval $[0, 1]$ or to the trivial real interval $[0, 0] = \{0\}$.

Let I be a set with an interval structure. A *subinterval* of I is a subset $J \subseteq I$ that becomes a subinterval of $[0, 1]$ or $[0, 0]$ under such an isomorphism. The *opposite* interval structure on I is obtained by reversing the partial ordering.

Definition 2.2. A *tree* is a set X together with the following data. For each $x, y \in X$, there exists a subset $[x, y] \subseteq X$ containing x and y and equipped with an interval structure. Furthermore, we have:

- (T1) $[x, x] = \{x\}$;
- (T2) if $x \neq y$, then $[x, y]$ and $[y, x]$ are equal as subsets of X but equipped with opposite interval structures; they have x and y as minimal elements, respectively;
- (T3) if $z \in [x, y]$ then $[x, z]$ and $[z, y]$ are subintervals of $[x, y]$ such that $[x, y] = [x, z] \cup [z, y]$ and $[x, z] \cap [z, y] = \{z\}$;
- (T4) for any $x, y, z \in X$ then there exists a unique element $x \wedge_z y \in [x, y]$ such that $[z, x] \cap [y, x] = [x \wedge_z y, x]$ and $[z, y] \cap [x, y] = [x \wedge_z y, y]$;
- (T5) if $x \in X$ and $(y_\alpha)_{\alpha \in A}$ is a net in X such that the segments $[x, y_\alpha]$ increase with α, then there exists $y \in X$ such that $\bigcup_\alpha [x, y_\alpha[= [x, y[$.

In (T5) we have used the convention $[x, y[:= [x, y] \setminus \{y\}$. Recall that a *net* is a sequence indexed by a directed (possibly uncountable) set. The subsets $[x, y]$ above will be called *intervals* or *segments*.

2.1.1. *Topology.* A tree as above carries a natural *weak topology*. Given a point $x \in X$, define two points $y, z \in X \setminus \{x\}$ to be equivalent if $]x, y] \cap]x, z] \neq \emptyset$. An equivalence class is called a *tangent direction* at x and the set of $y \in X$ representing a tangent direction $\vec{v}$ is denoted $U(\vec{v})$. The weak topology is generated by all such sets $U(\vec{v})$. Clearly X is arcwise connected and the connected components of $X \setminus \{x\}$

[7] Our definition of "tree" is not the same as the one used in set theory [Jec03] but we trust that no confusion will occur. The terminology "**R**-tree" would have been natural, but has already been reserved [GH90] for slightly different objects.

are exactly the sets $U(\vec{v})$ as $\vec{v}$ ranges over tangent directions at x. A tree is in fact uniquely arc connected in the sense that if $x \neq y$ and $\gamma : [0,1] \to X$ is an injective continuous map with $\gamma(0) = x$, $\gamma(1) = y$, then the image of γ equals $[x, y]$. Since the sets $U(\vec{v})$ are connected, any point in X admits a basis of connected open neighborhoods. We shall see shortly that X is compact in the weak topology.

If $\gamma = [x, y]$ is a nontrivial interval, then the *annulus* $A(\gamma) = A(x, y)$ is defined by $A(x, y) := U(\vec{v}_x) \cap U(\vec{v}_y)$, where $\vec{v}_x$ (resp., $\vec{v}_y$) is the tangent direction at x containing y (resp., at y containing x).

An *end* of X is a point admitting a unique tangent direction. A *branch point* is a point having at least three tangent directions.

2.1.2. *Subtrees.* A *subtree* of a tree X is a subset $Y \subseteq X$ such that the intersection $[x, y] \cap Y$ is either empty or a closed subinterval of $[x, y]$ for any $x, y \in X$. In particular, if $x, y \in Y$, then $[x, y] \subseteq Y$ and this interval is then equipped with the same interval structure as in X. It is easy to see that conditions (T1)–(T5) are satisfied so that Y is a tree. The intersection of any collection of subtrees of X is a subtree (if nonempty). The *convex hull* of any subset $Z \subseteq X$ is the intersection of all subtrees containing Z.

A subtree Y is a closed subset of X and the inclusion $Y \hookrightarrow X$ is an embedding. We can define a *retraction* $r : X \to Y$ as follows: for $x \in X$ and $y \in Y$ the intersection $[x, y] \cap Y$ is an interval of the form $[r(x), y]$; one checks that $r(x)$ does not depend on the choice of y. The map r is continuous and restricts to the identity on Y. A subtree of X is *finite* if it is the convex hull of a finite set.

Let $(Y_\alpha)_{\alpha \in A}$ be an increasing net of finite subtrees of X, indexed by a directed set A (i.e. $Y_\alpha \subseteq Y_\beta$ when $\alpha \leq \beta$). Assume that the net is *rich* in the sense that for any two distinct points $x_1, x_2 \in X$ there exists $\alpha \in A$ such that the retraction $r_\alpha : X \to Y_\alpha$ satisfies $r_\alpha(x_1) \neq r_\alpha(x_2)$. For example, A could be the set of *all* finite subtrees, partially ordered by inclusion. The trees (Y_α) form an inverse system via the retraction maps $r_{\alpha\beta} : Y_\beta \to Y_\alpha$ for $a \leq \beta$ defined by $r_{\alpha\beta} = r_\alpha|_{Y_\beta}$, and we can form the inverse limit $\varprojlim Y_\alpha$, consisting of points $(y_\alpha)_{\alpha \in A}$ in the product space $\prod_\alpha Y_\alpha$ such that $r_{\alpha\beta}(y_\beta) = y_\alpha$ for all $\alpha \leq \beta$. This inverse limit is a compact Hausdorff space. Since X retracts to each Y_α we get a continuous map

$$r : X \to \varprojlim Y_\alpha,$$

which is injective by the assumption that A is rich. That r is surjective is a consequence of condition (T5). Let us show that the inverse of r is also continuous. This will show that r is a homeomorphism, so that X is compact. (Of course, if we knew that X was compact, the continuity of r^{-1} would be immediate.)

Fix a point $x \in X$ and a tangent direction $\vec{v}$ at x. It suffices to show that $r(U(\vec{v}))$ is open in $\varprojlim Y_\alpha$. Pick a sequence $(x_n)_{n \geq 1}$ in $U(\vec{v})$ such that $[x_{n+1}, x] \subseteq [x_n, x]$ and $\bigcap_n [x_n, x[= \emptyset$. By richness there exists $\alpha_n \in A$ such that $r_{\alpha_n}(x_n) \neq r_{\alpha_n}(x)$. Let $\vec{v}_n$ be the tangent direction in X at $r_{\alpha_n}(x)$ represented by $r_{\alpha_n}(x_n)$. Then $r(U(\vec{v}_n))$ is open in $\varprojlim Y_\alpha$ and hence so is $r(U(\vec{v})) = \bigcup_n r(U(\vec{v}_n))$.

Remark 2.3. One may form the inverse limit of any inverse system of finite trees (not necessarily subtrees of a given tree). However, such an inverse limit may contain a "compactified long line" and hence not be a tree!

2.2. **Metric trees.** Let I be a set with an interval structure. A *generalized metric* on I is a function $d : I \times I \to [0, +\infty]$ satisfying:

(GM1) $d(x, y) = d(y, x)$ for all x, y, and $d(x, y) = 0$ iff $x = y$;
(GM2) $d(x, y) = d(x, z) + d(z, y)$ whenever $x \le z \le y$
(GM3) $d(x, y) < \infty$ if neither x nor y is an endpoint of I.
(GM4) if $0 < d(x, y) < \infty$, then for every $\varepsilon > 0$ there exists $z \in I$ such that $x \le z \le y$ and $0 < d(x, z) < \varepsilon$.

A *metric tree* is a tree X together with a choice of generalized metric on each interval $[x, y]$ in X such that whenever $[z, w] \subseteq [x, y]$, the inclusion $[z, w] \hookrightarrow [x, y]$ is an isometry in the obvious sense.

It is an interesting question whether or not every tree is *metrizable* in the sense that it can be equipped with a generalized metric. See Remark 2.6 below.

2.2.1. *Hyperbolic space.* Let X be a metric tree containing more than one point and let $x_0 \in X$ be a point that is not an end. Define *hyperbolic space* $\mathbf{H}$ to be the set of points $x \in X$ having finite distance from x_0. This definition does not depend on the choice of x_0. Note that all points in $X \setminus \mathbf{H}$ are ends, but that some ends in X may be contained in $\mathbf{H}$.

The generalized metric on X restricts to a *bona fide* metric on $\mathbf{H}$. One can show that $\mathbf{H}$ is complete in this metric and that $\mathbf{H}$ is an $\mathbf{R}$-tree in the usual sense [GH90]. In general, even if $\mathbf{H} = X$, the topology generated by the metric may be strictly stronger than the weak topology. In fact, the weak topology on X may not be metrizable. This happens, for example, when there is a point with uncountable tangent space: such a point does not admit a countable basis of open neighborhoods.

2.2.2. *Limit of finite trees.* As noted in Remark 2.3, the inverse limit of finite trees may fail to be a tree. However, this cannot happen in the setting of metric trees. A *finite metric tree* is a finite tree equipped with a generalized metric in which all distances are finite. Suppose we are given a directed set A, a finite metric tree Y_α for each $\alpha \in A$ and, for $\alpha \le \beta$:

- an isometric embedding $\iota_{\beta\alpha} : Y_\alpha \to Y_\beta$; this means that each interval in Y_α maps isometrically onto an interval in Y_β;
- a continuous map $r_{\alpha\beta} : Y_\beta \to Y_\alpha$ such that $r_{\alpha\beta} \circ \iota_{\beta\alpha} = \mathrm{id}_{Y_\alpha}$ and such that $r_{\alpha\beta}$ maps each connected component of $Y_\beta \setminus Y_\alpha$ to a single point in Y_α.

We claim that the space

$$X := \varprojlim_{\alpha} Y_\alpha$$

is naturally a metric tree. Recall that X is the set of points $(x_\alpha)_{\alpha \in A}$ in the product space $\prod_\alpha Y_\alpha$ such that $r_{\alpha\beta}(x_\beta) = x_\alpha$ for all $\alpha \le \beta$. It is a compact Hausdorff space. For each α we have an injective map $\iota_\alpha : Y_\alpha \to X$ mapping $x \in Y_\alpha$ to $(x_\beta)_{\beta \in A}$, where $x_\beta \in Y_\beta$ is defined as follows: $x_\beta = r_{\beta\gamma}\iota_{\gamma\alpha}(x)$, where $\gamma \in A$ dominates both α and β. Abusing notation, we view Y_α as a subset of X. For distinct points $x, y \in X$ define

$$[x, y] := \{x\} \cup \bigcup_{\alpha \in A} [x_\alpha, y_\alpha] \cup \{y\}.$$

We claim that $[x, y]$ naturally carries an interval structure as well as a generalized metric. To see this, pick α_0 such that $x_{\alpha_0} \neq y_{\alpha_0}$ and $z = (z_\alpha) \in]x_{\alpha_0}, y_{\alpha_0}[$. Then $d_\alpha(x_\alpha, z_\alpha)$ and $d_\alpha(y_\alpha, z_\alpha)$ are finite and increasing functions of α, hence converge to $\delta_x, \delta_y \in [0, +\infty]$, respectively. This gives rise to an isometry of $[x, y]$ onto the interval $[-\delta_x, \delta_y] \subseteq [-\infty, +\infty]$.

2.3. **Rooted and parametrized trees.** Sometimes there is a point in a tree that plays a special role. This leads to the following notion.

Definition 2.4. A *rooted tree* is a partially ordered set $(X, \leq)$ satisfying the following properties:

(RT1) X has a unique minimal element x_0;
(RT2) for any $x \in X \setminus \{x_0\}$, the set $\{z \in X \mid z \leq x\}$ is isomorphic (as a partially ordered set) to the real interval $[0, 1]$;
(RT3) any two points $x, y \in X$ admit an infimum $x \wedge y$ in X, that is, $z \leq x$ and $z \leq y$ iff $z \leq x \wedge y$;
(RT4) any totally ordered subset of X has a least upper bound in X.

Sometimes it is natural to reverse the partial ordering so that the root is the unique *maximal* element.

Remark 2.5. In [FJ04] it was claimed that (RT3) follows from the other three axioms but this is not true. A counterexample is provided by two copies of the interval $[0, 1]$ identified along the half-open subinterval $[0, 1[$. I am grateful to Josnei Novacoski and Franz-Viktor Kuhlmann for pointing this out.

Let us compare this notion with the definition of a tree above. If $(X, \leq)$ is a rooted tree, then we can define intervals $[x, y] \subseteq X$ as follows. First, when $x \leq y \in X$, set $[x, y] := \{z \in X \mid x \leq z \leq y\}$ and $[y, x] := [x, y]$. For general $x, y \in X$ set $[x, y] := [x \wedge y, x] \cup [x \wedge y, y]$. We leave it to the reader to equip $[x, y]$ with an interval structure and to verify conditions (T1)–(T5). Conversely, given a tree X and a point $x_0 \in X$, define a partial ordering on X by declaring $x \leq y$ iff $x \in [x_0, y]$. One checks that conditions (RT1)–(RT4) are verified.

A *parametrization* of a rooted tree $(X, \leq)$ as above is a monotone function $\alpha : X \to [-\infty, +\infty]$ whose restriction to any segment $[x, y]$ with $x < y$ is a homeomorphism onto a closed subinterval of $[-\infty, +\infty]$. We also require $|\alpha(x_0)| < \infty$ unless x_0 is an endpoint of X. This induces a generalized metric on X by setting

$$d(x, y) = |\alpha(x) - \alpha(x \wedge y)| + |\alpha(y) - \alpha(x \wedge y)|$$

for distinct points $x, y \in X$. The set $\mathbf{H}$ is exactly the locus where $|\alpha| < \infty$. Conversely given a generalized metric d on a tree X, a point $x_0 \in \mathbf{H}$ and a real number $\alpha_0 \in \mathbf{R}$, we obtain an increasing parametrization α of the tree X rooted in x_0 by setting $\alpha(x) = \alpha_0 + d(x, x_0)$.

Remark 2.6. A natural question is whether or not every rooted tree admits a parametrization. In personal communication to the author, Andreas Blass has outlined an example of a rooted tree that cannot be parametrized. His construction relies on Suslin trees [Jec03], the existence of which cannot be decided from the ZFC axioms. It would be interesting to have a more explicit example.

2.4. **Radon measures on trees.** Let us review the notions of Borel and Radon measures on compact topological spaces and, more specifically, on trees.

2.4.1. *Radon and Borel measures on compact spaces.* A reference for the material in this section is [Fol99, §7.1-2]. Let X be a compact (Hausdorff) space and $\mathcal{B}$ the associated Borel σ-algebra. A *Borel measure* on X is a function $\rho : \mathcal{B} \to [0, +\infty]$ satisfying the usual axioms. A Borel measure ρ is *regular* if for every Borel set $E \subseteq X$ and every $\varepsilon > 0$ there exists a compact set F and an open set U such that $F \subseteq E \subseteq U$ and $\rho(U \setminus F) < \varepsilon$.

A *Radon measure* on X is a positive linear functional on the vector space $C^0(X)$ of continuous functions on X. By the Riesz representation theorem, Radon measures can be identified with regular Borel measures.

If X has the property that every open set of X is σ-compact, that is, a countable union of compact sets, then every Borel measure on X is Radon. However, many Berkovich spaces do not have this property. For example, the Berkovich projective line over any non-Archimedean field K is a tree, but if the residue field of K is uncountable, then the complement of any Type 2 point (see §3.3.4) is an open set that is not σ-compact.

We write $\mathcal{M}^+(X)$ for the set of positive Radon measures on X and endow it with the topology of weak (or vague) convergence. By the Banach-Alaoglu Theorem, the subspace $\mathcal{M}_1^+(X)$ of Radon probability measure is compact.

A *finite atomic measure* on X is a Radon measure of the form $\rho = \sum_{i=1}^N c_i \delta_{x_i}$, where $c_i > 0$. A *signed* Radon measure is a real-valued linear functional on $C^0(X; \mathbf{R})$. The only signed measures that we shall consider will be of the form $\rho - \rho_0$, where ρ is a Radon measure and ρ_0 a finite atomic measure.

2.4.2. *Measures on finite trees.* Let X be a *finite* tree. It is then easy to see that every connected open set is of the form $\bigcap_{i=1}^n U(\vec{v}_i)$, where $\vec{v}_1, \dots, \vec{v}_n$ are tangent directions in X such that $U(\vec{v}_i) \cap U(\vec{v}_j) \neq \emptyset$ but $U(\vec{v}_i) \not\subseteq U(\vec{v}_j)$ for $i \neq j$. Each such set is a countable union of compact subsets so it follows from the above that every Borel measure is in fact a Radon measure.

2.4.3. *Radon measures on general trees.* Now let X be an arbitrary tree in the sense of Definition 2.2. It was claimed in [FJ04] and [BR10] that in this case, too, every Borel measure is Radon, but there is a gap in the proofs.

Example 2.7. Let Y be a set with the following property: there exists a probability measure μ on the maximal σ-algebra (that contains all subsets of Y) that gives zero mass to any finite set. The existence of such a set, whose cardinality is said to be a *real-valued measurable cardinal* is a well known problem in set theory [Fre93]: suffice it to say that its existence or nonexistence cannot be decided from the ZFC axioms. Now equip Y with the discrete topology and let X be the cone over Y, that is $X = Y \times [0, 1]/ \sim$, where $(y, 0) \sim (y', 0)$ for all $y, y' \in Y$. Let $\phi : Y \to X$ be the continuous map defined by $\phi(y) = (y, 1)$. Then $\rho := \phi_*\mu$ is a Borel measure on X which is not Radon. Indeed, the open set $U := X \setminus \{0\}$ has measure 1, but any compact subset of U is contained in a *finite* union of intervals $\{y\} \times]0, 1]$ and thus has measure zero.

Fortunately, this does not really lead to any problems. The message to take away is that on a general tree, one should systematically use Radon measures, and this is indeed what we shall do here.

2.4.4. *Coherent systems of measures.* The description of a general tree X as a profinite tree is well adapted to describe Radon measures on X. Namely, let $(Y_\alpha)_{\alpha\in A}$ be a rich net of finite subtrees of X, in the sense of §2.1.2. The homeomorphism $X \xrightarrow{\sim} \varprojlim Y_\alpha$ then induces a homeomorphism $\mathcal{M}_1^+(X) \xrightarrow{\sim} \varprojlim \mathcal{M}_1^+(Y_\alpha)$. Concretely, the right hand side consists of collections $(\rho_\alpha)_{\alpha\in A}$ of Radon measures on each Y_α satisfying $(r_{\alpha\beta})_*\rho_\beta = \rho_\alpha$ for $\alpha \le \beta$. Such a collection of measures is called a *coherent system of measures* in [BR10]. The homeomorphism above assigns to a Radon probability measure ρ on X the collection $(\rho_\alpha)_{\alpha\in A}$ defined by $\rho_\alpha := (r_\alpha)_*\rho$.

2.5. **Potential theory.** Next we outline how to do potential theory on a metric tree. The presentation is adapted to our needs but basically follows [BR10], especially §1.4 and §2.5. The Laplacian on a tree is a combination of the usual real Laplacian with the combinatorially defined Laplacian on a simplicial tree.

2.5.1. *Quasisubharmonic functions on finite metric trees.* Let X be a *finite* metric tree. The Laplacian Δ on X is naturally defined on the class $\mathrm{BDV}(X) \subseteq C^0(X)$ of functions with bounded differential variation, see [BR10, §3.5], but we shall restrict our attention to the subclass $\mathrm{QSH}(X) \subseteq \mathrm{BDV}(X)$ of *quasisubharmonic* functions.

Let $\rho_0 = \sum_{i=1}^N c_i\delta_{x_i}$ be a finite atomic measure on X. Define the class $\mathrm{SH}(X,\rho_0)$ of ρ_0-*subharmonic* functions as the set of continuous functions φ that are convex on any segment disjoint from the support of ρ_0 and such that, for any $x \in X$:

$$\rho_0\{x\} + \sum_{\vec{v}} D_{\vec{v}}\varphi \ge 0,$$

where the sum is over all tangent directions $\vec{v}$ at x. Here $D_{\vec{v}}\varphi$ denotes the directional derivative of φ in the direction $\vec{v}$ (outward from x): this derivative is well defined by the convexity of φ. We leave it to the reader to verify that

$$D_{\vec{v}}\varphi \le 0 \quad \text{whenever} \quad \rho_0(U(\vec{v})) = 0 \tag{2.1}$$

for any $\varphi \in \mathrm{SH}(X,\rho_0)$; this inequality is quite useful.

Define $\mathrm{QSH}(X)$ as the union of $\mathrm{SH}(X,\rho_0)$ over all finite atomic measures ρ_0. Note that if ρ_0, ρ_0' are two finite atomic measures with $\rho_0' \ge \rho_0$, then $\mathrm{SH}(X,\rho_0) \subseteq \mathrm{SH}(X,\rho_0')$. We also write $\mathrm{SH}(X,x_0) := \mathrm{SH}(X,\delta_{x_0})$ and refer to its elements as x_0-*subharmonic.*

Let $Y \subseteq X$ be a subtree of X containing the support of ρ_0. We have an injection $\iota : Y \hookrightarrow X$ and a retraction $r : X \to Y$. It follows easily from (2.1) that

$$\iota^*\,\mathrm{SH}(X,\rho_0) \subseteq \mathrm{SH}(Y,\rho_0) \quad \text{and} \quad r^*\,\mathrm{SH}(Y,\rho_0) \subseteq \mathrm{SH}(X,\rho_0).$$

Moreover, $\varphi \le r^*\iota^*\varphi$ for any $\varphi \in \mathrm{SH}(X,\rho_0)$.

2.5.2. *Laplacian.* For $\varphi \in \mathrm{QSH}(X)$, define $\Delta\varphi$ to be the signed (Borel) measure on X defined as follows: if $\vec{v}_1, \dots, \vec{v}_n$ are tangent directions in X such that $U(\vec{v}_i) \cap U(\vec{v}_j) \neq \emptyset$ but $U(\vec{v}_i) \not\subseteq U(\vec{v}_j)$ for $i \neq j$, then

$$\Delta\varphi(\bigcap_{i=1}^{n} U(\vec{v}_i)) = \sum_{i=1}^{n} D_{\vec{v}_i}\varphi.$$

This equation defines $\Delta\varphi$ uniquely as every open set in X is a countable disjoint union of open sets of the form $\bigcap U(\vec{v}_i)$. The mass of $\Delta\varphi$ at a point $x \in X$ is given by $\sum_{\vec{v}\in T_x} D_{\vec{v}}\varphi$ and the restriction of $\Delta\varphi$ to any open segment $I \subseteq X$ containing no branch point is equal to the usual real Laplacian of $\varphi|_I$.

The Laplace operator is essentially injective. Indeed, suppose $\varphi_1, \varphi_2 \in \mathrm{QSH}(X)$ and $\Delta\varphi_1 = \Delta\varphi_2$. We may assume $\varphi_1, \varphi_2 \in \mathrm{SH}(X, \rho_0)$ for a common positive measure ρ_0. If $\varphi = \varphi_1 - \varphi_2$, then φ is affine on any closed interval whose interior is disjoint from the support of ρ_0. Moreover, at any point $x \in X$ we have $\sum_{\vec{v}\in T_x} D_{\vec{v}}\varphi = 0$. These two conditions easily imply that φ is constant. (Too see this, first check that φ is locally constant at any end of X.)

If $\varphi \in \mathrm{SH}(X, \rho_0)$, then $\rho_0 + \Delta\varphi$ is a positive Borel measure on X of the same mass as ρ_0. In particular, when ρ_0 is a probability measure, we obtain a map

$$\mathrm{SH}(X, \rho_0) \ni \varphi \mapsto \rho_0 + \Delta\varphi \in \mathcal{M}_1^+(X), \tag{2.2}$$

where $\mathcal{M}_1^+(X)$ denotes the set of probability measures on X. We claim that this map is surjective. To see this, first note that the function $\varphi_{y,z}$ given by

$$\varphi_{y,z}(x) = -d(z, x \wedge_z y), \tag{2.3}$$

with $x \wedge_z y \in X$ as in (T4), belongs to $\mathrm{SH}(X, z)$ and satisfies $\Delta\varphi = \delta_y - \delta_z$. For a general probability measure ρ and finite atomic probability measure ρ_0, the function

$$\varphi(x) = \iint \varphi_{y,z}(x)\, d\rho(y) d\rho_0(z) \tag{2.4}$$

belongs to $\mathrm{SH}(X, \rho_0)$ and satisfies $\Delta\varphi = \rho - \rho_0$.

Let $Y \subseteq X$ be a subtree containing the support of ρ_0 and denote the Laplacians on X and Y by Δ_X and Δ_Y, respectively. Then, with notation as above,

$$\Delta_Y(\iota^*\varphi) = r_*(\Delta_X\varphi) \quad \text{for } \varphi \in \mathrm{SH}(X, \rho_0) \tag{2.5}$$

$$\Delta_X(r^*\varphi) = \iota_*(\Delta_Y\varphi) \quad \text{for } \varphi \in \mathrm{SH}(Y, \rho_0), \tag{2.6}$$

where $\iota : Y \hookrightarrow X$ and $r : X \to Y$ are the inclusion and retraction, respectively.

2.5.3. *Equicontinuity.* The spaces $\mathrm{SH}(X, \rho_0)$ have very nice compactness properties deriving from the fact that if ρ_0 is a probability measure then

$$|D_{\vec{v}}\varphi| \leq 1 \quad \text{for all tangent directions } \vec{v} \text{ and all } \varphi \in \mathrm{SH}(X, \rho_0). \tag{2.7}$$

Indeed, using the fact that a function in $\mathrm{QSH}(X)$ is determined, up to an additive constant, by its Laplacian (2.7) follows from (2.3) when ρ_0 and $\rho_0 + \Delta\varphi$ are Dirac masses, and from (2.4) in general.

As a consequence of (2.7), the functions in $\mathrm{SH}(X, \rho_0)$ are uniformly Lipschitz continuous and in particular equicontinuous. This shows that pointwise convergence in $\mathrm{SH}(X, \rho_0)$ implies uniform convergence.

The space $\mathrm{SH}(X,\rho_0)$ is easily seen to be closed in the C^0-topology, so we obtain several compactness assertions from the Arzela-Ascoli theorem. For example, the set $\mathrm{SH}^0(X,\rho_0)$ of $\varphi\in\mathrm{SH}(X,\rho_0)$ for which $\max\varphi=0$ is compact.

Finally, we have an exact sequence of topological vector spaces

$$0\to\mathbf{R}\to\mathrm{SH}(X,\rho_0)\to\mathcal{M}_1^+(X)\to 0; \tag{2.8}$$

here $\mathcal{M}_1^+(X)$ is equipped with the weak topology on measures. Indeed, the construction in (2.3)-(4.4) gives rise to a continuous bijection between $\mathcal{M}_1^+(X)$ and $\mathrm{SH}(X,\rho_0)/\mathbf{R}\simeq\mathrm{SH}^0(X,\rho_0)$. By compactness, the inverse is also continuous.

2.5.4. *Quasisubharmonic functions on general metric trees.* Now let X be a general metric tree and ρ_0 a finite atomic measure supported on the associated hyperbolic space $\mathbf{H}\subseteq X$.

Let A be the set of finite metric subtrees of X that contain the support of ρ_0. This is a directed set, partially ordered by inclusion. For $\alpha\in A$, denote the associated metric tree by Y_α. The net $(Y_\alpha)_{\alpha\in A}$ is rich in the sense of §2.1.2, so the retractions $r_\alpha:X\to Y_\alpha$ induce a homeomorphism $r:X\overset{\sim}{\to}\varprojlim Y_\alpha$.

Define $\mathrm{SH}(X,\rho_0)$ to be the set of functions $\varphi:X\to[-\infty,0]$ such that $\varphi|_{Y_\alpha}\in\mathrm{SH}(Y_\alpha,\rho_0)$ for all $\alpha\in A$ and such that $\varphi=\lim r_\alpha^*\varphi$. Notice that in this case $r_\alpha^*\varphi$ in fact decreases to φ. Since $r_\alpha^*\varphi$ is continuous for all α, this implies that φ is upper semicontinuous.

We define the topology on $\mathrm{SH}(X,\rho_0)$ in terms of pointwise convergence on $\mathbf{H}$. Thus a net φ_i converges to φ in $\mathrm{SH}(X,\rho_0)$ iff $\varphi_i|_{Y_\alpha}$ converges to $\varphi|_{Y_\alpha}$ for all α. Note, however, that the convergence $\varphi_i\to\varphi$ is not required to hold on all of X.

Since, for all α, $\mathrm{SH}(Y_\alpha,\rho_0)$ is compact in the topology of pointwise convergence on Y_α, it follows that $\mathrm{SH}(X,\rho_0)$ is also compact. The space $\mathrm{SH}(X,\rho_0)$ has many nice properties beyond compactness. For example, if $(\varphi_i)_i$ is a decreasing net in $\mathrm{SH}(X,\rho_0)$, and $\varphi:=\lim\varphi_i$, then either $\varphi_i\equiv-\infty$ on X or $\varphi\in\mathrm{SH}(X,\rho_0)$. Further, if $(\varphi_i)_i$ is a family in $\mathrm{SH}(X,\rho_0)$ with $\sup_i\max_X\varphi_i<\infty$, then the upper semicontinuous regularization of $\varphi:=\sup_i\varphi_i$ belongs to $\mathrm{SH}(X,\rho_0)$.

As before, we define $\mathrm{QSH}(X)$, the space of *quasisubharmonic functions*, to be the union of $\mathrm{SH}(X,\rho_0)$ over all finite atomic measures ρ_0 supported on $\mathbf{H}$.

2.5.5. *Laplacian.* Let X, ρ_0 and A be as above. Recall that a Radon probability measure ρ on X is given by a coherent system $(\rho_\alpha)_{\alpha\in A}$ of (Radon) probability measures on Y_α.

For $\varphi\in\mathrm{SH}(X,\rho_0)$ we define $\rho_0+\Delta\varphi\in\mathcal{M}_1^+(X)$ to be the unique Radon probability measure such that

$$(r_\alpha)_*(\rho_0+\Delta\varphi)=\rho_0+\Delta_{Y_\alpha}(\varphi|_{Y_\alpha})$$

for all $\alpha\in A$. This makes sense in view of (2.5).

The construction in (2.3)-(2.4) remains valid and the sequence (2.8) of topological vector spaces is exact. For future reference we record that if $(\varphi_i)_i$ is a net in $\mathrm{SH}^0(X,\rho_0)$, then $\varphi_i\to 0$ (pointwise on $\mathbf{H}$) iff $\Delta\varphi_i\to 0$ in $\mathcal{M}_1^+(X)$.

2.5.6. *Singularities of quasisubharmonic functions.* Any quasisubharmonic function on a metric tree X is bounded from above on all of X and Lipschitz continuous on hyperbolic space $\mathbf{H}$, but can take the value $-\infty$ at infinity. For example, if $x_0 \in \mathbf{H}$ and $y \in X \setminus \mathbf{H}$, then the function $\varphi(x) = -d_{\mathbf{H}}(x_0, x \wedge_{x_0} y)$ is x_0-subharmonic and $\varphi(y) = -\infty$. Note that $\Delta\varphi = \delta_y - \delta_{x_0}$. The following result allows us to estimate a quasisubharmonic function from below in terms of the mass of its Laplacian at infinity. It will be used in the proof of the equidistribution result in §5.7.

Proposition 2.8. *Let ρ_0 be a finite atomic probability measure on $\mathbf{H}$ and let $\varphi \in \mathrm{SH}(X, \rho_0)$. Pick $x_0 \in \mathbf{H}$ and any number $\lambda > \sup_{y \in X\setminus\mathbf{H}} \Delta\varphi\{y\}$. Then there exists a constant $C = C(x_0, \rho_0, \varphi, \lambda) > 0$ such that*

$$\varphi(x) \geq \varphi(x_0) - C - \lambda d_{\mathbf{H}}(x, x_0)$$

for all $x \in \mathbf{H}$.

We shall use the following estimates, which are of independent interest.

Lemma 2.9. *Let ρ_0 be a finite atomic probability measure on $\mathbf{H}$ and let $x_0 \in \mathbf{H}$. Pick $\varphi \in \mathrm{SH}(X, \rho_0)$ and set $\rho = \rho_0 + \Delta\varphi$. Then*

$$\varphi(x) - \varphi(x_0) \geq -\int_{x_0}^{x} \rho\{z \geq y\} d\alpha(y) \geq d_{\mathbf{H}}(x, x_0) \cdot \rho\{z \geq x\},$$

where $\leq$ is the partial ordering on X rooted in x_0.

Proof of Lemma 2.9. It follows from (2.4) that

$$\begin{aligned}\varphi(x) - \varphi(x_0) &= -\int_{x_0}^{x} (\Delta\varphi)\{z \geq y\} d\alpha(y) \\ &\geq -\int_{x_0}^{x} \rho\{z \geq y\} d\alpha(y) \geq -\int_{x_0}^{x} \rho\{z \geq x\} d\alpha(y) = d_{\mathbf{H}}(x, x_0) \cdot \rho\{z \geq x\},\end{aligned}$$

where we have used that $\rho \geq \Delta\varphi$ and $x \geq y$. □

Proof of Proposition 2.8. Let $\leq$ denote the partial ordering rooted in x_0 and set

$$Y_\lambda := \{y \in X \mid (\rho_0 + \Delta\varphi)\{z \geq y\} \geq \lambda\}.$$

Recall that $\rho_0 + \Delta\varphi$ is a probability measure. Thus $Y_\lambda = \emptyset$ if $\lambda > 1$. If $\lambda \leq 1$, then Y_λ is a finite subtree of X containing x_0 and having at most $1/\lambda$ ends. The assumption that $\lambda > \sup_{y \in X\setminus\mathbf{H}} \Delta\varphi\{y\}$ implies that Y_λ is in fact contained in $\mathbf{H}$. In particular, the number $C := \sup_{y \in Y_\lambda} d_{\mathbf{H}}(x_0, y)$ is finite.

It now follows from Lemma 2.9 that

$$\varphi(x) - \varphi(x_0) \geq -\int_{x_0}^{x} (\rho_0 + \Delta\varphi)\{z \geq y\} d\alpha(y) \geq -C - \lambda d_{\mathbf{H}}(x, x_0),$$

completing the proof. □

2.5.7. *Regularization.* In complex analysis, it is often useful to approximate a quasisubharmonic function by a decreasing sequence of smooth quasisubharmonic functions. In higher dimensions, regularization results of this type play a fundamental role in pluripotential theory, as developed by Bedford and Taylor [BT82, BT87]. They are also crucial to the approach to non-Archimedean pluripotential theory in [BFJ08b, BFJ11a, BFJ11b].

Let us say that a function $\varphi \in \mathrm{SH}(X, \rho_0)$ is *regular* if it is piecewise affine in the sense that $\Delta\varphi = \rho - \rho_0$, where ρ is a finite atomic measure supported on $\mathbf{H}$.

Theorem 2.10. *For any $\varphi \in \mathrm{SH}(X, \rho_0)$ there exists a decreasing sequence $(\varphi_n)_{n=1}^{\infty}$ of regular functions in $\mathrm{SH}(X, \rho_0)$ such that φ_n converges pointwise to φ on X.*

Proof. Let $Y_0 \subset X$ be a finite tree containing the support of ρ and pick a point $x_0 \in Y_0$. Set $\rho = \rho_0 + \Delta\varphi$.

First assume that ρ is supported on a finite subtree contained in $\mathbf{H}$. We may assume $Y_0 \subseteq Y$. For each $n \geq 1$, write $Y \setminus \{x_0\}$ as a finite disjoint union of half-open segments $\gamma_i =]x_i, y_i]$, $i \in I_n$, called segments of order n, in such a way that each segment of order n has length at most 2^{-n} and is the disjoint union of two segments of order $n + 1$. Define finite atomic measures ρ_n by

$$\rho_n = \rho\{x_0\}\delta_{x_0} + \sum_{i \in I_n} \rho(\gamma_i)\delta_{y_i}$$

and define $\varphi_n \in \mathrm{SH}(X, x_0)$ by $\Delta\varphi_n = \rho_n - \rho_0$, $\varphi_n(x_0) = \varphi(x_0)$. From (2.3) and (2.4) it follows that φ_n decreases to φ pointwise on X, as $n \to \infty$. Since $\varphi = r_Y^*\varphi$ is continuous, the convergence is in fact uniform by Dini's Theorem.

Now consider a general $\varphi \in \mathrm{SH}(X, \rho_0)$. For $n \geq 1$, define $Y_n' \subseteq X$ by

$$Y_n' := \{y \in X \mid \rho\{z \geq y\} \geq 2^{-n} \quad \text{and} \quad d_{\mathbf{H}}(x_0, y) \leq 2^n\},$$

where $\leq$ denotes the partial ordering rooted in x_0. Then Y_n' is a finite subtree of X and $Y_n' \subseteq Y_{n+1}'$ for $n \geq 1$. Let Y_n be the convex hull of the union of Y_n' and Y_0 and set $\psi_n = r_{Y_n}^*\varphi_n$. Since $Y_n \subseteq Y_{n+1}$, we have $\varphi \leq \psi_{n+1} \leq \psi_n$ for all n. We claim that $\psi_n(x)$ converges to $\varphi(x)$ as $n \to \infty$ for every $x \in X$. Write $x_n := r_{Y_n}(x)$ so that $\psi_n(x) = \varphi(x_n)$. The points x_n converge to a point $y \in [x_0, x]$ and $\lim_n \psi_n(x) = \varphi(y)$. If $y = x$, then we are done. But if $y \neq x$, then by construction of Y_n', the measure ρ puts no mass on the interval $]y, x]$, so it follows from (2.3) and (2.4) that $\varphi(x) = \varphi(y)$.

Hence ψ_n decreases to φ pointwise on X as $n \to \infty$. By the first part of the proof, we can find a regular $\varphi_n \in \mathrm{SH}(X, \rho_0)$ such that $\psi_n \leq \varphi_n \leq \psi_n + 2^{-n}$ on X. Then φ_n decreases to φ pointwise on X, as desired. □

Remark 2.11. A different kind of regularization is used in [FR06, §4.6]. Fix a point $x_0 \in \mathbf{H}$ and for each $n \geq 1$ let $X_n \subseteq X$ be the (a priori not finite) subtree defined by $X_n = \{x \in X \mid d_{\mathbf{H}}(x_0, x) \leq n^{-1}\}$. Let $\varphi_n \in \mathrm{SH}(X, \rho_0)$ be defined by $\rho_0 + \Delta\varphi_n = (r_n)_*(\rho_0 + \Delta\varphi)$ and $\varphi_n(x_0) = \varphi(x_0)$, where $r_n : X \to X_n$ is the retraction. Then φ_n is bounded and φ_n decreases to φ as $n \to \infty$.

2.6. **Tree maps.** Let X and X' be trees in the sense of §2.2. We say that a continuous map $f : X \to X'$ is a *tree map* if it is open, surjective and finite in the sense that there exists a number d such that every point in X' has at most d preimages in X. The smallest such number d is the *topological degree* of f.

Proposition 2.12. *Let $f : X \to X'$ be a tree map of topological degree d.*

(i) *if $U \subseteq X$ is a connected open set, then so is $f(U)$ and $\partial f(U) \subseteq f(\partial U)$;*

(ii) *if $U' \subseteq X'$ is a connected open set and U is a connected component of $f^{-1}(U')$, then $f(U) = U'$ and $f(\partial U) = \partial U'$; as a consequence, $f^{-1}(U')$ has at most d connected components;*

(iii) *if $U \subseteq X$ is a connected open set and $U' = f(U)$, then U is a connected component of $f^{-1}(U')$ iff $f(\partial U) \subseteq \partial U'$.*

The statement is valid for finite surjective open continuous maps $f : X \to X'$ between compact Hausdorff spaces, under the assumption that every point of X admits a basis of connected open neighborhoods. We omit the elementary proof; see Lemma 9.11, Lemma 9.12 and Proposition 9.15 in [BR10] for details.

Corollary 2.13. *Consider a point $x \in X$ and set $x' := f(x) \in X'$. Then there exists a connected open neighborhood V of x with the following properties:*

(i) *if $\vec{v}$ is a tangent direction at x, then there exists a tangent direction $\vec{v}'$ at x' such that $f(V \cap U(\vec{v})) \subseteq U(\vec{v}')$; furthermore, either $f(U(\vec{v})) = U(\vec{v}')$ or $f(U(\vec{v})) = X'$;*

(ii) *if $\vec{v}'$ is a tangent direction at x' then there exists a tangent direction $\vec{v}$ at x such that $f(V \cap U(\vec{v})) \subseteq U(\vec{v}')$.*

Definition 2.14. The *tangent map* of f at x is the map that associates $\vec{v}'$ to $\vec{v}$.

The tangent map is surjective and every tangent direction has at most d preimages. Since the ends of X are characterized by the tangent space being a singleton, it follows that f maps ends to ends.

Proof of Corollary 2.13. Pick V small enough so that it contains no preimage of x' besides x. Note that (ii) follows from (i) and the fact that $f(V)$ is an open neighborhood of x'.

To prove (i), note that $V \cap U(\vec{v})$ is connected for every $\vec{v}$. Hence $f(V \cap U(\vec{v}))$ is connected and does not contain x', so it must be contained in $U(\vec{v}')$ for some $\vec{v}'$. Moreover, the fact that f is open implies $\partial f(U(\vec{v})) \subseteq f(\partial U(\vec{v})) = \{x'\}$. Thus either $f(U(\vec{v})) = X'$ or $f(U(\vec{v}))$ is a connected open set with boundary $\{x'\}$. In the latter case, we must have $f(U(\vec{v})) = U(\vec{v}')$. □

2.6.1. *Images and preimages of segments.* The following result makes the role of the tangent map more precise.

Corollary 2.15. *Let $f : X \to X'$ be a tree map as above. Then:*

(i) *if $\vec{v}$ is a tangent direction at a point $x \in X$, then there exists a point $y \in U(\vec{v})$ such that f is a homeomorphism of the interval $[x, y] \subseteq X$ onto the interval $[f(x), f(y)] \subseteq X'$; furthermore, f maps the annulus $A(x, y)$ onto the annulus $A(f(x), f(y))$;*

(ii) *if $\vec{v}'$ is a tangent direction at a point $x' \in X'$, then there exists $y' \in U(\vec{v}')$ such that if $\gamma' := [x', y']$ then $f^{-1}\gamma' = \bigcup_i \gamma_i$, where the $\gamma_i = [x_i, y_i]$ are closed intervals in X with pairwise disjoint interiors and f maps γ_i homeomorphically onto γ' for all i; furthermore we have $f(A(x_i, y_i)) = A(x', y')$ for all i and $f^{-1}(A(x', y')) = \bigcup_i A(x_i, y_i)$.*

Proof. We first prove (ii). Set $U' = U(\vec{v}')$ and let U be a connected component of $f^{-1}(U')$. By Proposition 2.12 (ii), the boundary of U consists of finitely many preimages $x_1, \dots x_m$ of x'. (The same preimage of x' can lie on the boundary of several connected components U.) Since U is connected, there exists, for $1 \le i \le m$, a unique tangent direction $\vec{v}_i$ at x_i such that $U \subseteq U(\vec{v}_i)$.

Pick any point $z' \in U'$. Also pick points $z_1, \dots, z_m$ in U such that the segments $[x_i, z_i]$ are pairwise disjoint. Then $f(]x_i, z_i]) \cap]x', z'] \ne \emptyset$ for all i, so we can find $y' \in]x', z']$ and $y_i \in]x_i, z_i]$ arbitrarily close to x_i such that $f(y_i) = y'$ for all i. In particular, we may assume that the annulus $A_i := A(x_i, y_i)$ contains no preimage of z'. By construction it contains no preimage of x' either. Proposition 2.12 (i) first shows that $\partial f(A_i) \subseteq \{x', y'\}$, so $f(A_i) = A' := A(x', y')$ for all i. Proposition 2.12 (iii) then implies that A_i is a connected component of $f^{-1}(A')$. Hence $f^{-1}(A') \cap U = \bigcup_i A_i$.

Write $\gamma_i = [x_i, y_i]$ and $\gamma' = [x', y']$. Pick any $\xi \in]x_i, y_i[$ and set $\xi' := f(\xi)$. On the one hand, $f(A(\xi, y_i)) \subseteq f(A_i) = A'$. On the other hand, $\partial f(A(\xi, y_i)) \subseteq \{\xi', y'\}$ so we must have $f(A(\xi, y_i)) = A(\xi', y')$ and $\xi' \in \gamma'$. We conclude that $f(\gamma_i) = \gamma'$ and that $f : \gamma_i \to \gamma'$ is injective, hence a homeomorphism.

The same argument gives $f(A(x_i, \xi)) = A(x', \xi)$. Consider any tangent direction $\vec{w}$ at ξ such that $U(\vec{w}) \subseteq A_i$. As above we have $f(U(\vec{w})) \subseteq A'$ and $\partial f(U(\vec{w})) \subseteq \{\xi'\}$, which implies $f(U(\vec{w})) = U(\vec{w}')$ for some tangent direction $\vec{w}$ at ξ' for which $U(\vec{w}) \subseteq A'$. We conclude that $f^{-1}(\gamma') \cap A_i \subseteq \gamma_i$.

This completes the proof of (ii), and (i) is an easy consequence. □

Using compactness, we easily deduce the following result from Corollary 2.15. See the proof of Theorem 9.35 in [BR10].

Corollary 2.16. *Let $f : X \to X'$ be a tree map as above. Then:*

(i) *any closed interval γ in X can be written as a finite union of closed intervals γ_i with pairwise disjoint interiors, such that $\gamma_i' := f(\gamma_i) \subseteq X'$ is an interval and $f : \gamma_i \to \gamma_i'$ is a homeomorphism for all i; furthermore, f maps the annulus $A(\gamma_i)$ onto the annulus $A(\gamma_i')$;*

(ii) *any closed interval γ' in X' can be written as a union of finitely many intervals γ_i' with pairwise disjoint interiors, such that, for all i, $f^{-1}(\gamma_i')$ is a finite union of closed intervals γ_{ij} with pairwise disjoint interiors, such that $f : \gamma_{ij} \to \gamma_i'$ is a homeomorphism for each j; furthermore, f maps the annulus $A(\gamma_{ij})$ onto the annulus $A(\gamma_i')$; and $A(\gamma_{ij})$ is a connected component of $f^{-1}(A(\gamma_i'))$.*

2.6.2. *Fixed point theorem.* It is an elementary fact that any continuous selfmap of a *finite* tree admits a fixed point. This can be generalized to arbitrary trees. Versions of the following fixed point theorem can be found in [FJ04, Riv04, BR10].

Proposition 2.17. *Any tree map $f : X \to X$ admits a fixed point $x = f(x) \in X$. Moreover, we can assume that one of the following two conditions hold:*

(i) *x is not an end of X;*

(ii) *x is an end of X and x is an attracting fixed point: there exists an open neighborhood $U \subseteq X$ of x such that $f(U) \subseteq U$ and $\bigcap_{n \ge 0} f^n(U) = \{x\}$.*

In the proof we will need the following easy consequence of Corollary 2.16 (i).

Lemma 2.18. *Suppose there are points $x, y \in X$, $x \neq y$, with $r(f(x))) = x$ and $r(f(y)) = y$, where r denotes the retraction of X onto the segment $[x, y]$. Then f has a fixed point on $[x, y]$.*

Proof of Proposition 2.17. We may suppose that f does not have any fixed point that is not an end of X, or else we are in case (i). Pick any non-end $x_0 \in X$ and pick a finite subtree X_0 that contains x_0, all preimages of x_0, but does not contain any ends of X. Let A be the set of finite subtrees of X that contain X_0 but does not contain any end of X. For $\alpha \in A$, let Y_α be the corresponding subtree. Then $(Y_\alpha)_{\alpha \in A}$ is a rich net of subtrees in the sense of §2.1.2, so $X \stackrel{\sim}{\to} \varprojlim Y_\alpha$.

For each α, define $f_\alpha : Y_\alpha \to Y_\alpha$ by $f_\alpha = f \circ r_\alpha$. This is a continuous selfmap of a finite tree so the set F_α of its fixed points is a nonempty compact set. We will show that $r_\alpha(F_\beta) = F_\alpha$ when $\beta \geq \alpha$. This will imply that there exists $x \in X$ such that $r_\alpha(f(r_\alpha(x)) = r_\alpha(x)$ for all α. By assumption, x is an end in X. Pick a sequence $(x_n)_{n=0}^\infty$ of points in X such that $x_{n+1} \in]x_n, x[$ and $x_n \to x$ as $n \to \infty$. Applying what precedes to the subtrees $Y_{\alpha_n} = X_0 \cup [x_0, x_n]$ we easily conclude that x is an attracting fixed point.

It remains to show that $r_\alpha(F_\beta) = F_\alpha$ when $\beta \geq \alpha$. First pick $x_\beta \in F_\beta$. We will show that $x_\alpha := r_\alpha(x_\beta) \in F_\alpha$. This is clear if $x_\beta \in Y_\alpha$ since $r_\alpha = r_{\alpha\beta} \circ r_\beta$, so suppose $x_\beta \notin Y_\alpha$, By assumption, $f(x_\alpha) \neq x_\alpha$ and $f(x_\beta) \neq x_\beta$. Let $\vec{v}$ be the tangent direction at x_α represented by x_β. Then $U(\vec{v}) \cap Y_\alpha = \emptyset$ so $x_0 \notin f(U(\vec{v}))$ and hence $f(U(\vec{v})) = U(\vec{v}')$ for some tangent direction $\vec{v}'$ at $f(x_\alpha)$. Note that $f(x_\beta) \in U(\vec{v}')$. If $f(x_\alpha) \notin U(\vec{v})$, then Lemma 2.18 applied to $x = x_\alpha$, $y = x_\beta$ gives a fixed point for f in $[x_\alpha, x_\beta] \subseteq Y_\beta$, a contradiction. Hence $f(x_\alpha) \notin U(\vec{v})$, so that $r_\alpha(f(x_\alpha)) = x_\alpha$, that is, $x_\alpha \in F_\alpha$.

Conversely, pick $x_\alpha \in F_\alpha$. By assumption, $f(x_\alpha) \neq x_\alpha$. Let $\vec{v}$ be the tangent direction at x_α defined by $U(\vec{v})$. Then $U(\vec{v}) \cap Y_\alpha = \emptyset$ so $f(\overline{U(\vec{v})}) \subseteq U(\vec{v})$. Now $\overline{U(\vec{v})} \cap Y_\beta$ is a finite nonempty subtree of X that is invariant under f_β. Hence f_β admits a fixed point x_β in this subtree. Then $x_\beta \in Y_\beta$ and $r_\alpha(x_\beta) = x_\alpha$. □

2.7. Notes and further references.

Our definition of "tree" differs from the one in set theory, see [Jec03]. It is also not equivalent to the notion of "**R**-tree" that has been around for quite some time (see [GH90]) and found striking applications. An **R**-tree is a metric space and usually considered with its metric topology. On the other hand, the notion of the weak topology on an **R**-tree seems to have been rediscovered several times, sometimes under different names (see [CLM07]).

Our definitions of trees and metric trees are new but equivalent[8] to the ones given in [FJ04], where rooted trees are defined first and general (non-rooted) trees are defined as equivalence classes of rooted trees. The presentation here seems more natural. Following Baker and Rumely [BR10] we have emphasized viewing a tree as a pro-finite tree, that is, an inverse limit of finite trees.

Potential theory on simplicial graphs is a quite old subject but the possibility of doing potential theory on general metric trees seems to have been discovered

[8]Except for the missing condition (RT3), see Remark 2.5.

independently by Favre and myself [FJ04], Baker and Rumely [BR10] and Thuillier [Thu05]; see also [Fav05]. Our approach here follows [BR10] quite closely in how the Laplacian is extended from finite to general trees. The class of quasisubharmonic functions is modeled on its complex counterpart, where its compactness properties makes this class very useful in complex dynamics and geometry. It is sufficiently large for our purposes and technically easier to handle than the class of functions of bounded differential variations studied in [BR10].

Note that the interpretation of "potential theory" used here is quite narrow; for further results and questions we refer to [BR10, Thu05]. It is also worth mentioning that while potential theory on the Berkovich projective line can be done in a purely tree theoretic way, this approach has its limitations. In other situations, and especially in higher dimensions, it seems advantageous to take a more geometric approach. This point of view is used already in [Thu05] and is hinted at in our exposition of the valuative tree in §7 and §9. We should remark that Thuillier in [Thu05] does potential theory on general Berkovich curves. These are not always trees in our sense as they can contain loops.

Most of the results on tree maps in §2.6 are well known and can be found in [BR10] in the context of the Berkovich projective line. I felt it would be useful to isolate some properties that are purely topological and only depend on the map between trees being continuous, open and finite. In fact, these properties turn out to be quite plentiful.

As noted in the text, versions of the fixed point result in Proposition 2.17 can be found in the work of Favre and myself [FJ07] and of Rivera-Letelier [Riv04]. The proof here is new.

3. The Berkovich affine and projective lines

Let us briefly describe the Berkovich affine and projective lines. A comprehensive reference for this material is the recent book by Baker and Rumely [BR10]. See also Berkovich's original work [Ber90]. One minor difference to the presentation in [BR10] is that we emphasize working in a coordinate free way.

3.1. **Non-Archimedean fields.**

We start by recalling some facts about non-Archimedean fields. A comprehensive reference for this material is [BGR84].

3.1.1. *Seminorms and semivaluations.* Let R be a integral domain. A *multiplicative, non-Archimedean seminorm* on R is a function $|\cdot| : R \to \mathbf{R}_+$ satisfying $|0| = 0$, $|1| = 1$, $|ab| = |a||b|$ and $|a + b| \le \max\{|a|, |b|\}$. If $|a| > 0$ for all nonzero a, then $|\cdot|$ is a *norm*. In any case, the set $\mathfrak{p} \subseteq R$ consisting of elements of norm zero is a prime ideal and $|\cdot|$ descends to a norm on the quotient ring $R/\mathfrak{p}$ and in turn extends to a norm on the fraction field of the latter.

Sometimes it is more convenient to work additively and consider the associated *semi-valuation*[9] $v : R \to \mathbf{R} \cup \{+\infty\}$ defined by $v = -\log|\cdot|$. It satisfies the axioms $v(0) = +\infty$, $v(1) = 0$, $v(ab) = v(a) + v(b)$ and $v(a + b) \ge \min\{v(a), v(b)\}$. The prime ideal $\mathfrak{p}$ above is now given by $\mathfrak{p} = \{v = +\infty\}$ and v extends uniquely to a real-valued valuation on the fraction field of $R/\mathfrak{p}$.

Any seminorm on a field K is a norm. A *non-Archimedean field* is a field K equipped with a non-Archimedean, multiplicative norm $|\cdot| = |\cdot|_K$ such that K is complete in the induced metric. In general, we allow the norm on K be trivial: see Example 3.1. As a topological space, K is totally disconnected. We write $|K^*| = \{|a| \mid a \in K \setminus \{0\}\} \subseteq \mathbf{R}_+^*$ for the (multiplicative) *value group* of K.

3.1.2. *Discs.* A *closed disc* in K is a set of the form $D(a, r) = \{b \in K \mid |a - b| \le r\}$. This disc is *degenerate* if $r = 0$, *rational* if $r \in |K^*|$ and *irrational* otherwise. Similarly, $D^-(a, r) := \{b \in K \mid |a - b| < r\}$, $r > 0$, is an *open disc.*

The terminology is natural but slightly misleading since nondegenerate discs are both open and closed in K. Further, if $0 < r \notin |K^*|$, then $D^-(a, r) = D(a, r)$. Note that any point in a disc in K can serve as a center and that when two discs intersect, one must contain the other. As a consequence, any two closed discs admit a unique smallest closed disc containing them both.

3.1.3. *The residue field.* The *valuation ring* of K is the ring $\mathfrak{o}_K := \{|\cdot| \le 1\}$. It is a local ring with maximal ideal $\mathfrak{m}_K := \{|\cdot| < 1\}$. The *residue field* of K is $\tilde{K} := \mathfrak{o}_K/\mathfrak{m}_K$. We can identify $\mathfrak{o}_K$ and $\mathfrak{m}_K$ with the closed and open unit discs in K, respectively. The *residue characteristic* of K is the characteristic of $\tilde{K}$. Note that if $\tilde{K}$ has characteristic zero, then so does K.

Example 3.1. We can equip any field K with the *trivial* norm in which $|a| = 1$ whenever $a \ne 0$. Then $\mathfrak{o}_K = K$, $\mathfrak{m}_K = 0$ and $\tilde{K} = K$.

---

[9]Unfortunately, the terminology is not uniform across the literature. In [BGR84, Ber90] 'valuation' is used to denoted multiplicative norms. In [FJ04], 'valuation' instead of 'semi-valuation' is used even when the prime ideal $\{v = +\infty\}$ is nontrivial.

Example 3.2. The field $K = \mathbf{Q}_p$ of p-adic numbers is the completion of $\mathbf{Q}$ with respect to the p-adic norm. Its valuation ring $\mathfrak{o}_K$ is the ring of p-adic integers $\mathbf{Z}_p$ and the residue field $\tilde{K}$ is the finite field $\mathbf{F}_p$. In particular, $\mathbf{Q}_p$ has characteristic zero and residue characteristic $p > 0$.

Example 3.3. The algebraic closure of $\mathbf{Q}_p$ is not complete. Luckily, the completed algebraic closure $\mathbf{C}_p$ of $\mathbf{Q}_p$ is both algebraically closed and complete. Its residue field is $\overline{\mathbf{F}_p}$, the algebraic closure of $\mathbf{F}_p$. Again, $\mathbf{C}_p$ has characteristic zero and residue characteristic $p > 0$.

Example 3.4. Consider the field $\mathbf{C}$ of complex numbers (or any algebraically closed field of characteristic zero) *equipped with the trivial norm*. Let $K = \mathbf{C}((u))$ be the field of Laurent series with coefficients in $\mathbf{C}$. The norm $|\cdot|$ on K is given by $\log|\sum_{n\in\mathbf{Z}} a_n u^n| = -\min\{n \mid a_n \neq 0\}$. Then $\mathfrak{o}_K = \mathbf{C}[[u]]$, $\mathfrak{m}_K = u\mathfrak{o}_K$ and $\tilde{K} = \mathbf{C}$. We see that K is complete and of residue characteristic zero. However, it is not algebraically closed.

Example 3.5. Let $K = \mathbf{C}((u))$ be the field of Laurent series. By the Newton-Puiseux theorem, the algebraic closure K^a of K is the field of *Puiseux series*

$$a = \sum_{\beta\in B} a_\beta u^\beta, \tag{3.1}$$

where the sum is over a (countable) subset $B \subseteq \mathbf{Q}$ for which there exists $m, N \in \mathbf{N}$ (depending on a) such that $m + NB \subseteq \mathbf{N}$. This field is not complete; its completion $\widehat{K^a}$ is algebraically closed as well as complete. It has residue characteristic zero.

Example 3.6. A giant extension of $\mathbf{C}((u))$ is given by the field K consisting of series of the form (3.1), where B ranges over well-ordered subsets of $\mathbf{R}$. In this case, $|K^*| = \mathbf{R}^*$.

3.2. **The Berkovich affine line.** Write $R \simeq K[z]$ for the ring of polynomials in one variable with coefficients in K. The *affine line* $\mathbf{A}^1$ over K is the set of maximal ideals in R. Any choice of coordinate z (i.e. $R = K[z]$) defines an isomorphism $\mathbf{A}^1 \xrightarrow{\sim} K$. A (closed or open) disc in $\mathbf{A}^1$ is a disc in K under this isomorphism. This makes sense since any automorphism $z \mapsto az + b$ of K maps discs to discs. We can also talk about rational and irrational discs. However, the radius of a disc in $\mathbf{A}^1$ is not well defined.

Definition 3.7. The *Berkovich affine line* $\mathbf{A}^1_{\mathrm{Berk}} = \mathbf{A}^1_{\mathrm{Berk}}(K)$ is the set of multiplicative seminorms $|\cdot| : R \to \mathbf{R}_+$ whose restriction to the ground field $K \subseteq R$ is equal to the given norm $|\cdot|_K$.

Such a seminorm is necessarily non-Archimedean. Elements of $\mathbf{A}^1_{\mathrm{Berk}}$ are usually denoted x and the associated seminorm on R by $|\cdot|_x$. The topology on $\mathbf{A}^1_{\mathrm{Berk}}$ is the weakest topology in which all evaluation maps $x \mapsto |\phi|_x$, $\phi \in R$, are continuous. There is a natural partial ordering on $\mathbf{A}^1_{\mathrm{Berk}}$: $x \leq y$ iff $|\phi|_x \leq |\phi|_y$ for all $\phi \in R$.

3.3. **Classification of points.** One very nice feature of the Berkovich affine line is that we can completely and precisely classify its elements. The situation is typically much more complicated in higher dimensions. Following Berkovich [Ber90] we shall describe four types of points in $\mathbf{A}^1_{\mathrm{Berk}}$, then show that this list is in fact complete.

For simplicity we shall from now on and until §3.9 assume that K is *algebraically closed* and that the valuation on K is *nontrivial.* The situation when one or both of these conditions is not satisfied is discussed briefly in §3.9. See also §6.6 for a different presentation of the trivially valued case.

3.3.1. *Seminorms from points.* Any closed point $x \in \mathbf{A}^1$ defines a seminorm $|\cdot|_x$ on R through

$$|\phi|_x := |\phi(x)|.$$

This gives rise to an embedding $\mathbf{A}^1 \hookrightarrow \mathbf{A}^1_{\mathrm{Berk}}$. The images of this map will be called *classical points.*[10]

Remark 3.8. If we define $\mathbf{A}^1_{\mathrm{Berk}}$ as above when $K = \mathbf{C}$, then it follows from the Gel'fand-Mazur Theorem that all points are classical, that is, the map $\mathbf{A}^1 \to \mathbf{A}^1_{\mathrm{Berk}}$ is surjective. The non-Archimedean case is vastly different.

3.3.2. *Seminorms from discs.* Next, let $D \subseteq \mathbf{A}^1$ be a closed disc and define a seminorm $|\cdot|_D$ on R by

$$|\phi|_D := \max_{x \in D} |\phi(x)|.$$

It follows from Gauss' Lemma that this indeed defines a multiplicative seminorm on R. In fact, the maximum above is attained for a "generic" $x \in D$. We denote the corresponding element of $\mathbf{A}^1_{\mathrm{Berk}}$ by x_D. In the degenerate case $D = \{x\}$, $x \in \mathbf{A}^1$, this reduces to the previous construction: $x_D = x$.

3.3.3. *Seminorms from nested collections of discs.* It is clear from the construction that if D, D' are closed discs in $\mathbf{A}^1$, then

$$|\phi|_D \le |\phi|_{D'} \text{ for all } \phi \in R \text{ iff } D \subseteq D'. \tag{3.2}$$

Definition 3.9. A collection $\mathcal{E}$ of closed discs in $\mathbf{A}^1$ is *nested* if the following conditions are satisfied:

(a) if $D, D' \in \mathcal{E}$ then $D \subseteq D'$ or $D' \subseteq D$;
(b) if D and D' are closed discs in $\mathbf{A}^1$ with $D' \in \mathcal{E}$ and $D' \subseteq D$, then $D \in \mathcal{E}$;
(c) if $(D_n)_{n\ge1}$ is a decreasing sequence of discs in $\mathcal{E}$ whose intersection is a disc D in $\mathbf{A}^1$, then $D \in \mathcal{E}$.

In view of (3.2) we can associate a seminorm $x_{\mathcal{E}} \in \mathbf{A}^1_{\mathrm{Berk}}$ to a nested collection $\mathcal{E}$ of discs by

$$x_{\mathcal{E}} = \inf_{D \in \mathcal{E}} x_D;$$

indeed, the limit of an decreasing sequence of seminorms is a seminorm. When the intersection $\bigcap_{D\in\mathcal{E}} D$ is nonempty, it is a closed disc $D(\mathcal{E})$ (possibly of radius 0). In this case $x_{\mathcal{E}}$ is the seminorm associated to the disc $D(\mathcal{E})$. In general, however, the intersection above may be empty (the existence of a nested collection of discs with nonempty intersection is equivalent to the field K not being *spherically complete*).

[10] They are sometimes called *rigid points* as they are the points that show up rigid analytic geometry [BGR84].

The set of nested collections of discs is partially ordered by inclusion and we have $x_{\mathcal{E}} \le x_{\mathcal{E}'}$ iff $\mathcal{E}' \subseteq \mathcal{E}$.

3.3.4. *Classification.* Berkovich proved that all seminorms in $\mathbf{A}^1_{\mathrm{Berk}}$ arise from the construction above.

Theorem 3.10. *For any $x \in \mathbf{A}^1_{\mathrm{Berk}}$ there exists a unique nested collection $\mathcal{E}$ of discs in $\mathbf{A}^1$ such that $x = x_{\mathcal{E}}$. Moreover, the map $\mathcal{E} \to x_{\mathcal{E}}$ is an order-preserving isomorphism.*

Sketch of proof. The strategy is clear enough: given $x \in \mathbf{A}^1_{\mathrm{Berk}}$ define $\mathcal{E}(x)$ as the collection of discs D such that $x_D \ge x$. However, it requires a little work to show that the maps $\mathcal{E} \mapsto x_{\mathcal{E}}$ and $x \mapsto \mathcal{E}(x)$ are order-preserving and inverse one to another. Here we have to use the assumptions that K is algebraically closed and that the norm on K is nontrivial. The first assumption implies that x is uniquely determined by its values on *linear* polynomials in R. The second assumption is necessary to ensure surjectivity of $\mathcal{E} \mapsto x_{\mathcal{E}}$: if the norm on K is trivial, then there are too few discs in $\mathbf{A}^1$. See the proof of [BR10, Theorem 1.2] for details. □

3.3.5. *Tree structure.* Using the classification theorem above, we can already see that the Berkovich affine line is naturally a tree. Namely, let $\mathfrak{E}$ denote the set of nested collections of discs in $\mathbf{A}^1$. We also consider the empty collection as an element of $\mathfrak{E}$. It is then straightforward to verify that $\mathfrak{E}$, partially ordered by inclusion, is a rooted tree in the sense of §2.3. As a consequence, the set $\mathbf{A}^1_{\mathrm{Berk}} \cup \{\infty\}$ is a rooted metric tree. Here ∞ corresponds to the empty collection of discs in $\mathbf{A}^1$ and can be viewed as the function $|\cdot|_\infty : R \to [0, +\infty]$ given by $|\phi| = \infty$ for any nonconstant polynomial $\phi \in R$ and $|\cdot|_\infty = |\cdot|_K$ on K. Then $\mathbf{A}^1_{\mathrm{Berk}} \cup \{\infty\}$ is a rooted tree with the partial ordering $x \le x'$ iff $|\cdot|_x \ge |\cdot|_{x'}$ on R. See Figure 3.1.

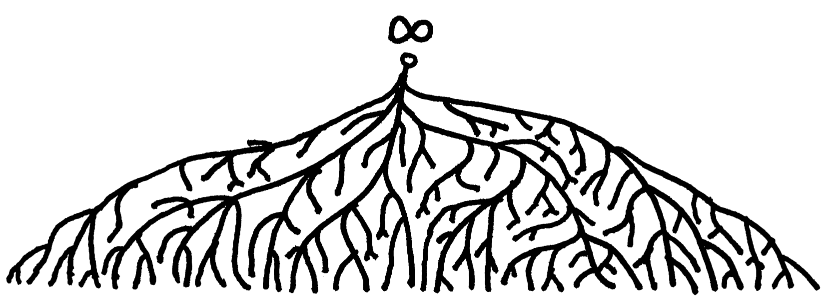


FIGURE 3.1. The Berkovich affine line.

3.3.6. *Types of points.* Using the identification with nested collections of discs, Berkovich classifies the points in $\mathbf{A}^1_{\mathrm{Berk}}$ as follows:

- a point of *Type 1* is a classical point, that is, a point in the image of the embedding $\mathbf{A}^1 \hookrightarrow \mathbf{A}^1_{\mathrm{Berk}}$;
- a point of *Type 2* is of the form x_D where D is a rational disc in $\mathbf{A}^1$;
- a point of *Type 3* is of the form x_D where D is an irrational disc in $\mathbf{A}^1$;

- a point of *Type 4* is of the form $x_{\mathcal{E}}$, where $\mathcal{E}$ is a nested collection of discs with empty intersection.

Note that Type 3 points exist iff $|K| \subsetneq \mathbf{R}_+$, while Type 4 points exist iff K is not spherically complete.

3.3.7. *Action by automorphisms.* Any automorphism $A \in \mathrm{Aut}(\mathbf{A}^1)$ arises from a K-algebra automorphism A^* of R, hence extends to an automorphism of $\mathbf{A}^1_{\mathrm{Berk}}$ by setting

$$|\phi|_{A(x)} := |A^*\phi|_x$$

for any polynomial $\phi \in R$. Note that A is order-preserving. If $\mathcal{E}$ is a nested collection of discs in $\mathbf{A}^1$, then so is $A(\mathcal{E})$ and $A(x_{\mathcal{E}}) = x_{A(\mathcal{E})}$. It follows that A preserves the type of a point in $\mathbf{A}^1_{\mathrm{Berk}}$.

Clearly $\mathrm{Aut}(\mathbf{A}^1)$ acts transitively on $\mathbf{A}^1$, hence on the Type 1 points in $\mathbf{A}^1_{\mathrm{Berk}}$. It also acts transitively on the rational discs in $\mathbf{A}^1$, hence the Type 2 points. In general, it will not act transitively on the set of Type 3 or Type 4 points, see §3.3.8.

3.3.8. *Coordinates, radii and the Gauss norm.* The description of $\mathbf{A}^1_{\mathrm{Berk}}$ above was coordinate independent. Now fix a coordinate $z : \mathbf{A}^1 \xrightarrow{\sim} K$. Using z, every disc $D \subseteq \mathbf{A}^1$ becomes a disc in K, hence has a well-defined *radius* $r_z(D)$. If D is a closed disc of radius $r = r_z(D)$ centered at point in $\mathbf{A}^1$ with coordinate $a \in K$, then

$$|z - b|_D = \max\{|a - b|, r\}. \tag{3.3}$$

We can also define the radius $r_z(\mathcal{E}) := \inf_{D\in\mathcal{E}} r_z(D)$ of a nested collection of discs. The completeness of K implies that if $r_z(\mathcal{E}) = 0$, then $\bigcap_{D\in\mathcal{E}} D$ is a point in $\mathbf{A}^1$.

The *Gauss norm* is the norm in $\mathbf{A}^1_{\mathrm{Berk}}$ defined by the unit disc in K. We emphasize that the Gauss norm depends on a choice of coordinate z. In fact, any Type 2 point is the Gauss norm in some coordinate.

The radius $r_z(D)$ of a disc depends on z. However, if we have two closed discs $D \subseteq D'$ in $\mathbf{A}^1$, then the ratio $r_z(D')/r_z(D)$ does *not* depend on z. Indeed, any other coordinate w is of the form $w = az+b$, with $a \in K^*$, $b \in K$ and so $r_w(D) = |a|r_z(D)$, $r_w(D') = |a|r_z(D')$. We think of the quantity $\log \frac{r_z(D')}{r_z(D)}$ as the modulus of the annulus $D' \setminus D$. It will play an important role in what follows.

In the same spirit, the class $[r_z(x)]$ of $r_z(x)$ in $\mathbf{R}^*_+/|K^*|$ does not depend on the choice of coordinate z. This implies that if $|K| \neq \mathbf{R}_+$, then $\mathrm{Aut}(\mathbf{A}^1)$ does not act transitively on Type 3 points. Indeed, if $|K| \neq \mathbf{R}_+$, then given any Type 3 point x we can find another Type 3 point $y \in [\infty, x]$ such that $[r_z(x)] \neq [r_z(y)]$. Then $A(x) \neq y$ for any $A \in \mathrm{Aut}(\mathbf{A}^1)$. The same argument shows that if K admits Type 4 points of any given radius, then A does not always act transitively on Type 4 points. For $K = \mathbf{C}_p$, there does indeed exist Type 4 points of any given radius, see [Rob00, p.143].

3.4. **The Berkovich projective line.** We can view the projective line $\mathbf{P}^1$ over K as the set of proper valuation rings A of F/K, where $F \simeq K(z)$ is the field of rational functions in one variable with coefficients in K. In other words, $A \subsetneq F$ is a subring containing K such that for every nonzero $\phi \in F$ we have $\phi \in A$ or $\phi^{-1} \in A$. Since $A \neq F$, there exists $z \in F \setminus A$ such that $F = K(z)$ and $z^{-1} \in A$. The other elements of $\mathbf{P}^1$ are then the localizations of the ring $R := K[z]$ at its maximal ideals. This gives rise to a decomposition $\mathbf{P}^1 = \mathbf{A}^1 \cup \{\infty\}$ in which A becomes the point $\infty \in \mathbf{P}^1$.

Given such a decomposition we define a *closed disc* in $\mathbf{P}^1$ to be a closed disc in $\mathbf{A}^1$, the singleton $\{\infty\}$, or the complement of an open disc in $\mathbf{A}^1$. Open discs are defined in the same way. A disc is *rational* if it comes from a rational disc in $\mathbf{A}^1$. These notions do not depend on the choice of point $\infty \in \mathbf{P}^1$.

Definition 3.11. The *Berkovich projective line* $\mathbf{P}^1_{\mathrm{Berk}}$ over K is the set of functions $|\cdot| : F \to [0, +\infty]$ extending the norm on $K \subseteq F$ and satisfying $|\phi+\psi| \le \max\{|\phi|, |\psi|\}$ for all $\phi, \psi \in F$, and $|\phi\psi| = |\phi||\psi|$ unless $|\phi| = 0$, $|\psi| = +\infty$ or $|\psi| = 0$, $|\phi| = +\infty$.

To understand this, pick a rational function $z \in F$ such that $F = K(z)$. Then $R := K[z]$ is the coordinate ring of $\mathbf{A}^1 := \mathbf{P}^1 \setminus \{z = \infty\}$. There are two cases. Either $|z| = \infty$, in which case $|\phi| = \infty$ for all nonconstant polynomials $\phi \in R$, or $|\cdot|$ is a seminorm on R, hence an element of $\mathbf{A}^1_{\mathrm{Berk}}$. Conversely, any element $x \in \mathbf{A}^1_{\mathrm{Berk}}$ defines an element of $\mathbf{P}^1_{\mathrm{Berk}}$ in the sense above. Indeed, every nonzero $\phi \in F$ is of the form $\phi = \phi_1/\phi_2$ with $\phi_1, \phi_2 \in R$ having no common factor. Then we can set $|\phi|_x := |\phi_1|_x/|\phi_2|_x$; this is well defined by the assumption on ϕ_1 and ϕ_2. Similarly, the function which is identically ∞ on all nonconstant polynomials defines a unique element of $\mathbf{P}^1_{\mathrm{Berk}}$: each $\phi \in F$ defines a rational function on $\mathbf{P}^1$ and $|\phi| := |\phi(\infty)| \in [0, +\infty]$. This leads to a decomposition

$$\mathbf{P}^1_{\mathrm{Berk}} = \mathbf{A}^1_{\mathrm{Berk}} \cup \{\infty\},$$

corresponding to the decomposition $\mathbf{P}^1 = \mathbf{A}^1 \cup \{\infty\}$.

We equip $\mathbf{P}^1_{\mathrm{Berk}}$ with the topology of pointwise convergence. By Tychonoff, $\mathbf{P}^1_{\mathrm{Berk}}$ is a compact Hausdorff space and as a consequence, $\mathbf{A}^1_{\mathrm{Berk}}$ is locally compact. The injection $\mathbf{A}^1 \hookrightarrow \mathbf{A}^1_{\mathrm{Berk}}$ extends to an injection $\mathbf{P}^1 \hookrightarrow \mathbf{P}^1_{\mathrm{Berk}}$ by associating the function $\infty \in \mathbf{P}^1_{\mathrm{Berk}}$ to the point $\infty \in \mathbf{P}^1$.

Any automorphism $A \in \mathrm{Aut}(\mathbf{P}^1)$ is given by an element $A^* \in \mathrm{Aut}(F/K)$. hence extends to an automorphism of $\mathbf{P}^1_{\mathrm{Berk}}$ by setting

$$|\phi|_{A(x)} := |A^*\phi|_x$$

for any rational function $\phi \in F$. As in the case of $\mathbf{A}^1_{\mathrm{Berk}}$, the type of a point is preserved. Further, $\mathrm{Aut}(\mathbf{P}^1)$ acts transitively on the set of Type 1 and Type 2 points, but not on the Type 3 or Type 4 points in general, see §3.3.8.

3.5. **Tree structure.** We now show that $\mathbf{P}^1_{\mathrm{Berk}}$ admits natural structures as a tree and a metric tree. See §2 for the relevant definitions.

Consider a decomposition $\mathbf{P}^1 = \mathbf{A}^1 \cup \{\infty\}$ and the corresponding decomposition $\mathbf{P}^1_{\mathrm{Berk}} = \mathbf{A}^1_{\mathrm{Berk}} \cup \{\infty\}$. The elements of $\mathbf{P}^1_{\mathrm{Berk}}$ define functions on the polynomial ring R with values in $[0, +\infty]$. This gives rise to a partial ordering on $\mathbf{P}^1_{\mathrm{Berk}}$: $x \le x'$ iff and only if $|\phi|_x \ge |\phi|_{x'}$ for all polynomials ϕ. As already observed in §3.3.5, $\mathbf{P}^1_{\mathrm{Berk}}$ then becomes a rooted tree in the sense of §2.3, with ∞ as its root. The partial ordering on $\mathbf{P}^1_{\mathrm{Berk}}$ depends on a choice of point $\infty \in \mathbf{P}^1$, but the associated (nonrooted) tree structure does not.

The ends of $\mathbf{P}^1_{\mathrm{Berk}}$ are the points of Type 1 and 4, whereas the branch points are the Type 2 points. See Figure 3.2.

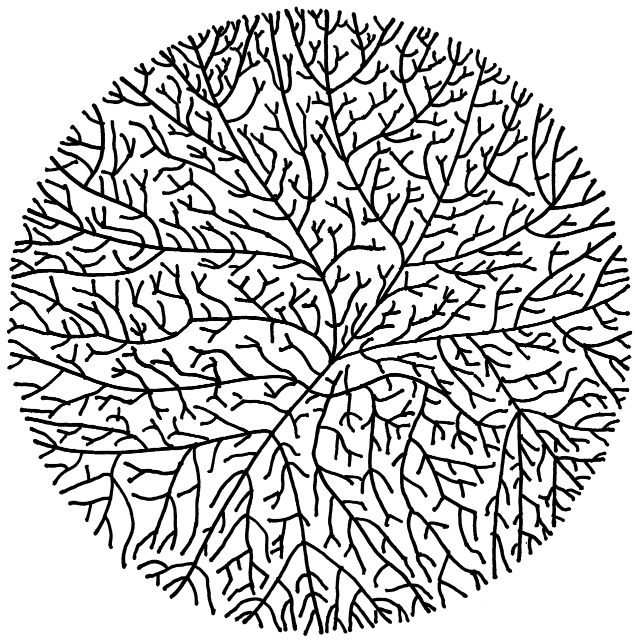


FIGURE 3.2. The Berkovich projective line.

Given a coordinate $z : \mathbf{A}^1 \xrightarrow{\sim} K$ we can parametrize $\mathbf{P}^1_{\mathrm{Berk}}$ rooted in ∞ using radii of discs. Instead of doing so literally, we define an decreasing parametrization $\alpha_z : \mathbf{P}^1_{\mathrm{Berk}} \to [-\infty, +\infty]$ using

$$\alpha_z(x_{\mathcal{E}}) := \log r_z(\mathcal{E}). \tag{3.4}$$

One checks that this is a parametrization in the sense of §2.3. The induced metric tree structure on $\mathbf{P}^1_{\mathrm{Berk}}$ does *not* depend on the choice of coordinate z and any automorphism of $\mathbf{P}^1$ induces an isometry of $\mathbf{P}^1_{\mathrm{Berk}}$ in this generalized metric. This is one reason for using the logarithm in (3.4). Another reason has to do with potential theory, see §3.6. Note that $\alpha_z(\infty) = \infty$ and $\alpha_z(x) = -\infty$ iff x is of Type 1.

The associated *hyperbolic space* in the sense of §2 is given by

$$\mathbf{H} := \mathbf{P}^1_{\mathrm{Berk}} \setminus \mathbf{P}^1.$$

The generalized metric on $\mathbf{P}^1_{\mathrm{Berk}}$ above induces a complete metric on $\mathbf{H}$ (in the usual sense). Any automorphism of $\mathbf{P}^1$ induces an isometry of $\mathbf{H}$.

3.6. **Topology and tree structure.** The topology on $\mathbf{P}^1_{\mathrm{Berk}}$ defined above agrees with the weak topology associated to the tree structure. To see this, note that $\mathbf{P}^1_{\mathrm{Berk}}$ is compact in both topologies. It therefore suffices to show that if $\vec{v}$ is a tree tangent direction $\vec{v}$ at a point $x \in \mathbf{P}^1_{\mathrm{Berk}}$, then the set $U(\vec{v})$ is open in the Berkovich topology. We may assume that x is of Type 2 or 3. In a suitable coordinate z, $x = x_{D(0,r)}$ and $\vec{v}$ is represented by the point x_0. Then $U(\vec{v}) = \{y \in \mathbf{P}^1_{\mathrm{Berk}} \mid |z|_x < r\}$, which is open in the Berkovich topology.

A *generalized open Berkovich disc* is a connected component of $\mathbf{P}^1_{\mathrm{Berk}} \setminus \{x\}$ for some $x \in \mathbf{P}^1_{\mathrm{Berk}}$. When x is of Type 2 or 3 we call it an *open Berkovich disc* and when x of Type 2 a *strict open Berkovich disc*. An (strict) *simple domain* is a finite intersection of (strict) open Berkovich discs. The collection of all (strict) simple domains is a basis for the topology on $\mathbf{P}^1_{\mathrm{Berk}}$.

3.7. **Potential theory.** As $\mathbf{P}^1_{\mathrm{Berk}}$ is a metric tree we can do potential theory on it, following §2.5. See also [BR10] for a comprehensive treatment, and the thesis of Thuillier [Thu05] for potential theory on general Berkovich analytic curves.

We shall not repeat the material in §2.5 here, but given a finite atomic probability measure ρ_0 on X with support on $\mathbf{H}$, we have a space $\mathrm{SH}(\mathbf{P}^1_{\mathrm{Berk}}, \rho_0)$ of ρ_0-subharmonic functions, as well as a homeomorphism

$$\rho_0 + \Delta : \mathrm{SH}(\mathbf{P}^1_{\mathrm{Berk}}, \rho_0)/\mathbf{R} \xrightarrow{\sim} \mathcal{M}^+_1(\mathbf{P}^1_{\mathrm{Berk}}).$$

Over the complex numbers, the analogue of $\mathrm{SH}(\mathbf{P}^1_{\mathrm{Berk}}, \rho_0)$ is the space $\mathrm{SH}(\mathbf{P}^1, \omega)$ of ω-subharmonic functions on $\mathbf{P}^1$, where ω is a Kähler form.

Lemma 3.12. *If* $\phi \in F\setminus\{0\}$ *is a rational function, then the function* $\log|\phi| : \mathbf{H} \to \mathbf{R}$ *is Lipschitz continuous with Lipschitz constant* $\deg(\phi)$*.*

Proof. Pick any coordinate z on $\mathbf{P}^1$ and write $\phi = \phi_1/\phi_2$, with ϕ_1, ϕ_2 polynomials. The functions $\log|\phi_1|$ and $\log|\phi_2|$ are decreasing in the partial ordering rooted at ∞ and $\log|\phi| = \log|\phi_1| - \log|\phi_2|$. Hence we may assume that ϕ is a polynomial. Using that K is algebraically closed we further reduce to the case $\phi = z - b$, where $b \in K$. But then the result follows from (3.3). □

Remark 3.13. The function $\log|\phi|$ belongs to the space $\mathrm{BDV}(\mathbf{P}^1_{\mathrm{Berk}})$ of functions of bounded differential variation and $\Delta \log|\phi|$ is the divisor of ϕ, viewed as a signed, finite atomic measure on $\mathbf{P}^1 \subseteq \mathbf{P}^1_{\mathrm{Berk}}$; see [BR10, Lemma 9.1]. Lemma 3.12 then also follows from a version of (2.7) for functions in $\mathrm{BDV}(\mathbf{P}^1_{\mathrm{Berk}})$. These considerations also show that the generalized metric on $\mathbf{P}^1_{\mathrm{Berk}}$ is the correct one from the point of potential theory.

3.8. **Structure sheaf and numerical invariants.** Above, we have defined the Berkovich projective line as a topological space, but it also an analytic space in the sense of Berkovich and carries a structure sheaf $\mathcal{O}$. The local rings $\mathcal{O}_x$ are useful for defining and studying the local degree of a rational map. They also allow us to recover Berkovich's classification via certain numerical invariants.

3.8.1. *Structure sheaf.* A holomorphic function on an open set $U \subseteq \mathbf{P}^1_{\mathrm{Berk}}$ is a locally uniform limit of rational functions without poles in U. To make sense of this, we first need to say where the holomorphic functions take their values: the value at a point $x \in \mathbf{P}^1_{\mathrm{Berk}}$ is in a non-Archimedean field $\mathcal{H}(x)$.

To define $\mathcal{H}(x)$, assume $x \in \mathbf{A}^1_{\mathrm{Berk}}$. The kernel of the seminorm $|\cdot|_x$ is a prime ideal in R and $|\cdot|_x$ defines a norm on the fraction field of $R/\ker(|\cdot|_x)$; the field $\mathcal{H}(x)$ is its completion.

When x is of Type 1, $\mathcal{H}(x) \simeq K$. If instead x is of Type 3, pick a coordinate $z \in R$ such that $r := |z|_x \notin |K|$. Then $\mathcal{H}(x)$ is isomorphic to the set of series $\sum_{-\infty}^{\infty} a_j z^j$ with $a_j \in K$ and $|a_j| r^j \to 0$ as $j \to \pm\infty$. For x of Type 2 or 4, I am not aware of a similar explicit description of $\mathcal{H}(x)$.

The pole set of a rational function $\phi \in F$ can be viewed as a set of Type 1 points in $\mathbf{P}^1_{\mathrm{Berk}}$. If x is not a pole of ϕ, then $\phi(x) \in \mathcal{H}(x)$ is well defined. The definition of a holomorphic function on an open subset $U \subseteq \mathbf{P}^1_{\mathrm{Berk}}$ now makes sense and gives rise to the structure sheaf $\mathcal{O}$.

3.8.2. *Local rings and residue fields.* The ring $\mathcal{O}_x$ for $x \in \mathbf{P}^1_{\mathrm{Berk}}$ is the ring of germs of holomorphic functions at x. Denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_x$ and by $\kappa(x) := \mathcal{O}_x/\mathfrak{m}_x$ the residue field. Note that the seminorm $|\cdot|_x$ on $\mathcal{O}_x$ induces a norm on $\kappa(x)$. The field $\mathcal{H}(x)$ above is the completion of $\kappa(x)$ with respect to the residue norm and is therefore called the completed residue field.

When x is of Type 1, $\mathcal{O}_x$ is isomorphic to the ring of power series $\sum_0^\infty a_j z^j$ such that $\limsup |a_j|^{1/j} < \infty$, and $\kappa(x) = \mathcal{H}(x) = K$.

If x is not of Type 1, then $\mathfrak{m}_x = 0$ and $\mathcal{O}_x = \kappa(x)$ is a field. This field is usually difficult to describe explicitly. However, when x is of Type 3 it has a description analogous to the one of its completion $\mathcal{H}(x)$ given above. Namely, pick a coordinate $z \in R$ such that $r := |z|_x \notin |K|$. Then $\mathcal{O}_x$ is isomorphic to the set of series $\sum_{-\infty}^\infty a_j z^j$ with $a_j \in K$ for which there exists $r' < r < r''$ such that $|a_j|(r'')^j, |a_{-j}|(r')^{-j} \to 0$ as $j \to +\infty$.

3.8.3. *Numerical invariants.* While the local rings $\mathcal{O}_x$ and the completed residue fields $\mathcal{H}(x)$ are not always easy to describe explicitly, certain numerical invariants of them are easily understood and allow us to recover Berkovich's classification.

First, x is of Type 1 iff the seminorm $|\cdot|_x$ has nontrivial kernel. Now suppose the kernel is trivial. Then $\mathcal{O}_x$ is a field and contains $F \simeq K(z)$ as a subfield. Both these fields are dense in $\mathcal{H}(x)$ with respect to the norm $|\cdot|_x$. In this situation we have two basic invariants.

First, the (additive) *value group* is defined by

$$\Gamma_x := \log|\mathcal{H}(x)^*|_x = \log|\mathcal{O}_x^*|_x = \log|F^*|_x.$$

This is an additive subgroup of $\mathbf{R}$ containing $\Gamma_K := \log|K^*|$. The *rational rank* $\mathrm{rat.rk}\, x$ of x is the dimension of the $\mathbf{Q}$-vector space $(\Gamma_x/\Gamma_K) \otimes_{\mathbf{Z}} \mathbf{Q}$.

Second, the three fields $\mathcal{H}(x)$, $\mathcal{O}_x$ and F have the same residue field with respect to the norm $|\cdot|_x$. We denote this field by $\widetilde{\mathcal{H}(x)}$; it contains the residue field $\tilde{K}$ of K as a subfield. The *transcendence degree* $\mathrm{tr.deg}\, x$ of x is the transcendence degree of the field extension $\widetilde{\mathcal{H}(x)}/\tilde{K}$.

One shows as in [BR10, Proposition 2.3] that

- if x is of Type 2, then $\mathrm{tr.deg}\, x = 1$ and $\mathrm{rat.rk}\, x = 0$; more precisely $\Gamma_x = \Gamma_K$ and $\widetilde{\mathcal{H}(x)} \simeq \tilde{K}(z)$;
- if x is of Type 3, then $\mathrm{tr.deg}\, x = 0$ and $\mathrm{rat.rk}\, x = 1$; more precisely, $\Gamma_x = \Gamma_K \oplus \mathbf{Z}\alpha$, where $\alpha \in \Gamma_x \setminus \Gamma_K$, and $\widetilde{\mathcal{H}(x)} \simeq \tilde{K}$;
- if x is of Type 4, then $\mathrm{tr.deg}\, x = 0$ and $\mathrm{rat.rk}\, x = 0$; more precisely, $\Gamma_x = \Gamma_K$ and $\widetilde{\mathcal{H}(x)} \simeq \tilde{K}$;

3.8.4. *Quasicompleteness of the residue field.* Berkovich proved in [Ber93, 2.3.3] that the residue field $\kappa(x)$ is *quasicomplete* in the sense that the induced norm $|\cdot|_x$ on

$\kappa(x)$ extends uniquely to any algebraic extension of $\kappa(x)$. This fact is true for any point of a "good" Berkovich space. It will be exploited (only) in §4.8.2.

3.8.5. *Weak stability of the residue field.* If x is of Type 2 or 3, then the residue field $\kappa(x) = \mathcal{O}_x$ is *weakly stable.* By definition [BGR84, 3.5.2/1] this means that any finite extension $L/\kappa(x)$ is *weakly Cartesian*, that is, there exists a linear homeomorphism $L \overset{\sim}{\to} \kappa(x)^n$, where $n = [L : \kappa(x)]$, see [BGR84, 2.3.2/4]. Here the norm on L is the unique extension of the norm on the quasicomplete field $\kappa(x)$. The homeomorphism above is not necessarily an isometry.

The only consequence of weak stability that we shall use is that if $L/\kappa(x)$ is a finite extension, then $[L : \kappa(x)] = [\hat{L} : \mathcal{H}(x)]$, where $\hat{L}$ denotes the completion of L, see [BGR84, 2.3.3/6]. This, in turn, will be used (only) in §4.8.2.

Let us sketch a proof that $\kappa(x) = \mathcal{O}_x$ is weakly stable when x is of Type 2 or 3. Using the remark at the end of [BGR84, 3.5.2] it suffices to show that the field extension $\mathcal{H}(x)/\mathcal{O}_x$ is separable. This is automatic if the ground field K has characteristic zero, so suppose K has characteristic $p > 0$. Pick a coordinate $z \in R$ such that x is associated to a disc centered at $0 \in K$. It is then not hard to see that $\mathcal{O}_x^{1/p} = \mathcal{O}_x[z^{1/p}]$ and it suffices to show that $z^{1/p} \notin \mathcal{H}(x)$. If x is of Type 3, then this follows from the fact that $\frac{1}{p}\log r = \log|z^{1/p}|_x \notin \Gamma_K + \mathbf{Z}\log r = \Gamma_x$. If instead x is of Type 2, then we may assume that x is the Gauss point with respect to the coordinate z. Then $\widetilde{\mathcal{H}(x)} \simeq \tilde{K}(z) \not\ni z^{1/p}$ and hence $z^{1/p} \notin \mathcal{H}(x)$.

3.8.6. *Stability of the completed residue field.* When x is a Type 2 or Type 3 point, the completed residue field $\mathcal{H}(x)$ is *stable* field in the sense of [BGR84, 3.6.1/1]. This means that any finite extension $L/\mathcal{H}(x)$ admits a basis $e_1, \dots, e_m$ such that $|\sum_i a_i e_i| = \max_i |a_i||e_i|$ for $a_i \in K$. Here the norm on L is the unique extension of the norm on the complete field $\mathcal{H}(x)$. The stability of $\mathcal{H}(x)$ is proved in [Tem10b, 6.3.6] (the case of a Type 2 point also follows from [BGR84, 5.3.2/1]).

Let x be of Type 2 or 3. The stability of $\mathcal{H}(x)$ implies that for any finite extension $L/\mathcal{H}(x)$ we have $[L : \mathcal{H}(x)] = [\Gamma_L : \Gamma_x] \cdot [\tilde{L} : \widetilde{\mathcal{H}(x)}]$, where Γ_L and $\tilde{L}$ are the value group and residue field of L, see [BGR84, 3.6.2/4].

3.8.7. *Tangent space and reduction map.* Fix $x \in \mathbf{P}^1_{\mathrm{Berk}}$. Using the tree structure, we define as in §2.1.1 the tangent space T_x of $\mathbf{P}^1_{\mathrm{Berk}}$ at x as well as a tautological "reduction" map from $\mathbf{P}^1_{\mathrm{Berk}} \setminus \{x\}$ onto T_x. Let us interpret this procedure algebraically in the case when x is a Type 2 point.

The tangent space T_x at a Type 2 point x is the set of valuation rings $A \subsetneq \tilde{H}(x)$ containing $\tilde{K}$. Fix a coordinate z such that x becomes the Gauss point. Then $\tilde{H}(x) \overset{\sim}{\to} \tilde{K}(z)$ and $T_x \simeq \mathbf{P}^1(\tilde{K})$. Let us define the reduction map r_x of $\mathbf{P}^1_{\mathrm{Berk}} \setminus \{x\}$ onto $T_x \simeq \mathbf{P}^1(\tilde{K})$. Pick a point $y \in \mathbf{P}^1_{\mathrm{Berk}} \setminus \{x\}$. If $|z|_y > 1$, then we declare $r_x(y) = \infty$. If $|z|_y \le 1$, then, since $y \neq x$, there exists $a \in \mathfrak{o}_K$ such that $|z - a|_y < 1$. The element a is not uniquely defined, but its class $\tilde{a} \in \tilde{K}$ is and we set $r_x(y) = \tilde{a}$. One can check that this definition does not depend on the choice of coordinate z and gives the same result as the tree-theoretic construction.

The reduction map can be naturally understood in the context of formal models, but we shall not discuss this here.

3.9. **Other ground fields.** Recall that from §3.4 onwards, we assumed that the field K was algebraically closed and nontrivially valued. These assumptions were used in the proof of Theorem 3.10. Let us briefly discuss what happens when they are removed.

As before, $\mathbf{A}^1_{\mathrm{Berk}}(K)$ is the set of multiplicative seminorms on $R \simeq K[z]$ extending the norm on K and $\mathbf{P}^1_{\mathrm{Berk}}(K) \simeq \mathbf{A}^1_{\mathrm{Berk}}(K) \cup \{\infty\}$. We can equip $\mathbf{A}^1_{\mathrm{Berk}}(K)$ and $\mathbf{P}^1_{\mathrm{Berk}}(K)$ with a partial ordering defined by $x \le x'$ iff $|\phi(x)| \ge |\phi(x')|$ for all polynomials $\phi \in R$.

3.9.1. *Non-algebraically closed fields.* First assume that K is nontrivially valued but not algebraically closed. Our discussion follows [Ber90, §4.2]; see also [Ked11b, §2.2], [Ked10, §5.1] and [Ked11a, §6.1].

Denote by K^a the algebraic closure of K and by $\widehat{K^a}$ its completion. Since K is complete, the norm on K has a unique extension to $\widehat{K^a}$.

The Galois group $G := \mathrm{Gal}(K^a/K)$ acts on $\widehat{K^a}$ and induces an action on $\mathbf{A}^1_{\mathrm{Berk}}(\widehat{K^a})$, which in turn extends to $\mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a}) = \mathbf{A}^1_{\mathrm{Berk}}(\widehat{K^a}) \cup \{\infty\}$ using $g(\infty) = \infty$ for all $g \in G$. It is a general fact that $\mathbf{P}^1_{\mathrm{Berk}}(K)$ is isomorphic to the quotient $\mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a})/G$. The quotient map $\pi : \mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a}) \to \mathbf{P}^1_{\mathrm{Berk}}(K)$ is continuous, open and surjective.

It is easy to see that g maps any segment $[x, \infty]$ homeomorphically onto the segment $[g(x), \infty]$. This implies that $\mathbf{P}^1_{\mathrm{Berk}}(K)$ is a tree in the sense of §2.1. In fact, the rooted tree structure on $\mathbf{P}^1_{\mathrm{Berk}}(K)$ is defined by the partial ordering above.

If $g \in G$ and $x \in \mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a})$, then x and $g(x)$ have the same type. This leads to a classification of points in $\mathbf{P}^1_{\mathrm{Berk}}(K)$ into Types 1-4. Note that since $\widehat{K^a} \ne K^a$ in general, there may exist Type 1 points $x \ne \infty$ such that $|\phi(x)| > 0$ for all polynomials $\phi \in R = K[z]$.

We can equip the Berkovich projective line $\mathbf{P}^1_{\mathrm{Berk}}(K)$ with a generalized metric. In fact, there are two natural ways of doing this. Fix a coordinate $z \in R$. Let $\hat{\alpha}_z : \mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a}) \to [-\infty, +\infty]$ be the parametrization defined in §3.5. It satisfies $\hat{\alpha}_z \circ g = \hat{\alpha}_z$ for all $g \in G$ and hence induces a parametrization $\hat{\alpha}_z : \mathbf{P}^1_{\mathrm{Berk}}(K) \to [-\infty, +\infty]$. The associated generalized metric on $\mathbf{P}^1_{\mathrm{Berk}}(K)$ does not depend on the choice of coordinate z and has the nice feature that the associated hyperbolic space consists exactly of points of Types 2–4.

However, for potential theoretic considerations, it is better to use a slightly different metric. For this, first define the *multiplicity*[11] $m(x) \in \mathbf{Z}_+ \cup \{\infty\}$ of a point $x \in \mathbf{P}^1_{\mathrm{Berk}}(K)$ as the number of preimages of x in $\mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a})$. The multiplicity of a Type 2 or Type 3 point is finite and if $x \le y$, then $m(x)$ divides $m(y)$. Note that $m(0) = 1$ so all points on the interval $[\infty, 0]$ have multiplicity 1. We now define a decreasing parametrization $\alpha_z : \mathbf{P}^1_{\mathrm{Berk}}(K) \to [-\infty, +\infty]$ as follows. Given $x \in \mathbf{P}^1_{\mathrm{Berk}}(K)$, set $x_0 := x \wedge 0$ and

$$\alpha_z(x) = \alpha_z(x_0) - \int_{x_0}^{x} \frac{1}{m(y)}\, d\hat{\alpha}_z(y) \tag{3.5}$$

[11] This differs from the "algebraic degree" used by Trucco, see [Tru09, Definition 5.1].

Again, the associated generalized metric on $\mathbf{P}^1_{\mathrm{Berk}}(K)$ does not depend on the choice of coordinate z. The hyperbolic space $\mathbf{H}$ now contains all points of Types 2–4 but may also contain some points of Type 1.

One nice feature of the generalized metric induced by α_z is that if ρ_0 is a finite positive measure on $\mathbf{P}^1_{\mathrm{Berk}}(K)$ supported on points of finite multiplicity and if $\varphi \in \mathrm{SH}(\mathbf{P}^1_{\mathrm{Berk}}(K), \rho_0)$, then $\pi^*\varphi \in \mathrm{QSH}(\mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a}))$ and

$$\Delta\varphi = \pi_*\Delta(\pi^*\varphi).$$

Furthermore, for any rational function $\phi \in F$, the measure $\Delta \log|\phi|$ on $\mathbf{P}^1_{\mathrm{Berk}}(K)$ can be identified with the divisor of ϕ, see Remark 3.13.

3.9.2. *Trivially valued fields.* Finally we discuss the case when K is trivially valued, adapting the methods above. A different approach is presented in §6.6.

First assume K is algebraically closed. Then a multiplicative seminorm on R is determined by its valued on linear polynomials. Given a coordinate $z \in R$ it is easy to see that any point $x \in \mathbf{A}^1_{\mathrm{Berk}}$ is of one of the following three types:

- we have $|z-a|_x = 1$ for all $a \in K$; this point x is the *Gauss point*;
- there exists a unique $a \in K$ such that $|z-a|_x < 1$;
- there exists $r > 1$ such that $|z-a|_x = r$ for all $a \in K$.

Thus we can view $\mathbf{A}^1_{\mathrm{Berk}}$ as the quotient $K \times [0,\infty[\,/\sim$, where $(a,r) \sim (b,s)$ iff $r = s$ and $|a-b| \le r$. Note that if $r \ge 1$, then $(a,r) \sim (b,r)$ for all r, whereas if $0 \le r < 1$, then $(a,r) \simeq (b,r)$ iff $a = b$.

We see that the Berkovich projective line $\mathbf{P}^1_{\mathrm{Berk}} = \mathbf{A}^1_{\mathrm{Berk}} \cup \{\infty\}$ is a tree naturally rooted at ∞ with the Gauss point as its only branch point. See Figure 3.3. The hyperbolic metric is induced by the parametrization $\alpha_z(a,r) = \log r$. In fact, this parametrization does not depend on the choice of coordinate $z \in R$.

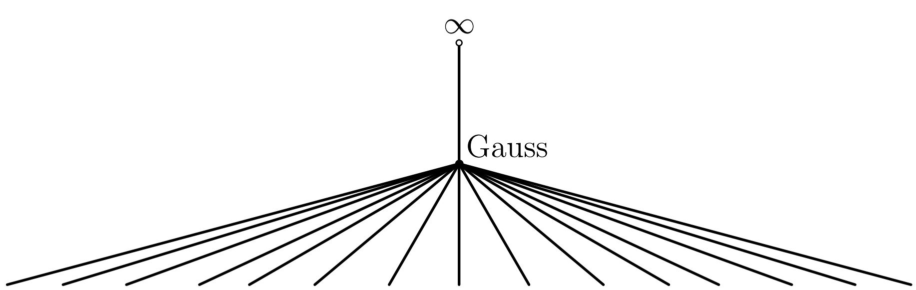


FIGURE 3.3. The Berkovich affine line over a trivially valued field.

If we instead choose the Gauss point as the root of the tree, then we can view the topological space underlying $\mathbf{P}^1_{\mathrm{Berk}}$ as the cone over $\mathbf{P}^1$, that is, as the quotient $\mathbf{P}^1 \times [0,\infty]$, where $(a,s) \sim (b,t)$ if $s = t = 0$. The Gauss point is the apex of the cone and its distance to (a,t) is t in the hyperbolic metric. See Figure 3.4.

Just as in the nontrivially valued case, the generalized metric on $\mathbf{P}^1_{\mathrm{Berk}}$ is the correct one in the sense that Remark 3.13 holds also in this case.

Following the terminology of §3.3.4, a point of the form (a,t) is of Type 1 and Type 2 iff $t = 0$ and $t = \infty$, respectively. All other points are of Type 3; there are no Type 4 points.

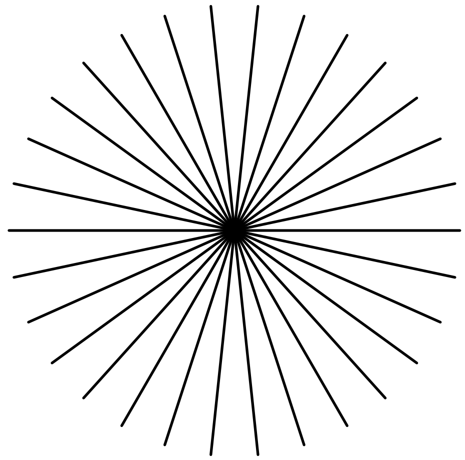


FIGURE 3.4. The Berkovich projective line over a trivially valued field.

We can also describe the structure sheaf $\mathcal{O}$. When x is the Gauss point, the local ring $\mathcal{O}_x$ is the field F of rational functions and $\mathcal{H}(x) = \mathcal{O}_x = F$ is equipped with the trivial norm. Further, $\Gamma_x = \Gamma_K = 0$, so $\operatorname{rat.rk} x = 0$ and $\operatorname{tr.deg} x = 1$.

Now assume $x \in \mathbf{P}^1_{\mathrm{Berk}}$ is not the Gauss point and pick a coordinate $z \in F$ such that $|z|_x < 1$. If x is of Type 3, that is, $0 < |z|_x < 1$, then $\mathcal{O}_x = K[[z]]$ is the ring of formal power series and $\mathcal{H}(x) = \mathcal{O}_x$ is equipped with the norm $|\sum_{j=0}^{\infty} a_j z^j|_x = r^{\max\{j|a_j \neq 0\}}$. Further, $\Gamma_x = \mathbf{Z} \log r$, so $\operatorname{rat.rk} x = 1$, $\operatorname{tr.deg} x = 0$.

If instead $|z|_x = 0$ so that x is of Type 1, then we still have $\mathcal{O}_x = K[[z]]$, whereas $\mathcal{H}(x) \simeq K$ is equipped with the trivial norm.

Finally, when K is not algebraically closed, we view $\mathbf{P}^1_{\mathrm{Berk}}(K)$ as a quotient of $\mathbf{P}^1_{\mathrm{Berk}}(K^a)$, where K^a is the algebraic closure of K (note that K^a is already complete in this case). We can still view the Berkovich projective line as the quotient $\mathbf{P}^1(K) \times [0, \infty]/\sim$, with $\mathbf{P}^1(K)$ the set of closed (but not necessarily K-rational) points of the projective line over K and where $(a, 0) \sim (b, 0)$ for all a, b. The multiplicity (i.e. the number of preimages in $\mathbf{P}^1_{\mathrm{Berk}}(K^a)$ of the Gauss point is 1 and the multiplicity of any point (a, t) is equal to the degree $[K(a) : K]$ if $t > 0$, where $K(a)$ is the residue field of a. We define a parametrization of $\mathbf{P}^1_{\mathrm{Berk}}(K)$ using (3.5). Then the result in Remark 3.13 remains valid.

3.10. **Notes and further references.** The construction of the Berkovich affine and projective lines is, of course, due to Berkovich and everything in this section is, at least implicitly, contained in his book [Ber90].

For general facts on Berkovich spaces we refer to the original works [Ber90, Ber93] or to some of the recent surveys, e.g. the ones by Conrad [Con08] and Temkin [Tem10a]. However, the affine and projective lines are very special cases of Berkovich spaces and in fact we need very little of the general theory in order to understand them. I can offer a personal testimony to this fact as I was doing dynamics on Berkovich spaces before I even knew what a Berkovich space was!

Having said that, it is sometimes advantageous to use some general notions, and in particular the structure sheaf, which will be used to define the local degree of a rational map in §4.6. Further, the stability of the residue field at Type 2 and 3 points is quite useful. In higher dimensions, simple arguments using the tree structure are probably less useful than in dimension 1.

The Berkovich affine and projective lines are studied in great detail in the book [BR10] by Baker and Rumely, to which we refer for more details. However, our presentation here is slightly different and adapted to our purposes. In particular, we insist on trying to work in a coordinate free way whenever possible. For example, the Berkovich unit disc and its associated Gauss norm play an important role in most descriptions of the Berkovich projective line, but they are only defined once we have chosen a coordinate; without coordinates all Type 2 points are equivalent. When studying the dynamics of rational maps, there is usually no canonical choice of coordinate and hence no natural Gauss point (the one exception being maps of simple reduction, see §5.5).

One thing that we do not talk about at all are formal models. They constitute a powerful geometric tool for studying Berkovich spaces, see [Ber99, Ber04] but we do not need them here. However, the corresponding notion for trivially valued fields is used systematically in §§6-10.

4. Action by polynomial and rational maps

We now study how a polynomial or a rational map acts on the Berkovich affine and projective lines, respectively. Much of the material in this chapter can be found with details in the literature. However, as a general rule our presentation is self-contained, the exception being when we draw more heavily on the general theory of Berkovich spaces or non-Archimedean geometry. As before, we strive to work in a coordinate free way whenever possible.

Recall that over the complex numbers, the projective line $\mathbf{P}^1$ is topologically a sphere. Globally a rational map $f : \mathbf{P}^1 \to \mathbf{P}^1$ is a branched covering. Locally it is of the form $z \mapsto z^m$, where $m \geq 1$ is the local degree of f at the point. In fact, $m = 1$ outside the ramification locus of f, which is a finite set.

The non-Archimedean case is superficially very different but in fact exhibits many of the same properties when correctly translated. The projective line is a tree and a rational map is a tree map in the sense of §2.6. Furthermore, there is a natural notion of local degree that we shall explore in some detail. The ramification locus can be quite large and has been studied in detail by Faber [Fab11a, Fab11b, Fab11c]. Finally, it is possible to give local normal forms, at least at points of Types 1-3.

4.1. **Setup.**

As before, K is a non-Archimedean field. We assume that the norm on K is non-trivial and that K is algebraically closed but of arbitrary characteristic. See §4.11 for extensions.

Recall the notation $R \simeq K[z]$ for the polynomial ring in one variable with coefficients in K, and $F \simeq K(z)$ for its fraction field.

4.2. **Polynomial and rational maps.**

We start by recalling some general algebraic facts about polynomial and rational maps. The material in §4.2.3–§4.2.5 is interesting mainly when the ground field K has positive characteristic. General references for that part are [Lan02, VII.7] and [Har77, IV.2].

4.2.1. *Polynomial maps.* A nonconstant polynomial map $f : \mathbf{A}^1 \to \mathbf{A}^1$ of the affine line over K is given by an injective K-algebra homomorphism $f^* : R \to R$. The *degree* $\deg f$ of f is the dimension of R as a vector space over f^*R. Given coordinates $z, w \in R$ on $\mathbf{A}^1$, f^*w is a polynomial in z of degree $\deg f$.

4.2.2. *Rational maps.* A nonconstant regular map $f : \mathbf{P}^1 \to \mathbf{P}^1$ of the projective line over K is is defined by an injective homomorphism $f^* : F \to F$ of fields over K, where $F \simeq K(z)$ is the fraction field of R. The degree of f is the degree of the field extension F/f^*F. Given coordinates $z, w \in F$ on $\mathbf{P}^1$, f^*w is a rational function of z of degree $d := \deg f$, that is, $f^*w = \phi/\psi$, where $\phi, \psi \in K[z]$ are polynomials without common factor and $\max\{\deg \phi, \deg \psi\} = d$. Thus we refer to f as a rational map, even though it is of course regular.

Any polynomial map $f : \mathbf{A}^1 \to \mathbf{A}^1$ extends to a rational map $f : \mathbf{P}^1 \to \mathbf{P}^1$ satisfying $f(\infty) = \infty$. In fact, polynomial maps can be identified with rational maps $f : \mathbf{P}^1 \to \mathbf{P}^1$ admitting a totally invariant point $\infty = f^{-1}(\infty)$.

4.2.3. *Separable maps.* We say that a rational map f is *separable* if the field extension F/f^*F is separable, see [Lan02, VII.4]. This is always the case if K has characteristic zero.

If f is separable, of degree d, then, by the Riemann-Hurwitz Theorem [Har77, IV.2/4] the *ramification divisor* R_f on $\mathbf{P}^1$ is well defined and of degree $2d-2$. In particular, all but finitely many points of $\mathbf{P}^1$ have exactly d preimages under f, so f has *topological degree* d.

4.2.4. *Purely inseparable maps.* We say that a rational map f is *purely inseparable* if the field extension F/f^*F is purely inseparable. Assuming $\deg f > 1$, this can only happen when K has characteristic $p > 0$ and means that for every $\phi \in F$ there exists $n \geq 0$ such that $\phi^{p^n} \in f^*F$, see [Lan02, VII.7]. Any purely inseparable map $f : \mathbf{P}^1 \to \mathbf{P}^1$ is bijective. We shall see in §5.3 that if f is purely inseparable of degree $d > 1$, then $d = p^n$ for some $n \geq 1$ and there exists a coordinate $z \in F$ on $\mathbf{P}^1$ such that $f^*z = z^d$.

4.2.5. *Decomposition.* In general, any algebraic field extension can be decomposed into a separable extension followed by a purely inseparable extension, see [Lan02, VII.7]. As a consequence, any rational map f can be factored as $f = g \circ h$, where g is separable and h is purely inseparable. The topological degree of f is equal to the degree of g or, equivalently, the separable degree of the field extension F/f^*F, see [Lan02, VII.4].

4.2.6. *Totally ramified points.* We say that a rational map $f : \mathbf{P}^1 \to \mathbf{P}^1$ is *totally ramified* at a point $x \in \mathbf{P}^1$ if $f^{-1}(f(x)) = \{x\}$.

Proposition 4.1. *Let $f : \mathbf{P}^1 \to \mathbf{P}^1$ be a rational map of degree $d > 1$.*

(i) *If f is purely inseparable, then f is totally ramified at every point $x \in \mathbf{P}^1$.*
(ii) *If f is not purely inseparable, then there are at most two points at which f is totally ramified.*

Proof. If f is purely inseparable, then $f : \mathbf{P}^1 \to \mathbf{P}^1$ is bijective and hence totally ramified at every point.

Now suppose f is not purely inseparable. Then $f = g \circ h$, where h is purely inseparable and g is separable, of degree $\deg g > 1$. If f is totally ramified at x, then so is g, so we may assume f is separable. In this case, a direct calculation shows that the ramification divisor has order $d-1$ at x. The result follows since the ramification divisor has degree $2(d-1)$. □

4.3. **Action on the Berkovich space.** Recall that the affine and projective line $\mathbf{A}^1$ and $\mathbf{P}^1$ embed in the corresponding Berkovich spaces $\mathbf{A}^1_{\mathrm{Berk}}$ and $\mathbf{P}^1_{\mathrm{Berk}}$, respectively.

4.3.1. *Polynomial maps.* Any nonconstant polynomial map $f : \mathbf{A}^1 \to \mathbf{A}^1$ extends to

$$f : \mathbf{A}^1_{\mathrm{Berk}} \to \mathbf{A}^1_{\mathrm{Berk}}$$

as follows. If $x \in \mathbf{A}^1_{\mathrm{Berk}}$, then $x' = f(x)$ is the multiplicative seminorm $|\cdot|_{x'}$ on R defined by

$$|\phi|_{x'} := |f^*\phi|_x.$$

It is clear that $f : \mathbf{A}^1_{\mathrm{Berk}} \to \mathbf{A}^1_{\mathrm{Berk}}$ is continuous, as the topology on $\mathbf{A}^1_{\mathrm{Berk}}$ was defined in terms of pointwise convergence. Further, f is order-preserving in the partial ordering on $\mathbf{A}^1_{\mathrm{Berk}}$ given by $x \le x'$ iff $|\phi|_x \le |\phi|_{x'}$ for all polynomials ϕ.

4.3.2. *Rational maps.* Similarly, we can extend any nonconstant rational map $f : \mathbf{P}^1 \to \mathbf{P}^1$ to a map

$$f : \mathbf{P}^1_{\mathrm{Berk}} \to \mathbf{P}^1_{\mathrm{Berk}}.$$

Recall that we defined $\mathbf{P}^1_{\mathrm{Berk}}$ as the set of generalized seminorms $|\cdot| : F \to [0, +\infty]$. If $x \in \mathbf{P}^1_{\mathrm{Berk}}$, then the value of the seminorm $|\cdot|_{f(x)}$ on a rational function $\phi \in F$ is given by

$$|\phi|_{f(x)} := |f^*\phi|_x.$$

On the Berkovich projective line $\mathbf{P}^1_{\mathrm{Berk}}$ there is no canonical partial ordering, so in general it does not make sense to expect f to be order preserving. The one exception to this is when there exist points $x, x' \in \mathbf{P}^1_{\mathrm{Berk}}$ such that $f^{-1}(x') = \{x\}$. In this case one can show that $f : \mathbf{P}^1_{\mathrm{Berk}} \to \mathbf{P}^1_{\mathrm{Berk}}$ becomes order preserving when the source and target spaces are equipped with the partial orderings rooted in x and x'. If x and x' are both of Type 2, we can find coordinates on the source and target in which x and x' are both equal to the Gauss point, in which case one says that f has *good reduction*, see §5.5.

4.4. **Preservation of type.** There are many ways of analyzing the mapping properties of a rational map $f : \mathbf{P}^1_{\mathrm{Berk}} \to \mathbf{P}^1_{\mathrm{Berk}}$. First we show the type of a point is invariant under f. For this, we use the numerical classification in §3.8.3.

Lemma 4.2. *The map $f : \mathbf{P}^1_{\mathrm{Berk}} \to \mathbf{P}^1_{\mathrm{Berk}}$ sends a point of Type 1-4 to a point of the same type.*

Proof. We follow the proof of [BR10, Proposition 2.15]. Fix $x \in \mathbf{P}^1_{\mathrm{Berk}}$ and write $x' = f(x)$.

If $|\cdot|_{x'}$ has nontrivial kernel, then clearly so does $|\cdot|_x$ and it is not hard to prove the converse, using that K is algebraically closed.

Now suppose $|\cdot|_x$ and $|\cdot|_{x'}$ have trivial kernels. In this case, the value group $\Gamma_{x'}$ is a subgroup of Γ_x of finite index. As a consequence, x and x' have the same rational rank. Similarly, $\widetilde{\mathcal{H}(x)}/\widetilde{\mathcal{H}(x')}$ is a finite field extension, so x and x' have the same transcendence degree. In view of the numerical classification, x and x' must have the same type. □

4.5. **Topological properties.** Next we explore the basic topological properties of a rational map.

Proposition 4.3. *The map $f : \mathbf{P}^1_{\mathrm{Berk}} \to \mathbf{P}^1_{\mathrm{Berk}}$ is continuous, finite, open and surjective. Any point in $\mathbf{P}^1_{\mathrm{Berk}}$ has at least one and at most d preimages, where $d = \deg f$.*

We shall see shortly that any point has *exactly* d preimages, counted with multiplicity. However, note that for a purely inseparable map, this multiplicity is equal to $\deg f$ at every point.

Proof. All the properties follow easily from more general results in [Ber90, Ber93], but we recall the proof from [FR10, p.126].

Continuity of f is clear from the definition, as is the fact that a point of Type 1 has at least one and at most d preimages. A point in $\mathbf{H} = \mathbf{P}^1_{\mathrm{Berk}} \setminus \mathbf{P}^1$ defines a norm on F, hence also on the subfield f^*F. The field extension F/f^*F has degree d, so by [ZS75] a valuation on f^*F has at least one and at most d extensions to F. This means that a point in $\mathbf{H}$ also has at least one and at most d preimages.

In particular, f is finite and surjective. By general results about morphisms of Berkovich spaces, this implies that f is open, see [Ber90, 3.2.4]. □

Since $\mathbf{P}^1_{\mathrm{Berk}}$ is a tree, Proposition 4.3 shows that all the results of §2.6 apply and give rather strong information on the topological properties of f.

One should note, however, that these purely topological results seem very hard to replicated for Berkovich spaces of higher dimensions. The situation over the complex numbers is similar, where the one-dimensional and higher-dimensional analyses are quite different.

4.6. **Local degree.** It is reasonable to expect that any point in $\mathbf{P}^1_{\mathrm{Berk}}$ should have exactly $d = \deg f$ preimages under f counted with multiplicity. This is indeed true, the only problem being to define this multiplicity. There are several (equivalent) definitions in the literature. Here we shall give the one spelled out by Favre and Rivera-Letelier [FR10], but also used by Thuillier [Thu05]. It is the direct translation of the corresponding notion in algebraic geometry.

Fix a point $x \in \mathbf{P}^1_{\mathrm{Berk}}$ and write $x' = f(x)$. Let $\mathfrak{m}_x$ be the maximal ideal in the local ring $\mathcal{O}_x$ and $\kappa(x) := \mathcal{O}_x/\mathfrak{m}_x$ the residue field. Using f we can $\mathcal{O}_x$ as an $\mathcal{O}_{x'}$-module and $\mathcal{O}_x/\mathfrak{m}_{x'}\mathcal{O}_x$ as a $\kappa(x')$-vector space.

Definition 4.4. The local degree of f at x is $\deg_x f = \dim_{\kappa(x')}(\mathcal{O}_x/\mathfrak{m}_{x'}\mathcal{O}_x)$.

Alternatively, since f is finite, it follows [Ber90, 3.1.6] that $\mathcal{O}_x$ is a *finite* $\mathcal{O}_{x'}$-module. The local degree $\deg_x f$ is therefore also equal to the rank of the module $\mathcal{O}_x$ viewed as $\mathcal{O}_{x'}$-module, see [Mat89, Theorem 2.3]. From this remark it follows that if $f, g : \mathbf{P}^1 \to \mathbf{P}^1$ are nonconstant rational maps, then

$$\deg_x(f \circ g) = \deg_x g \cdot \deg_{g(x)} f$$

for any $x \in \mathbf{P}^1_{\mathrm{Berk}}$.

The definition above of the local degree works also over the complex numbers. A difficulty in the non-Archimedean setting is that the local rings $\mathcal{O}_x$ are not as concrete as in the complex case, where they are isomorphic to the ring of convergent power series.

The following result shows that that local degree behaves as one would expect from the complex case. See [FR10, Proposition-Definition 2.1].

Proposition 4.5. *For every simple domain V and every connected component U of $f^{-1}(V)$, the integer*

$$\sum_{f(y)=x, y \in U} \deg_y f \tag{4.1}$$

is independent of the point $x \in V$.

Recall that a simple domain is a finite intersection of open Berkovich discs; see §3.6. The integer in (4.1) should be interpreted as the degree of the map from U to V. If we put $U = V = \mathbf{P}^1_{\mathrm{Berk}}$, then this degree is d.

We refer to [FR10, p.126] for a proof. The idea is to view $f : U \to V$ as a map between Berkovich analytic curves. In fact, this is one of the few places in these notes where we draw more heavily on the general theory of Berkovich spaces.

We would like to give a more concrete interpretation of the local degree. First, at a Type 1 point, it can be read off from a local expansion of f:

Proposition 4.6. *Let $x \in \mathbf{P}^1_{\mathrm{Berk}}$ be a Type 1 point and pick coordinates z, w on $\mathbf{P}^1$ such that $x = f(x) = 0$. Then $\mathcal{O}_x \simeq K\{z\}$, $\mathcal{O}_{f(x)} = K\{w\}$ and we have*

$$f^*w = az^k(1 + h(z)), \tag{4.2}$$

where $a \neq 0$, $k = \deg_x(f)$ and $h(0) = 0$.

Proof. The only thing that needs to be checked is that $k = \deg_x(f)$. We may assume $a = 1$. First suppose $\operatorname{char} K = 0$. Then we can find $\phi(z) \in K\{z\}$ such that $1+h(z) = (1+\phi(z))^k$ in $K\{z\}$. It is now clear that $\mathcal{O}_x \sim K\{z\}$ is a free module over $f^*\mathcal{O}_{f(x)}$ of rank k, with basis given by $(z(1+\phi(z)))^j$, $0 \leq j \leq k-1$, so $\deg_x(f) = k$. A similar argument can be used the case when K has characteristic $p > 0$; we refer to [FR10, p.126] for the proof. □

We shall later see how the local degree at a Type 2 or Type 3 points also appears in a suitable local expansion of f.

The following crucial result allows us to interpret the local degree quite concretely as a local expansion factor in the hyperbolic metric.

Theorem 4.7. *Let $f : \mathbf{P}^1_{\mathrm{Berk}} \to \mathbf{P}^1_{\mathrm{Berk}}$ be as above.*

(i) *If x is a point of Type 1 or 4 and $\gamma = [x, y]$ is a sufficiently small segment, then f maps γ homeomorphically onto $f(\gamma)$ and expands the hyperbolic metric on γ by a factor* $\deg_x(f)$.

(ii) *If x is a point of Type 3 and γ is a sufficiently small segment containing x in its interior, then f maps γ homeomorphically onto $f(\gamma)$ and expands the hyperbolic metric on γ by a factor* $\deg_x(f)$.

(iii) *If x is a point of Type 2, then for every tangent direction $\vec{v}$ at x there exists an integer $m_{\vec{v}}(f)$ such that the following holds:*
   (a) *for any sufficiently small segment $\gamma = [x, y]$ representing $\vec{v}$, f maps γ homeomorphically onto $f(\gamma)$ and expands the hyperbolic metric on γ by a factor $m_{\vec{v}}(f)$;*
   (b) *if $\vec{v}$ is any tangent direction at x and $\vec{v}_1, \dots, \vec{v}_m$ are the preimages of $\vec{v}$ under the tangent map, then $\sum_i m_{\vec{v}_i}(f) = \deg_x(f)$.*

Theorem 4.7 is due to Rivera-Letelier [Riv05, Proposition 3.1] (see also [BR10, Theorem 9.26]). However, in these references, different (but equivalent) definitions of local degree were used. In §4.8 below we will indicate a direct proof of Theorem 4.7 using the above definition of the local degree.

Since the local degree is bounded by the algebraic degree, we obtain as an immediate consequence

Corollary 4.8. *If $f : \mathbf{P}^1_{\mathrm{Berk}} \to \mathbf{P}^1_{\mathrm{Berk}}$ is as above, then*

$$d_{\mathbf{H}}(f(x), f(y)) \le \deg f \cdot d_{\mathbf{H}}(x, y)$$

for all $x, y \in \mathbf{H}$.

Using Theorem 4.7 we can also make Corollary 2.16 more precise:

Corollary 4.9. *Let $\gamma \subseteq \mathbf{P}^1_{\mathrm{Berk}}$ be a segment such that the local degree is constant on the interior of γ. Then f maps γ homeomorphically onto $\gamma' := f(\gamma)$.*

Proof. By Corollary 2.16 the first assertion is a local statement: it suffices to prove that if x belongs to the interior of γ then the tangent map of f is injective on the set of tangent directions at x defined by γ. But if this were not the case, the local degree at x would be too high in view of assertion (iii) (b) in Theorem 4.7. □

Remark 4.10. Using similar arguments, Rivera-Letelier was able to improve Proposition 2.12 and describe $f(U)$ for a simple domain U. For example, he described when the image of an open disc is an open disc as opposed to all of $\mathbf{P}^1_{\mathrm{Berk}}$ and similarly described the image of an annulus. See Theorems 9.42 and 9.46 in [BR10] and also the original papers [Riv03a, Riv03b].

4.7. **Ramification locus.** Recall that over the complex numbers, a rational map has local degree 1 except at finitely many points. In the non-Archimedean setting, the situation is more subtle.

Definition 4.11. The *ramification locus* R_f of f is the set of $x \in \mathbf{P}^1_{\mathrm{Berk}}$ such that $\deg_x(f) > 1$. We say that f is *tame*[12] if R_f is contained in the convex hull of a finite subset of $\mathbf{P}^1$.

Lemma 4.12. *If K has residue characteristic zero, then f is tame. More precisely, R_f is a finite union of segments in $\mathbf{P}^1_{\mathrm{Berk}}$ and is contained in the convex hull of the critical set of $f : \mathbf{P}^1 \to \mathbf{P}^1$. As a consequence, the local degree is one at all Type 4 points.*

We will not prove this lemma here. Instead we refer to the papers [Fab11a, Fab11b] by X. Faber for a detailed analysis of the ramification locus, including the case of positive residue characteristic. The main reason why the zero residue characteristic case is easier stems from the following version of Rolle's Theorem (see e.g. [BR10, Proposition A.20]): if $\mathrm{char}\, \tilde{K} = 0$ and $D \subseteq \mathbf{P}^1$ is an open disc such that $f(D) \ne \mathbf{P}^1$ and f is not injective on D, then f has a critical point in D.

Se §4.10 below for some examples of ramification loci.

4.8. **Proof of Theorem 4.7.** While several proofs of Theorem 4.7 exist in the literature, I am not aware of any that directly uses Definition 4.4 of the local degree. Instead, they use different definitions, which in view of Proposition 4.5 are equivalent to the one we use. Our proof of Theorem 4.7 uses some basic non-Archimedean analysis in the spirit of [BGR84].

[12] The terminology "tame" follows Trucco [Tru09].

4.8.1. *Type 1 points.* First suppose $x \in \mathbf{P}^1$ is a classical point. As in the proof of Proposition 4.6, we find coordinates z and w on $\mathbf{P}^1$ vanishing at x and x', respectively, such that $f^*w = az^k(1 + h(z))$, where $a \neq 0$, $k = \deg_x(f) \geq 1$ and $h(0) = 0$. In fact, we may assume $a = 1$. Pick $r_0 > 0$ so small that $|h(z)|_{D(0,r)} < 1$ for $r \leq r_0$. It then follows easily that $f(x_{D(0,r)}) = x_{D(0,r^k)}$ for $0 \leq r \leq r_0$. Thus f maps the segment $[x_0, x_{D(0,r_0)}]$ homeomorphically onto the segment $[x_0, x_{D(0,r_0^k)}]$ and the hyperbolic metric is expanded by a factor k.

4.8.2. *Completion.* Suppose x is of Type 2 or 3. Then the seminorm $|\cdot|_x$ is a norm, $\mathcal{O}_x$ is a field having $\mathcal{O}_{x'}$ as a subfield and $\deg_x(f)$ is the degree $[\mathcal{O}_x : \mathcal{O}_{x'}]$ of the field extension $\mathcal{O}_x/\mathcal{O}_{x'}$. Recall that $\mathcal{H}(x)$ is the completion of $\mathcal{O}_x$.

In general, the degree of a field extension can change when passing to the completion. However, we have

Proposition 4.13. *For any point $x \in \mathbf{P}^1_{\mathrm{Berk}}$ of Type 2 or 3 we have*

$$\deg_x(f) = [\mathcal{O}_x : \mathcal{O}_{x'}] = [\mathcal{H}(x) : \mathcal{H}(x')] = [\Gamma_x : \Gamma_{x'}] \cdot [\widetilde{\mathcal{H}(x)} : \widetilde{\mathcal{H}(x')}], \tag{4.3}$$

where Γ and $\widetilde{H}$ denotes the value groups and residue fields of the norms under consideration.

Proof. Recall from §3.8.4 that the field $\mathcal{O}_{x'}$ is quasicomplete in the sense that the norm $|\cdot|_{x'}$ on $\mathcal{O}_{x'}$ extends uniquely to any algebraic extension. In particular, the norm $|\cdot|_x$ is the unique extension of this norm to $\mathcal{O}_x$. Also recall from §3.8.5 that the field $\mathcal{O}_{x'}$ is weakly stable. Thus $\mathcal{O}_x$ is weakly Cartesian over $\mathcal{O}_{x'}$, which by [BGR84, 2.3.3/6] implies the second equality in (4.3).

Finally recall from §3.8.6 that the field $\mathcal{H}(x')$ is stable. The third equality in (4.3) then follows from [BGR84, 3.6.2/4]. □

4.8.3. *Approximation.* In order to understand the local degree of a rational map, it is useful to simplify the map in a way similar to (4.2). Suppose x and $x' = f(x)$ are Type 2 or Type 3 points. In suitable coordinates on the source and target, we can write $x = x_{D(0,r)}$ and $x' = x_{D(0,r')}$, where $0 < r, r' \leq 1$. If x and x' are Type 2 points, we can further assume $r = r' = 1$.

Write $f^*w = f(z)$ for some rational function $f(z) \in F \simeq K(z)$. Suppose we can find a decomposition in F of the form

$$f(z) = g(z)(1 + h(z)), \quad \text{where } |h(z)|_x < 1.$$

The rational function $g(z) \in F$ induces a rational map $g : \mathbf{P}^1 \to \mathbf{P}^1$, which extends to $g : \mathbf{P}^1_{\mathrm{Berk}} \to \mathbf{P}^1_{\mathrm{Berk}}$.

Lemma 4.14. *There exists $\delta > 0$ such that $g(y) = f(y)$ and $\deg_y(g) = \deg_y(f)$ for all $y \in \mathbf{H}$ with $d_{\mathbf{H}}(y, x) \leq \delta$.*

Proof. We may assume that $h(z) \not\equiv 0$, or else there is nothing to prove. Thus we have $|h(z)|_x > 0$. Pick $0 < \varepsilon < 1$ such that $|h(z)|_x \leq \varepsilon^3$, set

$$\delta = (1 - \varepsilon) \min\left\{\frac{|h(z)|_x}{\deg h(z)}, \frac{r'}{2 \deg f}\right\}$$

and assume $d_{\mathbf{H}}(y,x)\le\delta$. We claim that

$$|f^*\phi-g^*\phi|_y\le\varepsilon|f^*\phi|_y\quad\text{for all }\phi\in F.\tag{4.4}$$

Granting (4.4), we get $|g^*\phi|_y=|f^*\phi|_y$ for all ϕ and hence $g(y)=f(y)=:y'$. Furthermore, f and g give rise to isometric embeddings $f^*,g^*:\mathcal{H}(y')\to\mathcal{H}(y)$. By Proposition 4.13, the degrees of the two induced field extensions $\mathcal{H}(y)/\mathcal{H}(y')$ are equal to $\deg_y f$ and $\deg_y g$, respectively. By continuity, the inequality (4.4) extends to all $\phi\in\mathcal{H}(y')$. It then follows from [Tem10b, 6.3.3] that $\deg_y f=\deg_y g$.

We also remark that (4.4) implies

$$f^*\Gamma_{y'}=g^*\Gamma_{y'}\quad\text{and}\quad f^*\widetilde{\mathcal{H}(y')}=g^*\widetilde{\mathcal{H}(y')}.\tag{4.5}$$

Thus f and g give the same embeddings of $\Gamma_{y'}$ and $\widetilde{\mathcal{H}(y')}$ into Γ_y and $\widetilde{\mathcal{H}(y)}$, respectively. When y, and hence y' is of Type 2 or 3, the field $\mathcal{H}(y')$ is stable, and so (4.3) gives another proof of the equality $\deg_y f=\deg_y g$.

It remains to prove (4.4). A simple calculation shows that if (4.4) holds for $\phi,\psi\in F$, then it also holds for $\phi\psi$, $1/\phi$ and $a\phi$ for any $a\in K$. Since K is algebraically closed, it thus suffices to prove (4.4) for $\phi=w-b$, where $b\in K$.

Using Lemma 3.12 and the fact that $f(x)=x_{D(0,r')}$, we get

$$\begin{aligned}|f(z)-b|_y\ge|f(z)-b|_x-\delta\deg f&=|w-b|_{f(x)}-\delta\deg f\ge\\ &\ge r'-\delta\deg f\ge\varepsilon(r'+\delta\deg f)=\varepsilon(|f(z)|_x+\delta\deg f)\ge\varepsilon|f(z)|_y.\end{aligned}$$

Now Lemma 3.12 and the choice of δ imply $|h(z)|_y\le\varepsilon^2<1$. As a consequence, $|g(z)|_y=|f(z)|_y$. We conclude that

$$|f^*(w-b)-g^*(w-b)|_y=|h(z)|_y|g(z)|_y\le\varepsilon^2|f(z)|_y\le\varepsilon|f(z)-b|_y=\varepsilon|f^*(w-b)|_y,$$

establishing (4.4) and completing the proof of Lemma 4.14. □

4.8.4. *Type 3 points.* Now consider a point x of Type 3. In suitable coordinates z, w we may assume that x and $x'=f(x)$ are associated to irrational closed discs $D(0,r)$ and $D(0,r')$, respectively. In these coordinates, f is locally approximately monomial at x; there exists $\theta\in K^*$ and $k\in\mathbf{Z}\setminus\{0\}$ such that $f^*w=\theta z^k(1+h(z))$, where $h(z)\in K(z)$ satisfies $|h(z)|_x<1$. Replacing w by $(\theta^{-1}w)^{\pm1}$ we may assume $\theta=1$ and $k>0$. In particular, $r'=r^k$.

Let $g:\mathbf{P}^1\to\mathbf{P}^1$ be defined by $g^*w=z^k$. We claim that $\deg_x(g)=k$. Indeed, the field $\mathcal{H}(x)$ (resp. $\mathcal{H}(x')$) can be concretely described as the set of formal series $\sum_{-\infty}^{\infty}a_jz^j$ (resp. $\sum_{-\infty}^{\infty}b_jw^j$) with $|a_j|r^j\to0$ as $|j|\to\infty$ (resp. $|b_j|r^{kj}\to0$ as $|j|\to\infty$). Then $1,z,\dots,z^{k-1}$ form a basis for $\mathcal{H}(x)/\mathcal{H}(x')$. We can also see that $\deg_x(g)=k$ from (4.3) using that $\widetilde{\mathcal{H}(x)}=\widetilde{\mathcal{H}(x')}=\tilde{K}$, $\Gamma_x=\Gamma_K+\mathbf{Z}\log r$ and $\Gamma_{x'}=\Gamma_K+k\mathbf{Z}\log r$.

Lemma 4.14 gives $\deg_x(f)=\deg_x(g)$. Moreover, we must have $f(x_{D(0,s)})=x_{D(0,s^k)}$ for $s\approx r$, so f expands the hyperbolic metric by a factor $k=\deg_x(f)$. Thus we have established all statements in Theorem 4.7 for Type 3 points.

4.8.5. *Type 2 points.* Now suppose x and hence $x' = f(x)$ is of Type 2. Then $\Gamma_x = \Gamma_{x'} = \Gamma_K$. We may assume x and x' both equal the Gauss point in suitable coordinates z and w. The algebraic tangent spaces $T_x, T_{x'} \simeq \mathbf{P}^1(\tilde{K})$ defined in §3.8.7 have $\widetilde{\mathcal{H}(x)} \simeq \tilde{K}(z)$ and $\widetilde{\mathcal{H}(x')} \simeq \tilde{K}(w)$ as function fields. Now f induces a map $f^* : \widetilde{\mathcal{H}(x')} \to \widetilde{\mathcal{H}(x)}$ and hence a map $T_x \to T_{x'}$. By (4.3), the latter has degree $\deg_x(f)$.

As opposed to the Type 3 case, we cannot necessarily approximate f by a monomial map. However, after applying a coordinate change of the form $z \mapsto (\theta z)^{\pm 1}$, we can find $g(z) \in F = K(z)$ of the form

$$g(z) = z^m \frac{\prod_{i=1}^{l-m}(z - a_i)}{\prod_{j=1}^{k}(z - b_j)}, \tag{4.6}$$

with $m \geq 0$, $|a_i| = |b_j| = 1$, $a_i \neq b_j$ and $a_i b_j \neq 0$ for all i, j, such that

$$f^* w = g(z)(1 + h(z)),$$

in F, where $|h(z)|_x < 1 = |g(z)|_x$.

On the one hand, $g(z)$ induces a map $g : \mathbf{P}^1(K) \to \mathbf{P}^1(K)$ and hence also a map $g : \mathbf{P}^1_{\text{Berk}} \to \mathbf{P}^1_{\text{Berk}}$. We clearly have $g(x) = x'$ and Lemma 4.14 gives $\deg_x(g) = \deg_x(f)$. On the other hand, $g(z)$ also induces a map $g : \mathbf{P}^1(\tilde{K}) \to \mathbf{P}^1(\tilde{K})$, which can be identified with the common tangent map $T_x \to T_{x'}$ of f and g. Both these maps g have degree $\max\{l, k\}$, so in accordance with (4.3), we see that $\deg_x(f) = [\widetilde{\mathcal{H}(x)} : \widetilde{\mathcal{H}(x')}]$.

To prove the remaining statements in Theorem 4.7 (iii), define $m_{\vec{v}}(f)$ as the local degree of the algebraic tangent map $T_x \to T'_x$ at the tangent direction $\vec{v}$. Statement (a) in Theorem 4.7 (iii) is then clear, so it suffices to show (b). We may assume that $\vec{v}$ and its image $\vec{v}'$ are both represented by x_0. Then $m(\vec{v})$ is the integer m in (4.6). We see from (4.6) and from Lemma 4.14 that $f(x_{D(0,r)}) = x_{D(0,r^m)}$ when $0 \ll 1 - r < 1$. Thus (b) holds.

4.8.6. *Type 4 points.* Finally suppose x is a Type 4 point. By Corollary 2.15 we can find $y \in \mathbf{P}^1_{\text{Berk}}$ such that f is a homeomorphism of the segment $\gamma = [x, y]$ onto $f(\gamma)$. We first claim that by moving y closer to x, f will expand the hyperbolic metric on γ by a fixed integer constant $m \geq 1$.

Let $\vec{w}$ be the tangent direction at y represented by x. By moving y closer to x, if necessary, we may assume that x is the unique preimage of x' in $U(\vec{w})$.

Consider a point $\xi \in]x, y[$. If ξ is of Type 3, then we know that f locally expands the hyperbolic metric along γ by a factor $m(\xi)$. Now suppose ξ is a Type 2 point and let $\vec{v}_+$ and $\vec{v}_-$ be the tangent directions at ξ represented by x and y, respectively. Then f locally expands the hyperbolic metric along $\vec{v}_\pm$ by factors $m(\vec{v}_\pm)$. Suppose that $m(\vec{v}_+) < m(\vec{v}_-)$. Then there must exist a tangent direction $\vec{v}$ at ξ different from $\vec{v}_+$ but having the same image as $\vec{v}_+$ under the tangent map. By Corollary 2.13 this implies that $x' \in f(U(\vec{v})) \subseteq f(U(\vec{w}) \setminus \{x\})$, a contradiction. Hence $m(\vec{v}_+) \geq m(\vec{v}_-)$. Since $m(\vec{v}_+)$ is bounded from above by $d = \deg f$, we may assume that $m(\vec{v}_+) = m(\vec{v}_-)$ at all Type 2 points on γ. This shows that f expands the hyperbolic metric on γ by a constant factor m.

To see that $m = \deg_x(f)$, note that the above argument shows that $\deg_\xi(f) = m$ for all $\xi \in \gamma \setminus \{x\}$. Moreover, if $\vec{w}'$ is the tangent direction at $f(y)$ represented by $f(x)$, then the above reasoning shows that $U(\vec{w})$ is a connected component of $f^{-1}(U(\vec{w}'))$ and that ξ is the unique preimage of $f(\xi)$ in $U(\vec{w})$ for any $\xi \in \gamma$. It then follows from Proposition 4.5 that $\deg_x f = m$.

4.9. **Laplacian and pullbacks.** Using the local degree we can pull back Radon measures on $\mathbf{P}^1_{\mathrm{Berk}}$ by f. This we do by first defining a push-forward operator on continuous functions:

$$f_*H(x) = \sum_{f(y)=x} \deg_y(f)H(y)$$

for any $H \in C^0(\mathbf{P}^1_{\mathrm{Berk}})$. It follows from Proposition 4.5 that f_*H is continuous and it is clear that $\|f_*H\|_\infty \le d\|H\|_\infty$, where $d = \deg f$. We then define the pull-back of Radon measures by duality:

$$\langle f^*\rho, H\rangle = \langle \rho, f_*H\rangle.$$

The pull-back operator is continuous in the weak topology of measures. If ρ is a probability measure, then so is $d^{-1}f^*\rho$. Note that the pull-back of a Dirac mass becomes

$$f^*\delta_x = \sum_{f(y)=x} \deg_y(f)\delta_y.$$

Recall from §2.5 that given a positive Radon measure ρ on $\mathbf{P}^1_{\mathrm{Berk}}$ and a finite atomic measure ρ_0 supported on $\mathbf{H}$ of the same mass as ρ, we can write $\rho = \rho_0 + \Delta\varphi$ for a unique function $\varphi \in \mathrm{SH}^0(\mathbf{P}^1_{\mathrm{Berk}}, \rho_0)$. A key property is

Proposition 4.15. *If $\varphi \in \mathrm{SH}^0(\mathbf{P}^1_{\mathrm{Berk}}, \rho_0)$, then $f^*\varphi \in \mathrm{SH}^0(\mathbf{P}^1_{\mathrm{Berk}}, f^*\rho_0)$ and*

$$\Delta(f^*\varphi) = f^*(\Delta\varphi). \tag{4.7}$$

This formula, which will be crucial for the proof of the equidistribution in the next section, confirms that the generalized metric $d_{\mathbf{H}}$ on the tree $\mathbf{P}^1_{\mathrm{Berk}}$ is the correct one. See also Remark 3.13.

Proof. By approximating φ by its retractions $\varphi \circ r_X$, where X ranges over finite subtrees of $\mathbf{H}$ containing the support of ρ_0 we may assume that $\rho := \rho_0 + \Delta\varphi$ is supported on such a finite subtree X. This means that φ is locally constant outside X. By further approximation we reduce to the case when ρ is a finite atomic measure supported on Type 2 points of X.

Let $Y = f^{-1}(X)$. Using Corollary 2.15 and Theorem 4.7 we can write X (resp. Y) as a finite union γ_i (resp. γ_{ij}) of intervals with mutually disjoint interiors such that f maps γ_{ij} homeomorphically onto γ_i and the local degree is constant, equal to d_{ij} on the interior of γ_{ij}. We may also assume that the interior of each γ_i (resp. γ_{ij}) is disjoint from the support of ρ and ρ_0 (resp. $f^*\rho$ and $f^*\rho_0$). Since f expands the hyperbolic metric on each γ_{ij} with a constant factor d_{ij}, it follows that $\Delta(f^*\varphi) = 0$ on the interior of γ_{ij}.

In particular, $\Delta(f^*\varphi)$ is a finite atomic measure. Let us compute its mass at a point x. If $\vec{v}$ is a tangent direction at x and $\vec{v}' = Df(\vec{v})$ its image under the tangent map, then it follows from Theorem 4.7 (iii) that

$$D_{\vec{v}}(f^*\varphi) = m_{\vec{v}}(f)D_{\vec{v}'}(\varphi) \tag{4.8}$$

and hence

$$\Delta(f^*\varphi)\{x\} = \sum_{\vec{v}} D_{\vec{v}}(f^*\varphi) = \sum_{\vec{v}} m_{\vec{v}}(f)D_{\vec{v}'}(\varphi) = \sum_{\vec{v}'} D_{\vec{v}'}\varphi \sum_{Df(\vec{v})=\vec{v}'} m_{\vec{v}}(f)$$
$$= \deg_x(f)\sum_{\vec{v}'} D_{\vec{v}'}(\varphi) = \deg_x(f)(\Delta\varphi)\{f(x)\} = f^*(\Delta\varphi)\{x\},$$

which completes the proof. □

4.10. **Examples.** To illustrate the ideas above, let us study three concrete examples of rational maps. Fix a coordinate $z \in F$ on $\mathbf{P}^1$. Following standard practice we write $f(z)$ for the rational function f^*z.

Example 4.16. Consider the polynomial map defined by

$$f(z) = a(z^3 - 3z^2)$$

where $a \in K$. Here K has residue characteristic zero. The critical points of $f : \mathbf{P}^1 \to \mathbf{P}^1$ are $z = 0$, $z = 2$ and $z = \infty$, where the local degree is 2, 2 and 3, respectively. On $\mathbf{P}^1_{\mathrm{Berk}}$, the local degree is 3 on the interval $[x_G, \infty]$, where x_G is the Gauss norm. The local degree is 2 on the intervals $[0, x_G[$ and $[2, x_G[$ and it is 1 everywhere else. See Figure 4.1.

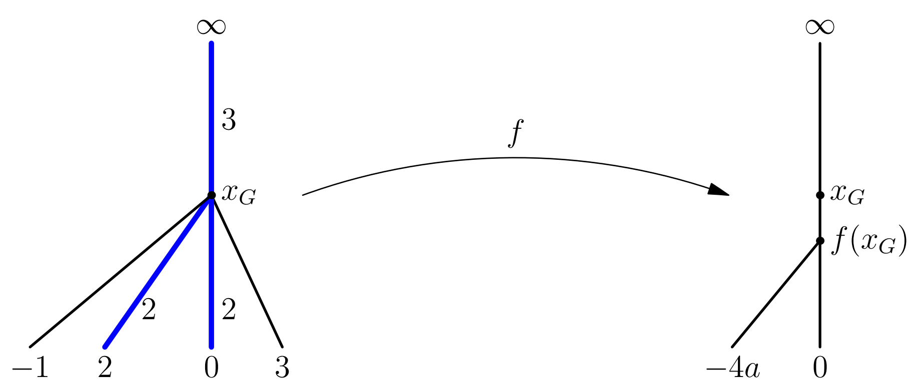


FIGURE 4.1. The ramification locus of the map $f(z) = a(z^3 - 3z^2)$ in Example 4.16 when $|a| < 1$. Here x_G is the Gauss point. The preimage of the interval $[0, f(x_G)]$ is $[0, x_G]$ (with multiplicity 2) and $[3, x_G]$. The preimage of the interval $[-4a, f(x_G)]$ is $[2, x_G]$ (with multiplicity 2) and $[-1, x_G]$. The preimage of the interval $[\infty, f(x_G)]$ is $[\infty, x_G]$ (with multiplicity 3).

Example 4.17. Next consider the polynomial map defined by

$$f(z) = z^p$$

for a prime p. Here the ground field K has characteristic zero. If the residue characteristic is different from p, then f is tamely ramified and the ramification locus is the segment $[0,\infty]$. On the other hand, if the residue characteristic is p, then f is not tamely ramified. A point in $\mathbf{A}^1_{\mathrm{Berk}}$ corresponding to a disc $D(a,r)$ belongs to the ramification locus iff $r \geq p^{-1}|a|$. The ramification locus is therefore quite large and can be visualized as an "inverted Christmas tree", as illustrated in Figure 4.2. It is the set of points in $\mathbf{P}^1_{\mathrm{Berk}}$ having hyperbolic distance at most $\log p$ to the segment $[0,\infty]$. See [BR10, Example 9.30] for more details.

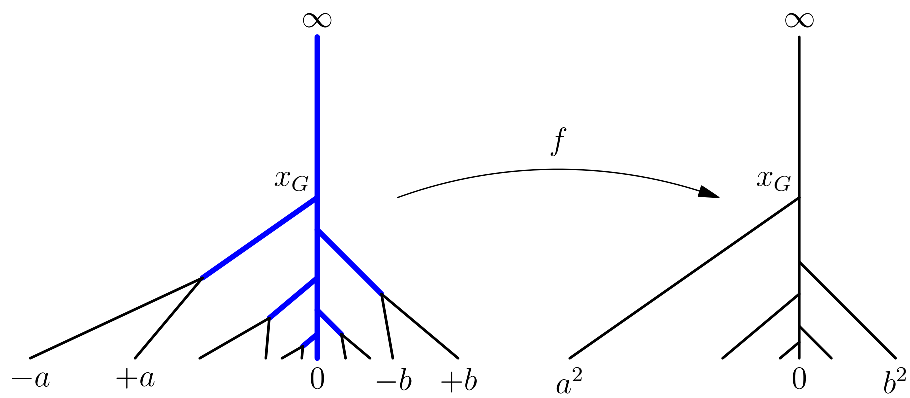


FIGURE 4.2. The ramification locus of the map $f(z) = z^2$ in residual characteristic 2. A point in $\mathbf{A}^1_{\mathrm{Berk}}$ corresponding to a disc $D(a,r)$ belongs to the ramification locus iff $r \geq 2|a|$. The point x_G is the Gauss point.

Example 4.18. Again consider the polynomial map defined by

$$f(z) = z^p$$

for a prime p, but now assume that K has characteristic $p > 0$. Then f is purely inseparable and usually called the *Frobenius map*. We will see in §5.3 that every purely inseparable map of degree > 1 is an iterate of the Frobenius map in some coordinate z.

The mapping properties of f on the Berkovich projective line are easy to describe. Since f is a bijection, the local degree is equal to p at *all* points of $\mathbf{P}^1_{\mathrm{Berk}}$. Hence the ramification locus is equal to $\mathbf{P}^1_{\mathrm{Berk}}$. The Gauss point x_G in the coordinate z is a fixed point: $f(x_G) = x_G$. If $x \in \mathbf{P}^1_{\mathrm{Berk}}$, then f maps the segment $[x_G, x]$ homeomorphically onto the segment $[x_G, f(x)]$ and expands the hyperbolic metric by a constant factor p.

For many more interesting examples, see [BR10, §10.10].

4.11. **Other ground fields.** Above we worked with the assumption that our non-Archimedean field K was algebraically closed and nontrivially valued. Let us briefly discuss what happens when one or both of these assumption is dropped.

4.11.1. *Non-algebraically closed fields.* First suppose K is nontrivially valued but not algebraically closed. Most of the results above remain true in this more general setting and can be proved by passing to the completed algebraic closure $\widehat{K^a}$ as in §3.9. Let us outline how to do this.

The definitions and results in §4.3 go through unchanged. Note that f induces a map $\hat{f} : \mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a}) \to \mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a})$ that is equivariant under the action of the Galois group $G = \mathrm{Gal}(K^a/K)$. Thus $f \circ \pi = \pi \circ \hat{f}$, where $\pi : \mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a}) \to \mathbf{P}^1_{\mathrm{Berk}}(K)$ is the projection. The fact that $\hat{f}$ preserves the type of a point (Lemma 4.2) implies that f does so as well. Proposition 4.3 remains valid and implies that f is a tree map in the sense of §2.6.

We define the local degree of f as in §4.6. Proposition 4.5 remains valid. The local degrees of f and $\hat{f}$ are related as follows. Pick a point $\hat{x} \in \mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a})$ and set $x = \pi(\hat{x})$, $\hat{x}' := f(\hat{x})$ and $x' := \pi(\hat{x}') = f(x)$. The stabilizer $G_{\hat{x}} := \{\sigma \in G \mid \sigma(\hat{x}) = \hat{x}\}$ is a subgroup of G and we have $G_{\hat{x}} \subseteq G_{\hat{x}'}$. The index of $G_{\hat{x}}$ in $G_{\hat{x}'}$ only depends on the projection $x = \pi(\hat{x})$ and we set

$$\delta_x(f) := [G_{\hat{x}'} : G_{\hat{x}}];$$

this is an integer bounded by the (topological) degree of f. We have $m(x) = \delta_x(f) m(f(x))$ for any $x \in \mathbf{P}^1_{\mathrm{Berk}}(K)$, where $m(x)$ is the multiplicity of x, i.e. the number of preimages of x under π. Now

$$\deg_x(f) = \delta_x(f) \deg_{\hat{x}}(\hat{f}).$$

Using this relation (and doing some work), one reduces the assertions in Theorem 4.7 to the corresponding statements for f. Thus the local degree can still be interpreted as a local expansion factor for the hyperbolic metric on $\mathbf{P}^1_{\mathrm{Berk}}(K)$, when this metric is defined as in §3.9. In particular, Corollaries 4.8 and 4.9 remain valid. Finally, the pullback of measures is defined using the local degree as in §4.9 and formulas (4.7)–(4.8) continue to hold.

4.11.2. *Trivially valued fields.* Finally, let us consider the case when K is trivially valued. First assume K is algebraically closed. The Berkovich projective line $\mathbf{P}^1_{\mathrm{Berk}}$ is discussed in §3.9.2 (see also §6.6 below). In particular, the Berkovich projective line is a cone over the usual projective line. In other words, $\mathbf{P}^1_{\mathrm{Berk}} \simeq \mathbf{P}^1 \times [0, \infty]/\sim$, where $(x, 0) \sim (y, 0)$ for any $x, y \in \mathbf{P}^1$. This common point $(x, 0)$ is the Gauss point in any coordinate. See Figure 6.4. The generalized metric on $\mathbf{P}^1_{\mathrm{Berk}}$ is induced by the parametrization $\alpha : \mathbf{P}^1_{\mathrm{Berk}} \to [0, +\infty]$ given by $\alpha(x, t) = t$.

Any rational map $f : \mathbf{P}^1 \to \mathbf{P}^1$ of degree $d \geq 1$ induces a selfmap of $\mathbf{P}^1_{\mathrm{Berk}}$ that fixes the Gauss point. The local degree is d at the Gauss point. At any point (x, t) with $t > 0$, the local degree is equal to the local degree of $f : \mathbf{P}^1 \to \mathbf{P}^1$ at x. Moreover, $f(x, t) = (f(x), t \deg_x(f))$, so f expands the hyperbolic metric by a factor equal to the local degree, in accordance with Theorem 4.7.

Finally, the case when K is trivially valued but not algebraically closed can be treated by passing to the algebraic closure K^a (which is of course already complete under the trivial norm).

4.12. **Notes and further references.** A rational map on the Berkovich projective line is a special case of a finite morphism between Berkovich curves, so various results from [Ber90, Ber93] apply. Nevertheless, it is instructive to see the mapping properties in more detail, in particular the interaction with the tree structure.

The fact that the Berkovich projective line can be understood from many different points of view means that there are several ways of defining the action of a rational map. In his thesis and early work, Rivera-Letelier viewed the action as an extension from $\mathbf{P}^1$ to the hyperbolic space $\mathbf{H}$, whose points he identified with nested collections of closed discs as in §3.3.4. The definition in [BR10, §2.3] uses homogeneous coordinates through a "Proj" construction of the Berkovich projective line whereas [FR10] simply used the (coordinate-dependent) decomposition $\mathbf{P}^1_{\mathrm{Berk}} = \mathbf{A}^1_{\mathrm{Berk}} \cup \{\infty\}$. Our definition here seems to be new, but it is of course not very different from what is already in the literature. As in §3, it is guided by the principle of trying to work without coordinates whenever possible.

There are some important techniques that we have not touched upon, in particular those that take place on the classical (as opposed to Berkovich) affine and projective lines. For example, we never employ Newton polygons even though these can be useful see [BR10, §A.10] or [Ben10, §3.2].

The definition of the local degree is taken from [FR10] but appears already in [Thu05] and is the natural one in the general context of finite maps between Berkovich spaces. In the early work of Rivera-Letelier, a different definition was used, modeled on Theorem 4.7. The definition of the local degree (called multiplicity there) in [BR10] uses potential theory and is designed to make (4.7) hold.

As noted by Favre and Rivera-Letelier, Proposition 4.5 implies that all these different definitions coincide. Having said that, I felt it was useful to have a proof of Theorem 4.7 that is directly based on the algebraic definition of the local degree. The proof presented here seems to be new although many of the ingredients are not.

The structure of the ramification locus in the case of positive residue characteristic is very interesting. We refer to [Fab11a, Fab11b, Fab11c] for details.

5. Dynamics of rational maps in one variable

Now that we have defined the action of a rational map on the Berkovich projective line, we would like to study the dynamical system obtained by iterating the map. While it took people some time to realize that considering the dynamics on $\mathbf{P}^1_{\mathrm{Berk}}$ (as opposed to $\mathbf{P}^1$) could be useful, it has become abundantly clear that this is the right thing to do for many questions.

It is beyond the scope of these notes to give an overview of all the known results in this setting. Instead, in order to illustrate some of the main ideas, we shall focus on an equidistribution theorem due to Favre and Rivera-Letelier [FR10], as well as some of its consequences. For these results we shall, on the other hand, give more or less self-contained proofs.

For results not covered here—notably on the structure of Fatou and Julia sets—we recommend the book [BR10] by Baker and Rumely and the survey [Ben10] by Benedetto.

5.1. **Setup.** We work over a fixed non-Archimedean field K, of any characteristic. For simplicity we shall assume that K is algebraically closed and nontrivially valued. The general case is discussed in §5.10.

Fix a rational map $f : \mathbf{P}^1 \to \mathbf{P}^1$ of degree $d > 1$. Our approach will be largely coordinate free, but in any case, note that since we are to study the dynamics of f, we must choose the same coordinates on the source and target. Given a coordinate z, f^*z is a rational function in z of degree d.

5.2. **Periodic points.** When analyzing a dynamical system, one of the first things to look at are periodic points. We say that $x \in \mathbf{P}^1_{\mathrm{Berk}}$ is a *fixed point* if $f(x) = x$ and a *periodic point* if $f^n(x) = x$ for some $n \geq 1$.

5.2.1. *Classical periodic points.* First suppose $x = f^n(x) \in \mathbf{P}^1$ is a classical periodic point and pick a coordinate z on $\mathbf{P}^1$ vanishing at x. Then

$$f^{*n}z = \lambda z + O(z^2)$$

where $\lambda \in K$ is the *multiplier* of the periodic point. We say that x is *attracting* if $|\lambda| < 1$, *neutral* if $|\lambda| = 1$ and *repelling* if $|\lambda| > 1$. The terminology is more or less self-explanatory. For example, if x is attracting, then there exists a small disc $D \subseteq \mathbf{P}^1$ containing x such that $f^n(D) \subseteq D$ and $f^{nm}(y) \to x$ as $m \to \infty$ for every $y \in D$.

The *multiplicity* of a periodic point $x = f^n(x)$ is the order of vanishing at x of the rational function $f^{n*}z - z$ for any coordinate $z \in F$ vanishing at x. It is easy to see that f has $d + 1$ fixed points counted with multiplicity. Any periodic point of multiplicity at least two must have multiplier $\lambda = 1$.

Proposition 5.1. *Let $f : \mathbf{P}^1 \to \mathbf{P}^1$ be a rational map of degree $d > 1$.*

(i) *There exist infinitely many distinct classical periodic points.*
(ii) *There exists at least one classical nonrepelling fixed point.*
(iii) *Any nonrepelling classical fixed point admits a basis of open neighborhoods $U \subseteq \mathbf{P}^1_{\mathrm{Berk}}$ that are invariant, i.e. $f(U) \subseteq U$.*

Statement (i) when $K = \mathbf{C}$ goes back at least to Julia. A much more precise result was proved by by I. N. Baker [Bak64]. Statements (ii) and (iii) are due to Benedetto [Ben98] who adapted an argument used by Julia.

Sketch of proof. To prove (i) we follow [Bea91, pp.102–103] and [Sil07, Corollary 4.7]. We claim that the following holds for all but at most $d+2$ primes q: any classical point x with $f(x) = x$ has the same multiplicity as a fixed point of f and as a fixed point of f^q. This will show that f^q has $d^q - d > 1$ fixed points (counted with multiplicity) that are not fixed points of f. In particular, f has infinitely many distinct classical periodic points.

To prove the claim, consider a fixed point $x \in \mathbf{P}^1$ and pick a coordinate $z \in F$ vanishing at x. We can write $f^*z = az + bz^{r+1} + O(z^{r+2})$, where $a, b \in K^*$ and $r > 0$. One proves by induction that

$$f^{n*}z = a^n z + b_n z^{r+1} + O(z^{r+2}),$$

where $b_n = a^{n-1}b(1 + a^r + \cdots + a^{(n-1)r})$. If $a \neq 1$, then for all but at most one prime q we have $a^q \neq 1$ and hence x is a fixed point of multiplicity one for both f and f^q. If instead $a = 1$, then $b_q = qb$, so if q is different from the characteristic of K, then x is a fixed point of multiplicity r for both f and f^q.

Next we prove (ii), following [Ben10, §1.3]. Any fixed point of f of multiplicity at least two is nonrepelling, so we may assume that f has exactly $d+1$ fixed points $(x_i)_{i=1}^{d+1}$. Let $(\lambda_i)_{i=1}^{d+1}$ be the corresponding multipliers. Hence $\lambda_i \neq 1$ for all i. it follows from the Residue Theorem (see [Ben10, Theorem 1.6]) that

$$\sum_{i=1}^{d+1} \frac{1}{1-\lambda_i} = 1.$$

If $|\lambda_i| > 1$ for all i, then the left hand side would have norm < 1, a contradiction. Hence $|\lambda_i| \leq 1$ for some i and then x_i is a nonrepelling fixed point.

Finally we prove (iii). Pick a coordinate $z \in F$ vanishing at x and write $f^*z = \lambda z + O(z^2)$, with $|\lambda| \leq 1$. For $0 < r \ll 1$ we have $f(x_{D(0,r)}) = x_{D(0,r')}$, where $r' = |\lambda| r \leq r$. Let $U_r := U(\vec{v}_r)$, where $\vec{v}_r$ is the tangent direction at $x_{D(0,r)}$ determined by x. The sets U_r form a basis of open neighborhoods of x and it follows from Corollary 2.13 (ii) that $f(U_r) \subseteq U_r$ for r small enough. □

5.2.2. *Nonclassical periodic points.* We say that a fixed point $x = f(x) \in \mathbf{H}$ is *repelling* if $\deg_x(f) > 1$ and *neutral* otherwise (points in $\mathbf{H}$ cannot be attracting). This is justified by the interpretation of the local degree as an expansion factor in the hyperbolic metric, see Theorem 4.7.

The following result is due to Rivera-Letelier [Riv03b, Lemme 5.4].

Proposition 5.2. *Any repelling fixed point $x \in \mathbf{H}$ must be of Type 2.*

Sketch of proof. We can rule out that x is of Type 3 using value groups. Indeed, by (4.3) the local degree of f at a Type 3 point is equal to index of the value group $\Gamma_{f(x)}$ as a subgroup of Γ_x, so if $f(x) = x$, then the local degree is one.

I am not aware of an argument of the same style to rule out repelling points of Type 4. Instead, Rivera-Letelier argues by contradiction. Using Newton polygons he shows that any neighborhood of a repelling fixed point of Type 4 would contain a classical fixed point. Since there are only finitely many classical fixed points, this gives a contradiction. See the original paper by Rivera-Letelier or [BR10, Lemma 10.80]. □

5.2.3. *Construction of fixed points.* Beyond Proposition 5.1 there are at least two other methods for producing fixed points.

First, one can use Newton polygons to produce classical fixed points. This was alluded to in the proof of Proposition 5.2 above. We shall not describe this further here but instead refer the reader to [Ben10, §3.2] and [BR10, §A.10].

Second, one can use topology. Since f can be viewed as a tree map, Proposition 2.17 applies and provides a fixed point in $\mathbf{P}^1_{\mathrm{Berk}}$. This argument can be refined, using that f expands the hyperbolic metric, to produce either attracting or repelling fixed points. See [BR10, §10.7].

5.3. **Purely inseparable maps.** Suppose f is purely inseparable of degree $d > 1$. In particular, $\operatorname{char} K = p > 0$. We claim that there exists a coordinate $z \in F$ and $n \geq 1$ such that $f^*z = z^{p^n}$. A rational map f such that $f^*z = z^p$ is usually called the *Frobenius map*, see [Har77, 2.4.1–2.4.2].

To prove the claim, we use the fact that f admits exactly $d + 1$ classical fixed points. Indeed, the multiplier of each fixed point is zero. Pick a coordinate $z \in F$ such that $z = 0$ and $z = \infty$ are fixed points of f. Since f is purely inseparable there exists $n \geq 0$ such that $z^{p^n} \in f^*F$. Choose n minimal with this property. Since $\deg f > 1$ we must have $n \geq 1$. On the other hand, the minimality of n shows that $z^{p^n} = f^*w$ for some coordinate $w \in F$. The fact that $z = 0$ and $z = \infty$ are fixed points imply that $z = aw$ for some $a \in K^*$, so $f^*z = az^{p^n}$. After multiplying z by a suitable power of a, we get $a = 1$, proving the claim.

5.4. **The exceptional set.** A classical point $x \in \mathbf{P}^1$ is called *exceptional* for f if its total backward orbit $\bigcup_{n\geq 0} f^{-n}(x)$ is finite. The *exceptional set* of f is the set of exceptional points and denoted E_f. Since f is surjective, it is clear that $E_{f^n} = E_f$ for any $n \geq 1$. We emphasize that E_f by definition consists of classical points only.

Lemma 5.3. *Let $f : \mathbf{P}^1 \to \mathbf{P}^1$ be a rational map of degree $d > 1$.*

- (i) *If f is not purely inseparable, then there are at most two exceptional points. Moreover:*
  - (a) *if there are two exceptional points, then $f(z) = z^{\pm d}$ in a suitable coordinate z on $\mathbf{P}^1$ and $E_f = \{0, \infty\}$;*
  - (b) *if there is exactly one exceptional point, then f is a polynomial in a suitable coordinate and $E_f = \{\infty\}$.*
- (ii) *If f is purely inseparable, then the exceptional set is countably infinite and consists of all periodic points of f.*

Case (ii) only occurs when $\operatorname{char} K = p > 0$ and f is an iterate of the Frobenius map: $f^*z = z^d$ for d a power of p in some coordinate $z \in F$, see §5.3.

Proof. For $x \in E_f$ set $F_x := \bigcup_{n\geq 0} f^{-n}(x)$. Then F_x is a finite set with $f^{-1}(F_x) \subseteq F_x \subseteq E_f$. Since f is surjective, $f^{-1}(F_x) = F_x = f(F_x)$. Hence each point in F_x must be totally ramified in the sense that $f^{-1}(f(x)) = \{x\}$.

If f is purely inseparable, then every point in $\mathbf{P}^1$ is totally ramified, so F_x is finite iff x is periodic.

If f is not purely inseparable, then it follows from Proposition 4.1 (i) that E_f has at most two elements. The remaining statements are easily verified. □

5.5. **Maps of simple reduction.** By definition, the exceptional set consists of classical points only. The following result by Rivera-Letelier [Riv03b] characterizes totally invariant points in hyperbolic space.

Proposition 5.4. *If $x_0 \in \mathbf{H}$ is a totally invariant point, $f^{-1}(x_0) = x_0$, then x_0 is a Type 2 point.*

Definition 5.5. A rational map $f : \mathbf{P}^1 \to \mathbf{P}^1$ has *simple reduction* if there exists a Type 2 point that is totally invariant for f.

Remark 5.6. Suppose f has simple reduction and pick a coordinate z in which the totally invariant Type 2 point becomes the Gauss point. Then we can write $f^*z = \phi/\psi$, where $\phi, \psi \in \mathfrak{o}_K[z]$ and where the rational function $\tilde{\phi}/\tilde{\psi} \in \tilde{K}(z)$ has degree $d = \deg f$. Such a map is usually said to have *good reduction* [MS95]. Some authors refer to simple reduction as *potentially good reduction.* One could argue that dynamically speaking, maps of good or simple reduction are not the most interesting ones, but they do play an important role. For more on this, see [Ben05b, Bak09, PST09].

Proof of Proposition 5.4. A totally invariant point in $\mathbf{H}$ is repelling so the result follows from Proposition 5.2. Nevertheless, we give an alternative proof.

Define a function $G : \mathbf{P}^1_{\mathrm{Berk}} \times \mathbf{P}^1_{\mathrm{Berk}} \to [-\infty, 0]$ by[13]

$$G(x, y) = -d_{\mathbf{H}}(x_0, x \wedge_{x_0} y).$$

It is characterized by the following properties: $G(y, x) = G(x, y)$, $G(x_0, y) = 0$ and $\Delta G(\cdot, y) = \delta_y - \delta_{x_0}$.

Pick any point $y \in \mathbf{P}^1_{\mathrm{Berk}}$. Let $(y_i)_{i=1}^m$ be the preimages of y under f and $d_i = \deg_{y_i}(f)$ the corresponding local degrees. We claim that

$$G(f(x), y) = \sum_{i=1}^m d_i G(x, y_i) \tag{5.1}$$

for any $x \in \mathbf{P}^1_{\mathrm{Berk}}$. To see this, note that since $f^*\delta_{x_0} = d\delta_{x_0}$ it follows from Proposition 4.15 that both sides of (5.1) are $d\delta_{x_0}$-subharmonic as a function of x, with Laplacian $f^*(\delta_y - \delta_{x_0}) = \sum_i d_i(\delta_{y_i} - \delta_{x_0})$. Now, the Laplacian determines a quasisubharmonic function up to a constant, so since both sides of (5.1) vanish when $x = x_0$ they must be equal for all x, proving the claim.

Now pick x and y as distinct classical fixed points of f. Such points exist after replacing f by an iterate, see Proposition 5.1. We may assume $y_1 = y$. Then (5.1) gives

$$(d_1 - 1)G(x, y) + \sum_{i \geq 2} d_i G(x, y_i) = 0 \tag{5.2}$$

Since $G \leq 0$, we must have $G(x, y_i) = 0$ for $i \geq 2$ and $(d_1 - 1)G(x, y) = 0$.

[13]In [Bak09, BR10], the function $-G$ is called the normalized Arakelov-Green's function with respect to the Dirac mass at x_0.

First assume x_0 is of Type 4. Then x_0 is an end in the tree $\mathbf{P}^1_{\mathrm{Berk}}$, so since $x \neq x_0$ and $y_i \neq x_0$ for all i, we have $x \wedge_{x_0} y_i \neq x_0$ and hence $G(x, y_i) < 0$. This contradicts (5.2).

Now assume x_0 is of Type 3. Then there are exactly two tangent directions at x_0 in the tree $\mathbf{P}^1_{\mathrm{Berk}}$. Replacing f by an iterate, we may assume that these are invariant under the tangent map. We may assume that the classical fixed points $x, y \in \mathbf{P}^1$ above represent the same tangent direction, so that $x \wedge_{x_0} y \neq x_0$. Since x_0 is totally invariant, it follows from Corollary 2.13 (i) that all the preimages y_i of y also represent this tangent vector at x_0. Thus $G(x, y_i) < 0$ for all i which again contradicts (5.2). □

Remark 5.7. The proof in [Bak09] also uses the function G above and analyzes the lifting of f as a homogeneous polynomial map of $K \times K$.

5.6. **Fatou and Julia sets.** In the early part of the 20th century, Fatou and Julia developed a general theory of iteration of rational maps on the Riemann sphere. Based upon some of those results, we make the following definition.

Definition 5.8. The *Julia set* $\mathcal{J} = \mathcal{J}_f$ is the set of points $x \in \mathbf{P}^1_{\mathrm{Berk}}$ such that for every open neighborhood U of x we have $\bigcup_{n \geq 0} f^n(U) \supseteq \mathbf{P}^1_{\mathrm{Berk}} \setminus E_f$. The *Fatou set* is the complement of the Julia set.

Remark 5.9. Over the complex numbers, one usually defines the Fatou set as the largest open subset of the Riemann sphere where the sequence of iterates is locally equicontinuous. One then shows that the Julia set is characterized by the conditions in the definition above. Very recently, a non-Archimedean version of this was found by Favre, Kiwi and Trucco, see [FKT11, Theorem 5.4]. Namely, a point $x \in \mathbf{P}^1_{\mathrm{Berk}}$ belongs to the Fatou set of f iff the family $\{f^n\}_{n \geq 1}$ is normal in a neighborhood of x in a suitable sense. We refer to [FKT11, §5] for the definition of normality, but point out that the situation is more subtle in the non-Archimedean case than over the complex numbers.

Theorem 5.10. *Let $f : \mathbf{P}^1 \to \mathbf{P}^1$ be any rational map of degree $d > 1$.*

(i) *The Fatou set $\mathcal{F}$ and Julia set $\mathcal{J}$ are totally invariant: $\mathcal{F} = f(\mathcal{J}) = f^{-1}(\mathcal{F})$ and $\mathcal{J} = f(\mathcal{F}) = f^{-1}(\mathcal{J})$.*

(ii) *We have $\mathcal{F}_f = \mathcal{F}_{f^n}$ and $\mathcal{J}_f = \mathcal{J}_{f^n}$ for all $n \geq 1$.*

(iii) *The Fatou set is open and dense in $\mathbf{P}^1_{\mathrm{Berk}}$. It contains any nonrepelling classical periodic point and in particular any exceptional point.*

(iv) *The Julia set is nonempty, compact and has empty interior. Further:*
   (a) *if f has simple reduction, then $\mathcal{J}$ consists of a single Type 2 point;*
   (b) *if f does not have simple reduction, then $\mathcal{J}$ is a perfect set, that is, it has no isolated points.*

Proof. It is clear that $\mathcal{F}$ is open. Since $f : \mathbf{P}^1_{\mathrm{Berk}} \to \mathbf{P}^1_{\mathrm{Berk}}$ is an open continuous map, it follows that $\mathcal{F}$ is totally invariant. Hence $\mathcal{J}$ is compact and totally invariant. The fact that $\mathcal{F}_{f^n} = \mathcal{F}_f$, and hence $\mathcal{J}_{f^n} = \mathcal{J}_f$, follow from the total invariance of $E_f = E_{f^n}$.

It follows from Proposition 5.1 that any nonrepelling classical periodic point is in the Fatou set. Since such points exist, the Fatou set is nonempty. This also implies that the Julia set has nonempty interior. Indeed, if U were an open set contained in the Julia set, then the set $U' := \bigcup_{n\ge1} f^n(U)$ would be contained in the Julia set for all $n \ge 1$. Since the Fatou set is open and nonempty, it is not contained in E_f, hence $\mathbf{P}^1_{\mathrm{Berk}} \setminus U' \not\subseteq E_f$, so that $U \subseteq \mathcal{F}$, a contradiction.

The fact that the Julia set is nonempty and that properties (a) and (b) hold is nontrivial and will be proved in §5.8 as a consequence of the equidistribution theorem below. See Propositions 5.14 and 5.16. □

Much more is known about the Fatou and Julia set than what is presented here. For example, as an analogue of the classical result by Fatou and Julia, Rivera-Letelier proved that $\mathcal{J}$ is the closure of the repelling periodic points of f.

For a polynomial map, the Julia set is also the boundary of the filled Julia set, that is, the set of points whose orbits are bounded in the sense that they are disjoint from a fixed open neighborhood of infinity. See [BR10, Theorem 10.91].

Finally, a great deal is known about the dynamics on the Fatou set. We shall not study this here. Instead we refer to [BR10, Ben10].

5.7. **Equidistribution theorem.** The following result that describes the distribution of preimages of points under iteration was proved by Favre and Rivera-Letelier [FR04, FR10]. The corresponding result over the complex numbers is due to Brolin [Bro65] for polynomials and to Lyubich [Lyu83] and Freire-Lopez-Mañé [FLM83] for rational functions.

Theorem 5.11. *Let $f : \mathbf{P}^1 \to \mathbf{P}^1$ be a rational map of degree $d > 1$. Then there exists a unique Radon probability measure ρ_f on $\mathbf{P}^1_{\mathrm{Berk}}$ with the following property: if ρ is a Radon probability measure on $\mathbf{P}^1_{\mathrm{Berk}}$, then*

$$\frac{1}{d^n} f^{n*}\rho \to \rho_f \quad \text{as } n \to \infty,$$

in the weak sense of measures, iff $\rho(E_f) = 0$. The measure ρ_f puts no mass on any classical point; in particular $\rho_f(E_f) = 0$. It is totally invariant in the sense that $f^*\rho_f = d\rho_f$.

Recall that we have assumed that the ground field K is algebraically closed and nontrivially valued. See §5.10 for the general case.

As a consequence of Theorem 5.11, we obtain a more general version of Theorem A from the introduction, namely

Corollary 5.12. *With f as above, we have*

$$\frac{1}{d^n} \sum_{f^n(y)=x} \deg_y(f^n)\delta_y \to \rho_f \quad \text{as } n \to \infty,$$

for any non-exceptional point $x \in \mathbf{P}^1_{\mathrm{Berk}} \setminus E_f$.

Following [BR10] we call ρ_f the *canonical measure* of f. It is clear that $\rho_f = \rho_{f^n}$ for $n \ge 1$. The proof of Theorem 5.11 will be given in §5.9.

Remark 5.13. Okuyama [Oku11b] has proved a quantitative strengthening of Corollary 5.12. The canonical measure is also expected to describe the distribution of repelling periodic points. This does not seem to be established full generality, but is known in many cases [Oku11a].

5.8. **Consequences of the equidistribution theorem.** In this section we collect some result that follow from Theorem 5.11.

Proposition 5.14. *The support of the measure ρ_f is exactly the Julia set $\mathcal{J} = \mathcal{J}_f$. In particular, $\mathcal{J}$ is nonempty.*

Proof. First note that the support of ρ_f is totally invariant. This follows formally from the total invariance of ρ_f. Further, the support of ρ_f cannot be contained in the exceptional set E_f since $\rho_f(E_f) = 0$.

Consider a point $x \in \mathbf{P}^1_{\mathrm{Berk}}$. If x is not in the support of ρ_f, let $U = \mathbf{P}^1_{\mathrm{Berk}} \setminus \operatorname{supp} \rho_f$. Then $f^n(U) = U$ for all n. In particular, $\bigcup_{n\geq 0} f^n(U)$ is disjoint from $\operatorname{supp} \rho_f$. Since $\operatorname{supp} \rho_f \not\subseteq E_f$, x must belong to the Fatou set.

Conversely, if $x \in \operatorname{supp} \rho_f$ and U is any open neighborhood of x, then $\rho_f(U) > 0$. For any $y \in \mathbf{P}^1_{\mathrm{Berk}} \setminus E_f$, Corollary 5.12 implies that $f^{-n}(y) \cap U \neq \emptyset$ for $n \gg 0$. We conclude that $\bigcup_{n\geq 0} f^n(U) \supseteq \mathbf{P}^1_{\mathrm{Berk}} \setminus E_f$, so x belongs to the Julia set. □

We will not study the equilibrium measure ρ_f in detail, but the following result is not hard to deduce from what we already know.

Proposition 5.15. *The following conditions are equivalent.*

(i) *ρ_f puts mass at some point in $\mathbf{P}^1_{\mathrm{Berk}}$;*
(ii) *ρ_f is a Dirac mass at a Type 2 point;*
(iii) *f has simple reduction;*
(iv) *f^n has simple reduction for all $n \geq 1$;*
(v) *f^n has simple reduction for some $n \geq 1$.*

Proof. If f has simple reduction then, by definition, there exists a totally invariant Type 2 point x_0. We then have $d^{-n} f^{n*} \delta_{x_0} = \delta_{x_0}$ so Corollary 5.12 implies $\rho_f = \delta_{x_0}$. Conversely, if $\rho_f = \delta_{x_0}$ for some Type 2 point x_0, then $f^* \rho_f = d\rho_f$ implies that x_0 is totally invariant, so that f has simple reduction. Thus (ii) and (iii) are equivalent. Since $\rho_f = \rho_{f^n}$, this implies that (ii)–(v) are equivalent.

Clearly (ii) implies (i). We complete the proof by proving that (i) implies (v). Thus suppose $\rho_f\{x_0\} > 0$ for some $x_0 \in \mathbf{P}^1_{\mathrm{Berk}}$. Since ρ_f does not put mass on classical points we have $x_0 \in \mathbf{H}$. The total invariance of ρ_f implies

$$0 < \rho_f\{x_0\} = \frac{1}{d}(f^* \rho_f)\{x_0\} = \frac{1}{d} \deg_{x_0}(f) \rho_f\{f(x_0)\} \leq \rho_f\{f(x_0)\},$$

with equality iff $\deg_{x_0}(f) = d$. Write $x_n = f^n(x_0)$ for $n \geq 0$. Now the total mass of ρ_f is finite, so after replacing x_0 by x_m for some $m \geq 0$ we may assume that $x_n = x_0$ and $\deg_{x_j}(f) = d$ for $0 \leq j < n$ and some $n \geq 1$. This implies that x_0 is totally invariant under f^n. By Proposition 5.4, x_0 is then a Type 2 point and f^n has simple reduction. □

With the following result we complete the proof of Theorem 5.10.

Proposition 5.16. *Let $f : \mathbf{P}^1 \to \mathbf{P}^1$ be a rational map of degree $d > 1$ and let $\mathcal{J} = \mathcal{J}_f$ be the Julia set of f.*

(i) *If f has simple reduction, then $\mathcal{J}$ consists of a single Type 2 point.*
(ii) *If f does not have simple reduction, then $\mathcal{J}$ is a perfect set.*

Proof. Statement (i) is a direct consequence of Proposition 5.15. Now suppose f does not have simple reduction. Pick any point $x \in \mathcal{J}$ and an open neighborhood U of x. It suffices to prove that there exists a point $y \in U$ with $y \neq x$ and $f^n(y) = x$ for some $n \geq 1$. After replacing f by an iterate we may assume that x is either fixed or not periodic. Set $m := \deg_x(f)$ if $f(x) = x$ and $m := 0$ otherwise. Note that $m < d$ as x is not totally invariant.

Since $x \notin E_f$, Corollary 5.12 shows that the measure $d^{-n} f^{n*}\delta_x$ converges weakly to ρ_f. Write $f^{n*}\delta_x = m^n \delta_x + \rho'_n$, where

$$\rho'_n = \sum_{y \neq x, f^n(y) = x} \deg_y(f^n)\delta_y.$$

We have $x \in \mathcal{J} = \operatorname{supp} \rho_f$ so $\rho_f(U) > 0$ and hence $\liminf_{n\to\infty}(d^{-n} f^{n*}\delta_x)(U) > 0$. Since $m < d$ it follows that $\rho'_n(U) > 0$ for $n \gg 0$. Thus there exist points $y \in U$ with $y \neq x$ and $f^n(y) = x$. □

5.9. **Proof of the equidistribution theorem.** To prove the equidistribution theorem we follow the approach of Favre and Rivera-Letelier [FR10], who in turn adapted potential-theoretic techniques from complex dynamics developed by Fornæss-Sibony and others. Using the tree Laplacian defined in §2.5 we can study convergence of measures in terms of convergence of quasisubharmonic functions, a problem for which there are good techniques. If anything, the analysis is easier in the nonarchimedean case. Our proof does differ from the one in [FR10] in that it avoids studying the dynamics on the Fatou set.

5.9.1. *Construction of the canonical measure.* Fix a point $x_0 \in \mathbf{H}$. Since $d^{-1} f^*\delta_{x_0}$ is a probability measure, we have

$$d^{-1} f^*\delta_{x_0} = \delta_{x_0} + \Delta u \tag{5.3}$$

for an x_0-subharmonic function u. In fact, (2.3) gives an explicit expression for u and shows that u is continuous, since $f^{-1}(x_0) \subseteq \mathbf{H}$.

Iterating (5.3) and using (4.7) leads to

$$d^{-n} f^{n*}\delta_{x_0} = \delta_{x_0} + \Delta u_n, \tag{5.4}$$

where $u_n = \sum_{j=0}^{n-1} d^{-j} u \circ f^j$. It is clear that the sequence u_n converges uniformly to a continuous x_0-subharmonic function u_∞. We set

$$\rho_f := \delta_{x_0} + \Delta u_\infty.$$

Since u_∞ is bounded, it follows from (2.4) that ρ_f does not put mass on any classical point. In particular, $\rho_f(E_f) = 0$, since E_f is at most countable.

5.9.2. *Auxiliary results.* Before starting the proof of equidistribution, let us record a few results that we need.

Lemma 5.17. *If $x_0, x \in \mathbf{H}$, then $d_{\mathbf{H}}(f^n(x), x_0) = O(d^n)$ as $n \to \infty$.*

Proof. We know that f expands the hyperbolic metric by a factor at most d, see Corollary 4.8. Using the triangle inequality and the assumption $d \geq 2$, this yields

$$d_{\mathbf{H}}(f^n(x), x) \leq \sum_{j=0}^{n-1} d_{\mathbf{H}}(f^{j+1}(x), f^j(x)) \leq \sum_{j=0}^{n-1} d^j d_{\mathbf{H}}(f(x), x) \leq d^n d_{\mathbf{H}}(f(x), x),$$

so that

$$\begin{aligned} d_{\mathbf{H}}(f^n(x), x_0) &\leq d_{\mathbf{H}}(f^n(x), f^n(x_0)) + d_{\mathbf{H}}(f^n(x_0), x_0) \\ &\leq d^n(d_{\mathbf{H}}(x, x_0) + d_{\mathbf{H}}(f(x_0), x_0)), \end{aligned}$$

completing the proof. □

Lemma 5.18. *Suppose that f is not purely inseparable. If ρ is a Radon probability measure on $\mathbf{P}^1_{\mathrm{Berk}}$ such that $\rho(E_f) = 0$ and we set $\rho_n := d^{-n} f^{n*}\rho$, then $\sup_{y \in \mathbf{P}^1} \rho_n\{y\} \to 0$ as $n \to \infty$.*

Note that the supremum is taken over classical points only. Also note that the lemma always applies if the ground field is of characteristic zero. However, the lemma is false for purely inseparable maps.

Proof. We have $\rho_n\{y\} = d^{-n} \deg_y(f^n)\rho\{f^n(y)\}$, so it suffices to show that

$$\sup_{y \in \mathbf{P}^1 \setminus E_f} \deg_y(f^n) = o(d^n). \tag{5.5}$$

For $y \in \mathbf{P}^1$ and $n \geq 0$, write $y_n = f^n(y)$. If $\deg_{y_n}(f) = d$ for $n = 0, 1, 2$, then Proposition 4.1 (i) implies $y \in E_f$. Thus $\deg_y(f^3) \leq d^3 - 1$ and hence $\deg_y(f^n) \leq d^2(d^3 - 1)^{n/3}$ for $y \in \mathbf{P}^1 \setminus E_f$, completing the proof. □

5.9.3. *Proof of the equidistribution theorem.* Let ρ be a Radon probability measure on $\mathbf{P}^1_{\mathrm{Berk}}$ and set $\rho_n = d^{-n} f^{n*}\rho$. If $\rho(E_f) > 0$, then $\rho_n(E_f) = \rho(E_f) > 0$ for all n. Any accumulation point of $\{\rho_n\}$ must also put mass on E_f, so $\rho_n \not\to \rho_f$ as $n \to \infty$.

Conversely, assume $\rho(E_f) = 0$ and let us show that $\rho_n \to \rho_f$ as $n \to \infty$. Let $\varphi \in \mathrm{SH}(\mathbf{P}^1_{\mathrm{Berk}}, x_0)$ be a solution to the equation $\rho = \delta_{x_0} + \Delta\varphi$. Applying $d^{-n} f^{n*}$ to both sides of this equation and using (4.7), we get

$$\rho_n = d^{-n} f^{n*}\delta_{x_0} + \Delta\varphi_n = \delta_{x_0} + \Delta(u_n + \varphi_n),$$

where $\varphi_n = d^{-n}\varphi \circ f^n$. Here $\delta_{x_0} + \Delta u_n$ tends to ρ_f by construction. We must show that $\delta_{x_0} + \Delta(u_n + \varphi_n)$ also tends to ρ_f. By §2.5.4, this amounts to showing that φ_n tends to zero pointwise on $\mathbf{H}$. Since φ is bounded from above, we always have $\limsup_n \varphi_n \leq 0$. Hence it remains to show that

$$\liminf_{n \to \infty} \varphi_n(x) \geq 0 \quad \text{for any } x \in \mathbf{H}. \tag{5.6}$$

To prove (5.6) we first consider the case when f is not purely inseparable. Set $\varepsilon_m = \sup_{y \in \mathbf{P}^1} \rho_m\{y\}$ for $m \geq 0$. Then $\varepsilon_m \to 0$ as $m \to \infty$ by Lemma 5.18. Using Lemma 5.17 and Proposition 2.8 we get, for $m, n \geq 0$

$$\begin{aligned} \varphi_{n+m}(x) &= d^{-n}\varphi_m(f^n(x)) \\ &\geq d^{-n}\varphi_m(x_0) - d^{-n}(C_m + \varepsilon_m d_{\mathbf{H}}(f^n(x), x_0)) \\ &\geq -D\varepsilon_m - C_m d^{-n} \end{aligned}$$

for some constant D independent of m and n and some constant C_m independent of n. Letting first $n \to \infty$ and then $m \to \infty$ yields $\liminf_n \varphi_n(x) \geq 0$, completing the proof.

Now assume f is purely inseparable. In particular, K has characteristic $p > 0$, f has degree $d = p^m$ for some $m \geq 1$ and there exists a coordinate $z \in F$ such that f becomes an iterate of the Frobenius map: $f^*z = z^d$.

In this case, we cannot use Lemma 5.18 since (5.5) is evidently false: the local degree is d everywhere on $\mathbf{P}^1_{\mathrm{Berk}}$. On the other hand, the dynamics is simple to describe, see Example 4.18. The Gauss point x_0 in the coordinate z is (totally) invariant. Hence $\rho_f = \delta_{x_0}$. The exceptional set E_f is countably infinite and consists of all classical periodic points. Consider the partial ordering on $\mathbf{P}^1_{\mathrm{Berk}}$ rooted in x_0. Then f is order preserving and $d_{\mathbf{H}}(f^n(x), x_0) = d^n d_{\mathbf{H}}(x, x_0)$ for any $x \in \mathbf{P}^1_{\mathrm{Berk}}$.

As above, write $\rho = \delta_{x_0} + \Delta\varphi$, with $\varphi \in \mathrm{SH}(\mathbf{P}^1_{\mathrm{Berk}}, x_0)$. Pick any point $x \in \mathbf{H}$. It suffices to prove that (5.6) holds, where $\varphi_n = d^{-n}\varphi(f^n(x))$. Using Lemma 2.9 and the fact that $d_{\mathbf{H}}(f^n(x), x_0) = d^n d_{\mathbf{H}}(x, x_0)$ it suffices to show that

$$\lim_{n\to\infty} \rho(Y_n) = 0, \quad \text{where } Y_n := \{y \geq f^n(x)\}. \tag{5.7}$$

Note that for $m, n \geq 1$, either $Y_{m+n} \subseteq Y_n$ or Y_n, Y_{n+m} are disjoint. If $\rho(Y_n) \not\to 0$, there must exist a subsequence $(n_j)_j$ such that $Y_{n_{j+1}} \subseteq Y_{n_j}$ for all j and $\rho(Y_{n_j}) \not\to 0$. Since $d_{\mathbf{H}}(f^n(x), x_0) \to \infty$ we must have $\bigcap_j Y_{n_j} = \{y_0\}$ for a classical point $y_0 \in \mathbf{P}^1$. Thus $\rho\{y_0\} > 0$. On the other hand, we claim that y_0 is periodic, hence exceptional, contradicting $\rho(E_f) = 0$.

To prove the claim, pick $m_1 \geq 1$ minimal such that $Y_{n_1+m_1} = f^{m_1}(Y_{n_1}) \subseteq Y_{n_1}$ and set $Z_r = Y_{n_1+rm_1} = f^{rm_1}(Y_{n_1})$ for $r \geq 0$. Then Z_r forms a decreasing sequence of compact sets whose intersection consists of a single classical point y, which moreover is periodic: $f^{m_1}(y) = y$. On the other hand, for $m \geq 1$ we have $Y_{n_1+m} \subseteq Y_{n_1}$ iff m_1 divides m. Thus we can write $n_j = n_1 + r_j m_1$ with $r_j \to \infty$. This implies that $\{y_0\} = \bigcap_j Y_{n_j} \subseteq \bigcap_r Z_r = \{y\}$ so that $y_0 = y$ is periodic.

The proof of Theorem 5.11 is now complete.

5.10. **Other ground fields.** Above we worked with the assumption that our non-Archimedean field K was algebraically closed and nontrivially valued. Let us briefly discuss what happens for other fields, focusing on the equidistribution theorem and its consequences.

5.10.1. *Non-algebraically closed fields.* Suppose K is of arbitrary characteristic and nontrivially valued but not algebraically closed. The Berkovich projective line $\mathbf{P}^1_{\mathrm{Berk}}(K)$ and the action by a rational map were outlined in §3.9.1 and §4.11.1, respectively. Let K^a be the algebraic closure of K and $\widehat{K^a}$ its completion. Denote by $\pi : \mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a}) \to \mathbf{P}^1_{\mathrm{Berk}}(K)$ the natural projection. Write $\hat{f} : \mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a}) \to \mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a})$ for the induced map. Define $E_{\hat{f}}$ as the exceptional set for $\hat{f}$ and set $E_f = \pi(E_{\hat{f}})$. Then $f^{-1}(E_f) = E_f$ and E_f has at most two elements, except if K has characteristic p and f is purely inseparable, in which case E_f is countable.

We will deduce the equidistribution result in Theorem 5.11 for f from the corresponding theorem for $\hat{f}$. Let $\rho_{\hat{f}}$ be the measure on $\mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a})$ given by Theorem 5.11 and set $\rho_f = \pi_*(\rho_{\hat{f}})$. Since $E_{\hat{f}} = \pi^{-1}(E_f)$, the measure ρ_f puts no mass on E_f.

Let ρ be a Radon probability measure on $\mathbf{P}^1_{\mathrm{Berk}}(K)$. If $\rho(E_f) > 0$, then any limit point of $d^{-n}f^{n*}\rho$ puts mass on E_f, hence $d^{-n}f^{n*}\rho \not\to \rho_f$. Now assume $\rho(E_f) = 0$. Write x_0 and $\hat{x}_0$ for the Gauss point on $\mathbf{P}^1_{\mathrm{Berk}}(K)$ and $\mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a})$, respectively, in some coordinate on K. We have $\rho = \delta_{x_0} + \Delta\varphi$ for some $\varphi \in \mathrm{SH}(\mathbf{P}^1_{\mathrm{Berk}}(K), x_0)$. The generalized metric on $\mathbf{P}^1_{\mathrm{Berk}}(K)$ was defined in such a way that $\pi^*\varphi \in \mathrm{SH}(\mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a}), x_0)$. Set $\hat{\rho} := \delta_{\hat{x}_0} + \Delta(\pi^*\varphi)$. Then $\hat{\rho}$ is a Radon probability measure on $\mathbf{P}^1_{\mathrm{Berk}}(\widehat{K^a})$ such that $\pi_*\hat{\rho} = \rho$. Since $E_{\hat{f}}$ is countable, $\pi(E_{\hat{f}}) = E_f$ and $\rho(E_f) = 0$ we must have $\hat{\rho}(E_f) = 0$. Theorem 5.11 therefore gives $d^{-n}\hat{f}^{n*}\hat{\rho} \to \rho_{\hat{f}}$ and hence $d^{-n}f^{n*}\rho \to \rho_f$ as $n \to \infty$.

5.10.2. *Trivially valued fields.* Finally let us consider the case when K is equipped with the trivial valuation. Then the Berkovich projective line is a cone over $\mathbf{P}^1(K)$, see §3.9.2. The equidistribution theorem can be proved essentially as above, but the proof is in fact much easier. The measure ρ_f is a Dirac mass at the Gauss point and the exceptional set consists of at most two points, except if f is purely inseparable, The details are left as an exercise to the reader.

5.11. **Notes and further references.** The equidistribution theorem is due to Favre and Rivera-Letelier. Our proof basically follows [FR10] but avoids studying the dynamics on the Fatou set and instead uses the hyperbolic metric more systematically through Proposition 2.8 and Lemmas 5.17 and 5.18. In any case, both the proof here and the one in [FR10] are modeled on arguments from complex dynamics. The remarks in §5.10 about general ground fields seem to be new.

The measure ρ_f is conjectured to describe the distribution of repelling periodic points, see [FR10, Question 1, p.119]. This is known in certain cases but not in general. In characteristic zero, Favre and Rivera-Letelier proved that the classical periodic points (a priori not repelling) are distributed according to ρ_f, see [FR10, Théorème B] as well as [Oku11a].

Again motivated by results over the complex numbers, Favre and Rivera also go beyond equidistribution and study the ergodic properties of ρ_f.

Needless to say, I have not even scratched the surface when describing the dynamics of rational maps. I decided to focus on the equidistribution theorem since its proof uses potential theoretic techniques related to some of the analysis in later sections.

One of the many omissions is the Fatou-Julia theory, in particular the classification of Fatou components, existence and properties of wandering components etc. See [BR10, §10] and [Ben10, §§6–7] for this.

Finally, we have said nothing at all about arithmetic aspects of dynamical systems. For this, see e.g. the book [Sil07] and lecture notes [Sil10] by Silverman.

6. The Berkovich affine plane over a trivially valued field

In the remainder of the paper we will consider polynomial dynamics on the Berkovich affine plane over a trivially valued field, at a fixed point and at infinity. Here we digress and discuss the general structure of the Berkovich affine space $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ in the case of a trivially valued field. While we are primarily interested in the case $n = 2$, many of the notions and results are valid in any dimension.

6.1. **Setup.** Let K be any field equipped with the trivial norm. (In §§6.10–6.11 we shall make further restriction on K.) Let $R \simeq K[z_1, \dots, z_n]$ denote the polynomial ring in n variables with coefficients in K. Thus R is the coordinate ring of the affine n-space $\mathbf{A}^n$ over K. We shall view $\mathbf{A}^n$ as a scheme equipped with the Zariski topology. Points of $\mathbf{A}^n$ are thus prime ideals of R and closed points are maximal ideals.

6.2. **The Berkovich affine space and analytification.** We start by introducing the basic object that we shall study.

Definition 6.1. The Berkovich affine space $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ of dimension n is the set of multiplicative seminorms on the polynomial ring R whose restriction to K is the trivial norm.

This definition is a special case of the *analytification* of a variety (or scheme) over K. Let $Y \subseteq \mathbf{A}^n$ be an irreducible subvariety defined by a prime ideal $I_Y \subseteq R$ and having coordinate ring $K[Y] = R/I_Y$. Then the analytification $\mathrm{Y}_{\mathrm{Berk}}$ of Y is the set of multiplicative seminorms on $K[Y]$ restricting to the trivial norm on K.[14] We equip $\mathrm{Y}_{\mathrm{Berk}}$ with the topology of pointwise convergence. The map $R \to R/I_Y$ induces a continuous injection $\mathrm{Y}_{\mathrm{Berk}} \hookrightarrow \mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$.

As before, points in $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ will be denoted x and the associated seminorm by $|\cdot|_x$. It is customary to write $|\phi(x)| := |\phi|_x$ for a polynomial $\phi \in R$. Let $\mathfrak{p}_x \subset R$ be the kernel of the seminorm $|\cdot|_x$. The completed residue field $\mathcal{H}(x)$ is the completion of the ring $R/\mathfrak{p}_x$ with respect to the norm induced by $|\cdot|_x$. The structure sheaf $\mathcal{O}$ on $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ can now be defined in the same way as in §3.8.1, following [Ber90, §1.5.3], but we will not directly us it.

Closely related to $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ is the *Berkovich unit polydisc* $\mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}}$. This is defined[15] in [Ber90, §1.5.2] as the spectrum of the Tate algebra over K. Since K is trivially valued, the Tate algebra is the polynomial ring R and $\mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}}$ is the set of multiplicative seminorms on R bounded by the trivial norm, that is, the set of points $x \in \mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ such that $|\phi(x)| \le 1$ for all polynomials $\phi \in R$.

6.3. **Home and center.** To a seminorm $x \in \mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ we can associate two basic geometric objects. First, the kernel $\mathfrak{p}_x$ of $|\cdot|_x$ defines a point in $\mathbf{A}^n$ that we call the *home* of x. Note that the home of x is equal to $\mathbf{A}^n$ iff $|\cdot|_x$ is a norm on R. We obtain a continuous *home map*

$$\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}} \to \mathbf{A}^n.$$

[14] The analytification of a general variety or scheme over K is defined by gluing the analytifications of open affine subsets, see [Ber90, §3.5].

[15] The unit polydisc is denoted by $E(0, 1)$ in [Ber90, §1.5.2].

Recall that $\mathbf{A}^n$ is viewed as a scheme with the Zariski topology.

Second, we define the *center* of x on $\mathbf{A}^n$ as follows. If there exists a polynomial $\phi \in R$ such that $|\phi(x)| > 1$, then we say that x has *center at infinity*. Otherwise x belongs to the Berkovich unit polydisc $\mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}}$, in which case we define the center of x to be the point of $\mathbf{A}^n$ defined by the prime ideal $\{\phi \in R \mid |\phi(x)| < 1\}$. Thus we obtain a *center map*[16]

$$\mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}} \to \mathbf{A}^n$$

which has the curious property of being *anticontinuous* in the sense that preimages of open/closed sets are closed/open.

The only seminorm in $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ whose center is all of $\mathbf{A}^n$ is the trivial norm on R. More generally, if $Y \subseteq \mathbf{A}^n$ is any irreducible subvariety, there is a unique seminorm in $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ whose home and center are both equal to Y, namely the image of the trivial norm on $K[Y]$ under the embedding $\mathrm{Y}_{\mathrm{Berk}} \hookrightarrow \mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$, see also (6.2) below. This gives rise to an embedding

$$\mathbf{A}^n \hookrightarrow \mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$$

and shows that the home and center maps are both surjective.

The home of a seminorm always contains the center, provided the latter is not at infinity. By letting the home and center vary over pairs of points of $\mathbf{A}^n$ we obtain various partitions of the Berkovich affine space, see §6.5.

It will occasionally be convenient to identify irreducible subvarieties of $\mathbf{A}^n$ with their generic points. Therefore, we shall sometimes think of the center and home of a seminorm as irreducible subvarieties (rather than points) of $\mathbf{A}^n$.

There is a natural action of $\mathbf{R}_+^*$ on $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ which to a real number $t > 0$ and a seminorm $|\cdot|$ associates the seminorm $|\cdot|^t$. The fixed points under this action are precisely the images under the embedding $\mathbf{A}^n \hookrightarrow \mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ above.

6.4. **Semivaluations.** In what follows, it will be convenient to work additively rather than multiplicatively. Thus we identify a seminorm $|\cdot| \in \mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ with the corresponding *semivaluation*

$$v = -\log|\cdot|. \tag{6.1}$$

The home of v is now given by the prime ideal $(v = +\infty)$ of R. We say that v is a *valuation* if the home is all of $\mathbf{A}^n$. If $v(\phi) < 0$ for some polynomial $\phi \in R$, then v has center at infinity; otherwise v belongs to the $\mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}}$ and its center is defined by the prime ideal $\{v > 0\}$. The action of $\mathbf{R}_+^*$ on $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ is now given by multiplication: $(t, v) \mapsto tv$. The image of an irreducible subvariety $Y \subseteq \mathbf{A}^n$ under the embedding $\mathbf{A}^n \hookrightarrow \mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ is the semivaluation triv_Y, defined by

$$\mathrm{triv}_Y(\phi) = \begin{cases} +\infty & \text{if } \phi \in I_Y \\ 0 & \text{if } \phi \notin I_Y, \end{cases} \tag{6.2}$$

where I_Y is the ideal of Y. Note that $\mathrm{triv}_{\mathbf{A}^n}$ is the trivial valuation on R.

For $v \in \mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}}$ we write

$$v(\mathfrak{a}) := \min_{p \in \mathfrak{a}} v(\phi)$$

[16]The center map is called the *reduction map* in [Ber90, §2.4]. We use the valuative terminology center as in [Vaq00, §6] since it will be convenient to view the elements of $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ as semivaluations rather than seminorms.

for any ideal $\mathfrak{a} \subseteq R$; here it suffices to take the minimum over any set of generators of $\mathfrak{a}$.

6.5. **Stratification.** Let $Y \subseteq \mathbf{A}^n$ be an irreducible subvariety. To Y we can associate two natural elements of $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$: the semivaluation triv_Y above and the valuation ord_Y[17] defined by

$$\mathrm{ord}_Y(\phi) = \max\{k \geq 0 \mid \phi \in I_Y^k\}.$$

As we explain next, Y also determines several natural subsets of $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$.

6.5.1. *Stratification by home.* Define

$$\mathcal{W}_{\supseteq Y}, \quad \mathcal{W}_{\subseteq Y} \quad \text{and} \quad \mathcal{W}_Y$$

as the set of semivaluations in $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ whose home in $\mathbf{A}^n$ contains Y, is contained in Y and is equal to Y, respectively. Note that $\mathcal{W}_{\subseteq Y}$ is closed by the continuity of the home map. We can identify $\mathcal{W}_{\subseteq Y}$ with the analytification $\mathrm{Y}_{\mathrm{Berk}}$ of the affine variety Y as defined in §6.2. In particular, $\mathrm{triv}_Y \in \mathcal{W}_{\subseteq Y}$ corresponds to the trivial valuation on $K[Y]$.

The set $\mathcal{W}_{\supseteq Y}$ is open, since it is the complement in $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ of the union of all $\mathcal{W}_{\subseteq Z}$, where Z ranges over irreducible subvarieties of $\mathbf{A}^n$ not containing Y. The set $\mathcal{W}_Y$, on the other hand, is neither open nor closed unless Y is a point or all of $\mathbf{A}^n$. It can be identified with the set of *valuations* on the coordinate ring $K[Y]$.

6.5.2. *Valuations centered at infinity.* We define $\hat{\mathcal{V}}_\infty$ to be the open subset of $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ consisting of semivaluations having center at infinity. Note that $\hat{\mathcal{V}}_\infty$ is the complement of $\mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}}$ in $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$:

$$\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}} = \mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}} \cup \hat{\mathcal{V}}_\infty \quad \text{and} \quad \mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}} \cap \hat{\mathcal{V}}_\infty = \emptyset.$$

The space $\hat{\mathcal{V}}_\infty$ is useful for the study of polynomial mappings of $\mathbf{A}^n$ at infinity and will be explored in §9 in the two-dimensional case. Notice that the action of $\mathbf{R}_+^*$ on $\hat{\mathcal{V}}_\infty$ is fixed point free. We denote the quotient by $\mathcal{V}_\infty$:

$$\mathcal{V}_\infty := \hat{\mathcal{V}}_\infty / \mathbf{R}_+^*.$$

If we write $R = K[z_1, \dots, z_n]$, then we can identify $\mathcal{V}_\infty$ with the set of semivaluations for which $\min_{1 \leq i \leq n}\{v(z_i)\} = -1$. However, this identification depends on the choice of coordinates, or at least on the embedding of $\mathbf{A}^n \hookrightarrow \mathbf{P}^n$.

6.5.3. *Stratification by center.* We can classify the semivaluations in the Berkovich unit polydisc $\mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}}$ according to their centers. Given an irreducible subvariety $Y \subseteq \mathbf{A}^n$ we define

$$\hat{\mathcal{V}}_{\supseteq Y}, \quad \hat{\mathcal{V}}_{\subseteq Y} \quad \text{and} \quad \hat{\mathcal{V}}_Y$$

as the set of semivaluations in $\mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}}$ whose center contains Y, is contained in Y and is equal to Y, respectively. By anticontinuity of the center map, $\hat{\mathcal{V}}_{\subseteq Y}$ is open and, consequently, $\hat{\mathcal{V}}_{\supseteq Y}$ closed in $\mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}}$. Note that $v \in \hat{\mathcal{V}}_{\subseteq Y}$ iff $v(I_Y) > 0$. As before, $\hat{\mathcal{V}}_Y$ is neither open nor closed unless Y is a closed point or all of $\mathbf{A}^n$.

[17] This is a divisorial valuation given by the order of vanishing along the exceptional divisor of the blowup of Y, see §6.10.

Note that $\mathcal{W}_{\subseteq Y} \cap \mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}} \subseteq \hat{\mathcal{V}}_{\subseteq Y}$. The difference $\hat{\mathcal{V}}_{\subseteq Y} \setminus \mathcal{W}_{\subseteq Y}$ is the open subset of $\mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}}$ consisting of semivaluations v satisfying $0 < v(I_Y) < \infty$. If we define

$$\mathcal{V}_Y := \{v \in \mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}} \mid v(I_Y) = 1\}, \tag{6.3}$$

then $\mathcal{V}_Y$ is a closed subset of $\mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}}$ (hence also of $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$) and the map $v \mapsto v/v(I_Y)$ induces a homeomorphism

$$(\hat{\mathcal{V}}_{\subseteq Y} \setminus \mathcal{W}_{\subseteq Y})/\mathbf{R}^*_+ \xrightarrow{\sim} \mathcal{V}_Y.$$

Remark 6.2. In the terminology of Thuillier [Thu07], $\hat{\mathcal{V}}_{\subseteq Y}$ is the Berkovich space associated to the completion of $\mathbf{A}^n$ along the closed subscheme Y. Similarly, the open subset $\hat{\mathcal{V}}_{\subseteq Y} \setminus \mathcal{W}_{\subseteq Y}$ is the generic fiber of this formal subscheme. This terminology differs slightly from that of Berkovich [Ber94] who refers to $\hat{\mathcal{V}}_{\subseteq Y}$ as the generic fiber, see [Thu07, p.383].

6.5.4. *Extremal cases.* Let us describe the subsets of $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ introduced above in the case when the subvariety Y has maximal or minimal dimension. First, it is clear that

$$\mathcal{W}_{\subseteq \mathbf{A}^n} = \mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}} \quad \text{and} \quad \hat{\mathcal{V}}_{\subseteq \mathbf{A}^n} = \mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}}.$$

Furthermore,

$$\hat{\mathcal{V}}_{\supseteq \mathbf{A}^n} = \hat{\mathcal{V}}_{\mathbf{A}^n} = \mathcal{W}_{\supseteq \mathbf{A}^n} = \mathcal{W}_{\mathbf{A}^n} = \{\mathrm{triv}_{\mathbf{A}^n}\},$$

the trivial valuation on R. Since $I_{\mathbf{A}^n} = 0$, we also have

$$\mathcal{V}_{\mathbf{A}^n} = \emptyset.$$

At the other extreme, for a closed point $\xi \in \mathbf{A}^n$, we have

$$\mathcal{W}_{\subseteq \xi} = \mathcal{W}_\xi = \{\mathrm{triv}_\xi\}.$$

The space $\mathcal{V}_\xi$ is a singleton when $n = 1$ (see §6.6) but has a rich structure when $n > 1$. We shall describe in dimension two in §7, in which case it is a tree in the sense of §2.1. See [BFJ08b] for the higher-dimensional case.

6.5.5. *Passing to the completion.* A semivaluation $v \in \mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}}$ whose center is equal to an irreducible subvariety Y extends uniquely to a semivaluation on the local ring $\mathcal{O}_{\mathbf{A}^n,Y}$ such that $v(\mathfrak{m}_Y) > 0$, where $\mathfrak{m}_Y$ is the maximal ideal. By $\mathfrak{m}_Y$-adic continuity, v further extends uniquely as a semivaluation on the completion and by Cohen's structure theorem, the latter is isomorphic to the power series ring $\kappa(Y)[\![z_1, \dots z_r]\!]$, where r is the codimension of Y. Therefore we can view $\hat{\mathcal{V}}_Y$ as the set of semivaluations v on $\kappa(Y)[\![z_1, \dots z_r]\!]$ whose restriction to $\kappa(Y)$ is trivial and such that $v(\mathfrak{m}_Y) > 0$. In particular, for a closed point ξ, we can view $\hat{\mathcal{V}}_\xi$ (resp., $\mathcal{V}_\xi$) as the set of semivaluations v on $\kappa(\xi)[\![z_1, \dots z_n]\!]$ whose restriction to $\kappa(\xi)$ is trivial and such that $v(\mathfrak{m}_\xi) > 0$ (resp., $v(\mathfrak{m}_\xi) = 1$). This shows that when K is algebraically closed, the set $\mathcal{V}_\xi$ above is isomorphic to the space considered in [BFJ08b]. This space was first introduced in dimension $n = 2$ in [FJ04] where it was called the valuative tree. We shall study it from a slightly different point of view in §7. Note that it may happen that a valuation $v \in \hat{\mathcal{V}}_\xi$ has home ξ but that the extension of v to $\widehat{\mathcal{O}}_{\mathbf{A}^n,\xi}$ is a semivaluation for which the ideal $\{v = \infty\} \subseteq \widehat{\mathcal{O}}_{\mathbf{A}^n,\xi}$ is nontrivial.

6.6. **The affine line.** Using the definitions above, let us describe the Berkovich affine line $\mathbf{A}^1_{\mathrm{Berk}}$ over a trivially valued field K.

An irreducible subvariety of $\mathbf{A}^1$ is either $\mathbf{A}^1$ itself or a closed point. As we noted in §6.5.4

$$\hat{\mathcal{V}}_{\subseteq \mathbf{A}^1} = \mathbf{D}_{\mathrm{Berk}}, \quad \mathcal{W}_{\subseteq \mathbf{A}^1} = \mathbf{A}^1_{\mathrm{Berk}}, \quad \hat{\mathcal{V}}_{\supseteq \mathbf{A}^1} = \hat{\mathcal{V}}_{\mathbf{A}^1} = \mathcal{W}_{\supseteq \mathbf{A}^1} = \mathcal{W}_{\mathbf{A}^1} = \{\mathrm{triv}_{\mathbf{A}^1}\}$$

whereas $\mathcal{V}_{\mathbf{A}^1}$ is empty.

Now suppose the center of $v \in \mathbf{A}^1_{\mathrm{Berk}}$ is a closed point $\xi \in \mathbf{A}^1$. If the home of v is also equal to ξ, then $v = \mathrm{triv}_\xi$. Now suppose the home of v is $\mathbf{A}^1$, so that $0 < v(I_\xi) < \infty$. After scaling we may assume $v(I_\xi) = 1$ so that $v \in \mathcal{V}_\xi$. Since $R \simeq K[z]$ is a PID is follows easily that $v = \mathrm{ord}_\xi$. This shows that

$$\mathcal{W}_{\subseteq \xi} = \mathcal{W}_\xi = \{\mathrm{triv}_\xi\} \quad \text{and} \quad \mathcal{V}_\xi = \{\mathrm{ord}_\xi\},$$

Similarly, if $v \in \mathbf{A}^1_{\mathrm{Berk}}$ has center at infinity, then, after scaling, we may assume that $v(z) = -1$, where $z \in R$ is a coordinate. It is then clear that $v = \mathrm{ord}_\infty$, where ord_∞ is the valuation on R defined by $\mathrm{ord}_\infty(\phi) = -\deg \phi$. Thus we have

$$\mathcal{V}_\infty = \{\mathrm{ord}_\infty\}.$$

Note that any polynomial $\phi \in R$ can be viewed as a rational function on $\mathbf{P}^1 = \mathbf{A}^1 \cup \{\infty\}$ and $\mathrm{ord}_\infty(\phi) \leq 0$ is the order of vanishing of ϕ at ∞.

We leave it as an exercise to the reader to compare the terminology above with the one in §3.9.2. See Figure 6.1 for a picture of the Berkovich affine line over a trivially valued field.

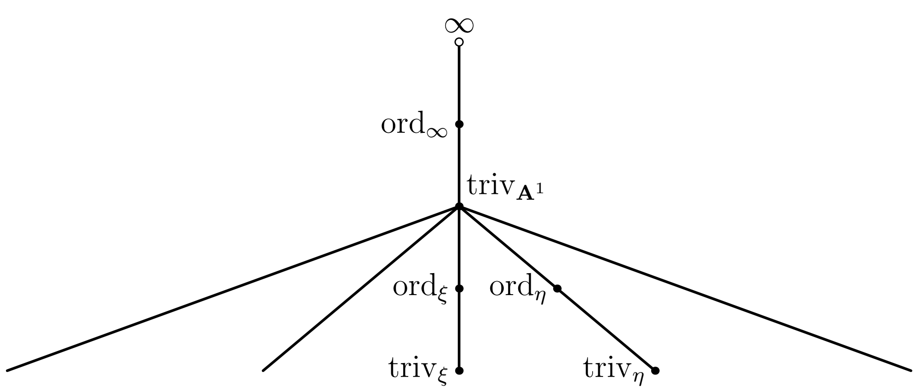


FIGURE 6.1. The Berkovich affine line over a trivially valued field. The trivial valuation $\mathrm{triv}_{\mathbf{A}^1}$ is the only point with center $\mathbf{A}^1$. The point triv_ξ for $\xi \in \mathbf{A}^1$ has home ξ. All the points on the open segment $]\,\mathrm{triv}_{\mathbf{A}^1}, \mathrm{triv}_\xi[$ have home $\mathbf{A}^1$ and center ξ and are proportional to the valuation ord_ξ. The point ∞ does not belong to $\mathbf{A}^1_{\mathrm{Berk}}$. The points on the open segment $]\,\mathrm{triv}_{\mathbf{A}^1}, \infty[$ have home $\mathbf{A}^1$, center at infinity and are proportional to the valuation ord_∞.

6.7. **The affine plane.** In dimension $n = 2$, the Berkovich affine space is significantly more complicated than in dimension one, but can still—with some effort—be visualized.

An irreducible subvariety of $\mathbf{A}^2$ is either all of $\mathbf{A}^2$, a curve, or a closed point. As we have seen,

$$\hat{\mathcal{V}}_{\subseteq\mathbf{A}^2} = \mathbf{D}^2_{\mathrm{Berk}}, \quad \mathcal{W}_{\subseteq\mathbf{A}^2} = \mathbf{A}^2_{\mathrm{Berk}}, \quad \hat{\mathcal{V}}_{\supseteq\mathbf{A}^2} = \hat{\mathcal{V}}_{\mathbf{A}^2} = \mathcal{W}_{\supseteq\mathbf{A}^2} = \mathcal{W}_{\mathbf{A}^2} = \{\mathrm{triv}_{\mathbf{A}^2}\}$$

whereas $\mathcal{V}_{\mathbf{A}^2}$ is empty.

Now let ξ be a closed point. As before, $\mathcal{W}_{\subseteq\xi} = \mathcal{W}_\xi = \{\mathrm{triv}_\xi\}$, where triv_ξ is the image of ξ under the embedding $\mathbf{A}^2 \hookrightarrow \mathbf{A}^2_{\mathrm{Berk}}$. The set $\hat{\mathcal{V}}_{\subseteq\xi} = \hat{\mathcal{V}}_\xi$ is open and $\hat{\mathcal{V}}_\xi \setminus \{\mathrm{triv}_\xi\} = \hat{\mathcal{V}}_\xi \setminus \mathcal{W}_\xi$ is naturally a punctured cone with base $\mathcal{V}_\xi$. The latter will be called *the valuative tree* (at the point ξ) and is studied in detail in §7. Suffice it here to say that it is a tree in the sense of §2.1. The whole space $\hat{\mathcal{V}}_\xi$ is a cone over the valuative tree with its apex at triv_ξ. The boundary of $\hat{\mathcal{V}}_\xi$ consists of all semivaluations whose center strictly contains ξ, so it is the union of $\mathrm{triv}_{\mathbf{A}^2}$ and $\hat{\mathcal{V}}_C$, where C ranges over curves containing C. As we shall see, the boundary therefore has the structure of a tree naturally rooted in $\mathrm{triv}_{\mathbf{A}^2}$. See Figure 6.2. If ξ and η are two different closed points, then the open sets $\hat{\mathcal{V}}_\xi$ and $\hat{\mathcal{V}}_\eta$ are disjoint.

Next consider a curve $C \subseteq \mathbf{A}^2$. By definition, the set $\mathcal{W}_{\subseteq C}$ consists all semivaluations whose home is contained in C. This means that $\mathcal{W}_{\subseteq C}$ is the image of the analytification $\mathrm{C}_{\mathrm{Berk}}$ of C under the embedding $\mathrm{C}_{\mathrm{Berk}} \hookrightarrow \mathbf{A}^2_{\mathrm{Berk}}$. As such, it looks quite similar to the Berkovich affine line $\mathbf{A}^1_{\mathrm{Berk}}$, see [Ber90, §1.4.2]. More precisely, the semivaluation triv_C is the unique semivaluation in $\mathcal{W}_{\subseteq C}$ having center C. All other semivaluations in $\mathcal{W}_{\subseteq C}$ have center at a closed point $\xi \in C$. The only such semivaluation having home ξ is triv_ξ; the other semivaluations in $\mathcal{W}_{\subseteq C} \cap \hat{\mathcal{V}}_\xi$ have home C and center ξ. We can normalize them by $v(I_\xi) = 1$. If ξ is a nonsingular point on C, then there is a unique normalized semivaluation $v_{C,\xi} \in \mathbf{A}^2_{\mathrm{Berk}}$ having home C and center ξ. When ξ is a singular point on C, the set of such semivaluations is instead in bijection with the set of local branches[18] of C at ξ. We see that $\mathcal{W}_{\subseteq C}$ looks like $\mathbf{A}^1_{\mathrm{Berk}}$ except that there may be several intervals joining triv_C and triv_ξ: one for each local branch of C at ξ. See Figure 6.4.

Now look at the closed set $\hat{\mathcal{V}}_{\supseteq C}$ of semivaluations whose center contains C. It consists of all semivaluations $t\,\mathrm{ord}_C$ for $0 \le t \le \infty$. Here $t = \infty$ and $t = 0$ correspond to triv_C and $\mathrm{triv}_{\mathbf{A}^2}$, respectively. As a consequence, for any closed point ξ, $\partial\hat{\mathcal{V}}_\xi$ has the structure of a tree, much like the Berkovich affine line $\mathbf{A}^1_{\mathrm{Berk}}$.

The set $\hat{\mathcal{V}}_{\subseteq C}$ is open and its boundary consists of semivaluations whose center strictly contains C. In other words, the boundary is the singleton $\{\mathrm{triv}_{\mathbf{A}^2}\}$. For two curves C, D, the intersection $\hat{\mathcal{V}}_{\subseteq C} \cap \hat{\mathcal{V}}_{\subseteq D}$ is the union of sets $\hat{\mathcal{V}}_\xi$ over all closed points $\xi \in C \cap D$.

The set $\mathcal{V}_C \simeq (\hat{\mathcal{V}}_{\subseteq C} \setminus \mathcal{W}_{\subseteq C})/\mathbf{R}^*_+$ looks quite similar to the valuative tree at a closed point. To see this, note that the valuation ord_C is the only semivaluation in $\mathcal{V}_C$ whose center is equal to C. All other semivaluations in $\mathcal{V}$ have center at a closed point $\xi \in C$. For each semivaluation $v \in \mathcal{V}_\xi$ whose home is not equal to

[18]A local branch is a preimage of a point of C under the normalization map.

C, there exists a unique $t = t(\xi, C) > 0$ such that $tv \in \mathcal{V}_C$; indeed, $t = v(I_C)$. Therefore, $\mathcal{V}_C$ can be obtained by taking the disjoint union of the trees $\mathcal{V}_\xi$ over all $\xi \in C$ and identifying the semivaluations having home C with the point ord_C. If C is nonsingular, then $\mathcal{V}_C$ will be a tree naturally rooted in ord_C.

We claim that if C is a line, then $\mathcal{V}_C$ can be identified with the Berkovich unit disc over the field of Laurent series in one variable with coefficients in K. To see this, pick affine coordinates (z_1, z_2) such that $C = \{z_1 = 0\}$. Then $\mathcal{V}_C$ is the set of semivaluations $v : K[z_1, z_2] \to \mathbf{R}_+ \cup \{\infty\}$ such that $v(z_1) = 1$. Let $L = K((z_1))$ be the field of Laurent series, equipped with the valuation v_L that is trivial on K and takes value 1 on z_1. Then the Berkovich unit disc $\mathbf{D}_{\mathrm{Berk}}$ over L is the set of semivaluations $L[z_2] \to \mathbf{R}_+ \cup \{\infty\}$ extending v_L. Every element of $\mathbf{D}_{\mathrm{Berk}}$ defines an element of $\mathcal{V}_C$ by restriction. Conversely, pick $v \in \mathcal{V}_C$. If $v = \mathrm{ord}_C$, then v extends uniquely to an element of $\mathbf{D}_{\mathrm{Berk}}$, namely the Gauss point. If $v \neq \mathrm{ord}_C$, then the center of v is a closed point $\xi \in C$ and v extends uniquely to the fraction field of the completion $\hat{\mathcal{O}}_\xi$. This fraction field contains $L[z_2]$.

The open subset $\hat{\mathcal{V}}_\infty = \mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}} \setminus \mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}}$ of semivaluations centered at infinity is a punctured cone over a base $\mathcal{V}_\infty$. The latter space is called *the valuative tree at infinity* and will be studied in detail in §9. Superficially, its structure is quite similar to the valuative tree at a closed point ξ. In particular it is a tree in the sense of §2.1. The boundary of $\hat{\mathcal{V}}_\infty$ is the union of $\hat{\mathcal{V}}_{\supseteq C}$ over *all* affine curves C, that is, the set of semivaluations in $\mathbf{D}^2_{\mathrm{Berk}}$ whose center is *not* a closed point. Thus the boundary has a structure of a tree rooted in $\mathrm{triv}_{\mathbf{A}^2}$. We emphasize that there is no point triv_∞ in $\hat{\mathcal{V}}_\infty$.

To summarize the discussion, $\mathbf{A}^2_{\mathrm{Berk}}$ contains a closed subset Σ with nonempty interior consisting of semivaluations having center of dimension one or two. This set is a naturally a tree, which can be viewed as the cone over the collection of all irreducible affine curves. The complement of Σ is an open dense subset whose connected components are $\hat{\mathcal{V}}_\infty$, and $\hat{\mathcal{V}}_\xi$, where ξ ranges over closed points of $\mathbf{A}^2$. The set $\hat{\mathcal{V}}_\infty$ is a punctured cone over a tree $\mathcal{V}_\infty$ and its boundary is all of Σ. For a closed point ξ, $\hat{\mathcal{V}}_\xi$ is a cone over a tree $\mathcal{V}_\xi$ and its boundary is a subtree of Σ, namely the cone over the collection of all irreducible affine curves containing ξ.

6.8. **Valuations.** A semivaluation v on $R \simeq K[z_1, \dots, z_n]$ is a *valuation* if the corresponding seminorm is a norm, that is, if $v(\phi) < \infty$ for all nonzero polynomials $\phi \in R$. A valuation v extends to the fraction field $F \simeq K(z_1, \dots, z_n)$ of R by setting $v(\phi_1/\phi_2) = v(\phi_1) - v(\phi_2)$.

Let X be a variety over K whose function field is equal to F. The *center* of a valuation v on X, if it exists, is the unique (not necessarily closed) point $\xi \in X$ defined by the properties that $v \geq 0$ on the local ring $\mathcal{O}_{X,\xi}$ and $\{v > 0\} \cap \mathcal{O}_{X,\xi} = \mathfrak{m}_{X,\xi}$. By the valuative criterion of properness, the center always exists and is unique when X is proper over K.

Following [JM10] we write Val_X for the set of valuations of F that admit a center on X. As usual, this set is endowed with the topology of pointwise convergence. Note that Val_X is a subset of $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ that can in fact be shown to be dense. One nice feature of Val_X is that any proper birational morphism $X' \to X$ induces an

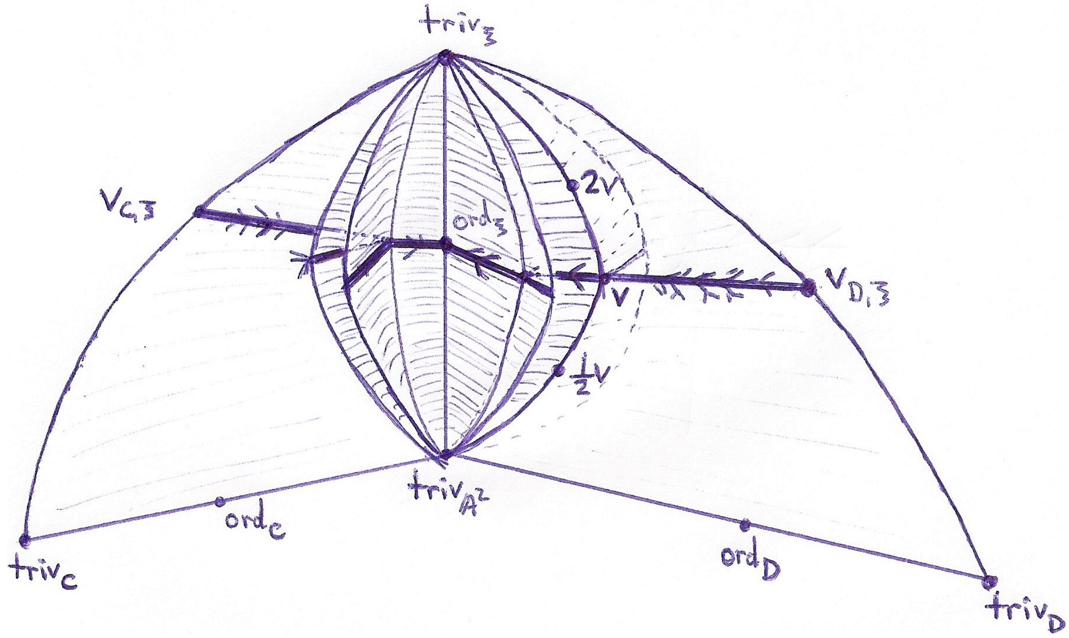


FIGURE 6.2. The Berkovich affine plane over a trivially valued field. The picture shows the closure of the set $\hat{\mathcal{V}}_\xi$ of semivaluations having center at a closed point $\xi \in \mathbf{A}^2$. Here C, D are irreducible curves containing ξ. The semivaluation $\mathrm{triv}_\xi \in \hat{\mathcal{V}}_\xi$ has home ξ. All semivaluations in $\hat{\mathcal{V}}_\xi \setminus \{\mathrm{triv}_\xi\}$ are proportional to a semivaluation v in the valuative tree $\mathcal{V}_\xi$ at ξ. We have $tv \to \mathrm{triv}_\xi$ as $t \to \infty$. As $t \to 0+$, tv converges to the semivaluation triv_Y, where Y is the home of v. The semivaluations $v_{C,\xi}$ and $v_{D,\xi}$ belong to $\mathcal{V}_\xi$ and have home C and D, respectively. The boundary of $\hat{\mathcal{V}}_\xi$ is a tree consisting of all segments $[\mathrm{triv}_{\mathbf{A}^2}, \mathrm{triv}_C]$ for all irreducible affine curves C containing both ξ. Note that the segment $[\mathrm{triv}_C, \mathrm{triv}_\xi]$ in the closure of $\hat{\mathcal{V}}_\xi$ is also a segment in the analytification $\mathrm{C}_{\mathrm{Berk}} \subseteq \mathbf{A}^2_{\mathrm{Berk}}$ of C, see Figure 6.4.

isomorphism $\mathrm{Val}_{X'} \xrightarrow{\sim} \mathrm{Val}_X$. (In the same situation, the analytification X'_{Berk} maps onto X_{Berk}, but this map is not injective.)

We can view the Berkovich unit polydisc $\mathbf{D}^{\mathrm{n}}_{\mathrm{Berk}}$ as the disjoint union of Val_Y, where Y ranges over irreducible subvarieties of X.

6.9. **Numerical invariants.** To a valuation $v \in \mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$ we can associate several invariants. First, the *value group* of v is defined by $\Gamma_v := \{v(\phi) \mid \phi \in F \setminus \{0\}\}$. The *rational rank* $\mathrm{rat.rk}\, v$ of v is the dimension of the $\mathbf{Q}$-vector space $\Gamma_v \otimes_{\mathbf{Z}} \mathbf{Q}$.

Second, the valuation ring $R_v = \{\phi \in F \mid v(\phi) \geq 0\}$ of v is a local ring with maximal ideal $\mathfrak{m}_v = \{v(\phi) > 0\}$. The *residue field* $\kappa(v) = R_v/\mathfrak{m}_v$ contains K as a subfield and the *transcendence degree* of v is the transcendence degree of the field extension $\kappa(v)/K$.

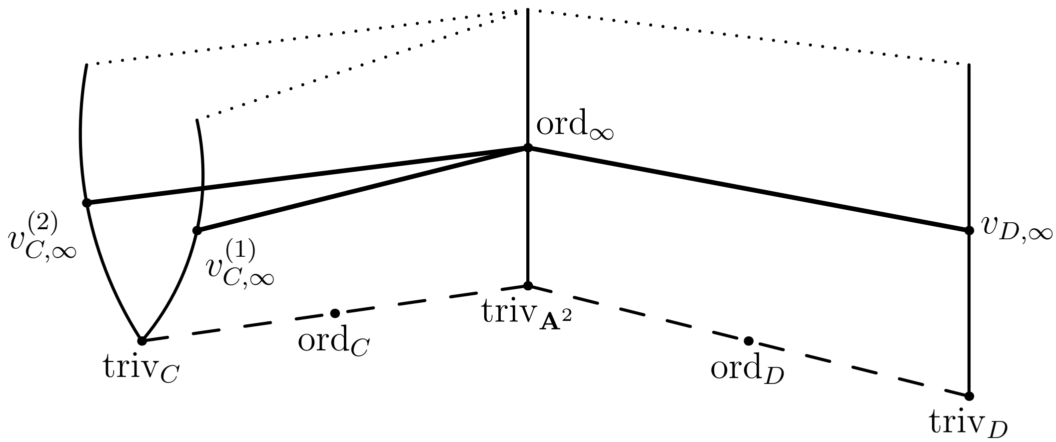


FIGURE 6.3. The Berkovich affine plane over a trivially valued field. The picture shows (part of) the closure of the set $\hat{\mathcal{V}}_\infty$ of semivaluations having center at infinity. Here C and D are affine curves having two and one places at infinity, respectively. The set $\hat{\mathcal{V}}_\infty$ is a cone whose base is $\mathcal{V}_\infty$, the valuative tree at infinity. Fixing an embedding $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$ allows us to identify $\mathcal{V}_\infty$ with a subset of $\hat{\mathcal{V}}_\infty$ and the valuation ord_∞ is the order of vanishing along the line at infinity in $\mathbf{P}^2$. The semivaluations $v_{D,\infty}$ and $v^{(i)}_{C,\infty}$, $i = 1, 2$ have home D and C, respectively; the segments $[\mathrm{ord}_\infty, v_{D,\infty}]$ and $[\mathrm{ord}_\infty, v^{(i)}_{C,\infty}]$, $i = 1, 2$ belong to $\mathcal{V}_\infty$. The segments $[\mathrm{triv}_{\mathbf{A}^2}, \mathrm{triv}_C]$ and $[\mathrm{triv}_{\mathbf{A}^2}, \mathrm{triv}_D]$ at the bottom of the picture belong to the boundary of $\hat{\mathcal{V}}_\infty$: the full boundary is a tree consisting of all such segments and whose only branch point is $\mathrm{triv}_{\mathbf{A}^2}$. The dotted segments in the top of the picture do not belong to the Berkovich affine plane.

In our setting, the fundamental *Abhyankar inequality* states that

$$\mathrm{rat.\,rk}\, v + \mathrm{tr.\,deg}\, v \le n. \tag{6.4}$$

The valuations for which equality holds are of particular importance. At least in characteristic zero, they admit a nice geometric description that we discuss next.

6.10. **Quasimonomial and divisorial valuations.** Let X be a smooth variety over K with function field F. We shall assume in this section that the field K has characteristic zero or that X has dimension at most two. This allows us to freely use resolutions of singularities.

Let $\xi \in X$ be a point (not necessarily closed) with residue field $\kappa(\xi)$. Let $(\zeta_1, \dots, \zeta_r)$ be a system of algebraic coordinates at ξ (i.e. a regular system of parameters of $\mathcal{O}_{X,\xi}$). We say that a valuation $v \in \mathrm{Val}_X$ is *monomial* in coordinates $(\zeta_1, \dots, \zeta_r)$ with weights $t_1, \dots, t_r \ge 0$ if the following holds: if we write $\phi \in \widehat{\mathcal{O}}_{X,\xi}$ as $\phi = \sum_{\beta \in \mathbf{Z}^m_{\ge 0}} c_\beta \zeta^\beta$ with each $c_\beta \in \widehat{\mathcal{O}}_{X,\xi}$ either zero or a unit, then

$$v(\phi) = \min\{\langle t, \beta\rangle \mid c_\beta \neq 0\},$$

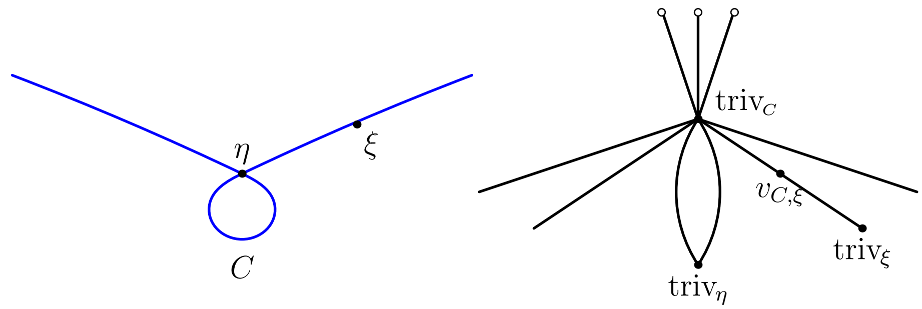


FIGURE 6.4. The analytification C_{Berk} of an affine curve C over a trivially valued field. The semivaluation triv_C is the only semivaluation in C_{Berk} having center C and home C. To each closed point $\xi \in C$ is associated a unique semivaluation $\mathrm{triv}_\xi \in C_{\mathrm{Berk}}$ with center and home ξ. The set of elements of C_{Berk} with home C and center at a given closed point ξ is a disjoint union of open intervals, one for each local branch of C at ξ. Similarly, the set of elements of C_{Berk} with home C and center at infinity is a disjoint union of open intervals, one for each branch of C at infinity. The left side of the picture shows a nodal cubic curve C and the right side shows its analytification C_{Berk}. Note that for a smooth point ξ on C, the segment $[\mathrm{triv}_C, \mathrm{triv}_\xi]$ in C_{Berk} also lies in the closure of the cone $\hat{\mathcal{V}}_\xi$, see Figure 6.2.

where $\langle t, \beta\rangle = t_1\beta_1 + \cdots + t_r\beta_r$. After replacing ξ by the (generic point of the) intersection of all divisors $\{\zeta_i = 0\}$ we may in fact assume that $t_i > 0$ for all i.

We say that a valuation $v \in \mathrm{Val}_X$ is *quasimonomial* (on X) if it is monomial in some birational model of X. More precisely, we require that there exists a proper birational morphism $\pi : X' \to X$, with X' smooth, such that v is monomial in some algebraic coordinates at some point $\xi \in X'$. As explained in [JM10], in this case we can assume that the divisors $\{\zeta_i = 0\}$ are irreducible components of a reduced, effective simple normal crossings divisor D on X' that contains the exceptional locus of π. (In the two-dimensional situation that we shall be primarily interested in, arranging this is quite elementary.)

It is a fact that a valuation $v \in \mathrm{Val}_X$ is quasimonomial iff equality holds in Abhyankar's inequality (6.4). For this reason, quasimonomial valuations are sometimes called Abhyankar valuations. See [ELS03, Proposition 2.8].

Furthermore, we can arrange the situation so that the weights t_i are all strictly positive and linearly independent over $\mathbf{Q}$: see [JM10, Proposition 3.7]. In this case the residue field of v is isomorphic to the residue field of ξ, and hence $\mathrm{tr.deg}\, v = \dim(\overline{\xi}) = n - r$. Furthermore, the value group of v is equal to

$$\Gamma_v = \sum_{i=1}^{r} \mathbf{Z} t_i, \tag{6.5}$$

so $\mathrm{rat.rk}\, v = r$.

A very important special case of quasimonomial valuations are given by *divisorial valuations*. Numerically, they are characterized by $\operatorname{rat.rk} = 1$, $\operatorname{tr.deg} = n-1$. Geometrically, they are described as follows: there exists a birational morphism $X' \to X$, a prime divisor $D \subseteq X'$ and a constant $t > 0$ such that $t^{-1}v(\phi)$ is the order of vanishing along D for all $\phi \in F$.

6.11. **The Izumi-Tougeron inequality.** Keep the same assumptions on K and X as in §6.10. Consider a valuation $v \in \mathrm{Val}_X$ and let ξ be its center on X. Thus ξ is a (not necessarily closed) point of X. By definition, v is nonnegative on the local ring $\mathcal{O}_{X,\xi}$ and strictly positive on the maximal ideal $\mathfrak{m}_{X,\xi}$. Let ord_ξ be the order of vanishing at ξ. It follows from the valuation axioms that

$$v \geq c \operatorname{ord}_\xi, \tag{6.6}$$

on $\mathcal{O}_{X,\xi}$, where $c = v(\mathfrak{m}_{X,\xi}) > 0$.

It will be of great importance to us that if $v \in \mathrm{Val}_X$ is quasimonomial then the reverse inequality holds in (6.6). Namely, there exists a constant $C = C(v) > 0$ such that

$$c \operatorname{ord}_\xi \leq v \leq C \operatorname{ord}_\xi \tag{6.7}$$

on $\mathcal{O}_{X,\xi}$. This inequality is often referred to as Izumi's inequality (see [Izu85, Ree89, HS01, ELS03]) but in the smooth case we are considering it goes back at least to Tougeron [Tou72, p.178]. More precisely, Tougeron proved this inequality for divisorial valuations, but that easily implies the general case.

As in §4.8.2, a valuation $v \in \mathrm{Val}_X$ having center ξ on X extends uniquely to a semivaluation on $\widehat{\mathcal{O}}_{X,\xi}$. The Izumi-Tougeron inequality (6.7) implies that if v is quasimonomial, then this extension is in fact a valuation. In general, however, the extension may not be a valuation, so the Izumi-Tougeron inequality certainly does not hold for *all* valuations in Val_X having center ξ on X. For a concrete example, let $X = \mathbf{A}^2$, let ξ be the origin in coordinates (z, w) and let $v(\phi)$ be defined as the order of vanishing at $u = 0$ of $\phi(u, \sum_{i=1}^\infty \frac{u^i}{i!})$. Then $v(\phi) < \infty$ for all nonzero polynomials ϕ, whereas $v(w - \sum_{i=1}^\infty \frac{u^i}{i!}) = 0$.

6.12. **Notes and further references.** It is a interesting feature of Berkovich's theory that one can work with trivially valued fields: this is definitely not possible in rigid geometry (see e.g. [Con08] for a general discussion of rigid geometry and various other aspects of non-Archimedean geometry).

In fact, Berkovich spaces over trivially valued fields have by now seen several interesting and unexpected applications. In these notes we focus on dynamics, but one can also study use Berkovich spaces to study the singularities of plurisubharmonic functions [FJ05a, BFJ08b] and various asymptotic singularities in algebraic geometry, such as multiplier ideals [FJ05b, JM10]. In other directions, Thuillier [Thu07] exploited Berkovich spaces to give a new proof of a theorem by Stepanov in birational geometry, and Berkovich [Ber09] has used them in the context of mixed Hodge structures.

The Berkovich affine space of course also comes with a structure sheaf $\mathcal{O}$. We shall not need use it in what follows but it is surely a useful tool for a more systematic study of polynomial mappings on the $\mathbf{A}^{\mathrm{n}}_{\mathrm{Berk}}$.

The spaces $\hat{\mathcal{V}}_\xi$, $\mathcal{V}_\xi$ and $\mathcal{V}_\infty$ were introduced (in the case of K algebraically closed of characteristic zero) and studied in [FJ04, FJ07, BFJ08b] but not explicitly identified as subset of the Berkovich affine plane. The structure of the Berkovich affine space does not seem to have been written down in detail before, but see [YZ09b].

The terminology "home" is not standard. Berkovich uses this construction in [Ber90, §1.2.5] but does not give it a name. The name "center" comes from valuation theory, see [Vaq00, §6] whereas non-Archimedean geometry tends to use the term "reduction". Our distinction between (additive) valuations and (multiplicative) norms is not always made in the literature. Furthermore, in [FJ04, BFJ08b], the term 'valuation' instead of 'semi-valuation' is used even when the prime ideal $\{v = +\infty\}$ is nontrivial.

The space Val_X was introduced in [JM10] for the study of asymptotic invariants of graded sequences of ideals. In *loc. cit.* it is proved that Val_X is an inverse limit of cone complexes, in the same spirit as §7.5.4 below.

7. The valuative tree at a point

Having given an overview of the Berkovich affine plane over a trivially valued field, we now study the set of semivaluations centered at a closed point. As indicated in §6.7, this is a cone over a space that we call the valuative tree.

The valuative tree is treated in detail in the monograph [FJ04]. However, the self-contained presentation here has a different focus. In particular, we emphasize aspects that generalize to higher dimension. See [BFJ08b] for some of these generalizations.

7.1. **Setup.** Let K be field equipped with the trivial norm. For now we assume that K is algebraically closed but of arbitrary characteristic. (See §7.11 for a more general case). In applications to complex dynamics we would of course pick $K = \mathbf{C}$, but we emphasize that the norm is then *not* the Archimedean one. As in §6 we work additively rather than multiplicatively and consider K equipped with the trivial valuation, whose value on nonzero elements is zero and whose value on 0 is $+\infty$.

Let R and F be the coordinate ring and function field of $\mathbf{A}^2$. Fix a closed point $0 \in \mathbf{A}^2$ and write $\mathfrak{m}_0 \subseteq R$ for the corresponding maximal ideal. If (z_1, z_2) are global coordinates on $\mathbf{A}^2$ vanishing at 0, then $R = K[z_1, z_2]$, $F = K(z_1, z_2)$ and $\mathfrak{m}_0 = (z_1, z_2)$. We say that an ideal $\mathfrak{a} \subseteq R$ is $\mathfrak{m}_0$-*primary* or simply *primary* if it contains some power of $\mathfrak{m}_0$.

Recall that the Berkovich affine plane $\mathbf{A}^2_{\mathrm{Berk}}$ is the set of semivaluations on R that restrict to the trivial valuation on K. Similarly, the Berkovich unit bidisc $\mathbf{D}^2_{\mathrm{Berk}}$ is the set of semivaluations $v \in \mathbf{A}^2_{\mathrm{Berk}}$ that are nonnegative on R. If $\mathfrak{a} \subseteq R$ is an ideal and $v \in \mathbf{D}^2_{\mathrm{Berk}}$, then we write $v(\mathfrak{a}) = \min\{v(\phi) \mid \phi \in \mathfrak{a}\}$. In particular, $v(\mathfrak{m}_0) = \min\{v(z_1), v(z_2)\}$.

7.2. **The valuative tree.** Let us recall some definitions from §6.5.3 and §6.7. Let $\hat{\mathcal{V}}_0 \subseteq \mathbf{D}^2_{\mathrm{Berk}}$ be the subset of semivaluations whose center on $\mathbf{A}^2$ is equal to the closed point $0 \in \mathbf{A}^2$. In other words, $\hat{\mathcal{V}}_0$ is the set of semivaluations $v : R \to [0, +\infty]$ such that $v|_{K^*} \equiv 0$ and $v(\mathfrak{m}_0) > 0$.

There are now two cases. Either $v(\mathfrak{m}_0) = +\infty$, in which case $v = \mathrm{triv}_0 \in \mathbf{A}^2_{\mathrm{Berk}}$ is the trivial valuation associated to the point $0 \in \mathbf{A}^2$, or $0 < v(\mathfrak{m}_0) < \infty$. Define $\hat{\mathcal{V}}_0^*$ as the set of semivaluations of the latter type. This set is naturally a pointed cone and admits the following set as a "section".

Definition 7.1. The *valuative tree* $\mathcal{V}_0$ at the point $0 \in \mathbf{A}^2$ is the set of semivaluations $v : R \to [0, +\infty]$ satisfying $v(\mathfrak{m}_0) = 1$.

To repeat, we have

$$\hat{\mathcal{V}}_0 = \{\mathrm{triv}_0\} \cup \hat{\mathcal{V}}_0^* \quad \text{and} \quad \hat{\mathcal{V}}_0^* = \mathbf{R}_+^* \mathcal{V}_0.$$

We equip $\mathcal{V}_0$ and $\hat{\mathcal{V}}_0$ with the subspace topology from $\mathbf{A}^2_{\mathrm{Berk}}$, that is, the weakest topology for which all evaluation maps $v \mapsto v(\phi)$ are continuous, where ϕ ranges over polynomials in R. It follows easily from Tychonoff's theorem that $\mathcal{V}_0$ is a compact Hausdorff space.

Equivalently, we could demand that $v \mapsto v(\mathfrak{a})$ be continuous for any primary ideal $\mathfrak{a} \subseteq R$. For many purposes it is indeed quite natural to evaluate semivaluations in $\hat{\mathcal{V}}_0^*$ on primary ideals rather than polynomials. For example, we have $v(\mathfrak{a} +$

$\mathfrak{b}) = \min\{v(\mathfrak{a}), v(\mathfrak{b})\}$ for any primary ideals $\mathfrak{a}$, $\mathfrak{b}$, whereas we only have $v(\phi+\psi) \geq \min\{v(\phi), v(\psi)\}$ for polynomials ϕ, ψ.

An important element of $\mathcal{V}_0$ is the valuation ord_0 defined by

$$\mathrm{ord}_0(\phi) = \max\{k \geq 0 \mid \phi \in \mathfrak{m}_0^k\}.$$

Note that $v(\phi) \geq \mathrm{ord}_0(\phi)$ for all $v \in \mathcal{V}_0$ and all $\phi \in R$.

Any semivaluation $v \in \mathbf{A}^2_{\mathrm{Berk}}$ extends as a function $v : F \to [-\infty, +\infty]$, where F is the fraction field of R, by setting $v(\phi_1/\phi_2) = v(\phi_1) - v(\phi_2)$; this is well defined since $\{v = +\infty\} \subseteq R$ is a prime ideal.

Our goal for now is to justify the name "valuative tree" by showing that $\mathcal{V}_0$ can be equipped with a natural tree structure, rooted at ord_0. This structure can be obtained from many different points of view, as explained in [FJ04]. Here we focus on a geometric approach that is partially generalizable to higher dimensions (see [BFJ08b]).

7.3. **Blowups and log resolutions.** We will consider birational morphisms

$$\pi : X_\pi \to \mathbf{A}^2,$$

with X_π smooth, that are isomorphisms above $\mathbf{A}^2 \setminus \{0\}$. Such a π is necessarily a finite composition of point blowups; somewhat sloppily we will refer to it simply as a *blowup*. The set $\mathfrak{B}_0$ of blowups is a partially ordered set: we say $\pi \leq \pi'$ if the induced birational map $X_{\pi'} \to X_\pi$ is a morphism (and hence itself a composition of point blowups). In fact, $\mathfrak{B}_0$ is a directed system: any two blowups can be dominated by a third.

7.3.1. *Exceptional primes.* An irreducible component $E \subseteq \pi^{-1}(0)$ is called an *exceptional prime* (divisor) of π. There are as many exceptional primes as the number of points blown up. We often identify an exceptional prime of π with its strict transform to any blowup $\pi' \in \mathfrak{B}_0$ dominating π. In this way we can identify an exceptional prime E (of some blowup π) with the corresponding divisorial valuation ord_E.

If π_0 is the simple blowup of the origin, then there is a unique exceptional prime E_0 of π_0 whose associated divisorial valuation is $\mathrm{ord}_{E_0} = \mathrm{ord}_0$. Since any blowup $\pi \in \mathfrak{B}_0$ factors through π_0, E_0 is an exceptional prime of any π.

7.3.2. *Free and satellite points.* The following terminology is convenient and commonly used in the literature. Consider a closed point $\xi \in \pi^{-1}(0)$ for some blowup $\pi \in \mathfrak{B}_0$. We say that ξ is a *free* point if it belongs to a unique exceptional prime; otherwise it is the intersection point of two distinct exceptional primes and is called a *satellite* point.

7.3.3. *Exceptional divisors.* A divisor on X_π is *exceptional* if its support is contained in $\pi^{-1}(0)$. We write $\mathrm{Div}(\pi)$ for the abelian group of exceptional divisors on X_π. If E_i, $i \in I$, are the exceptional primes of π, then $\mathrm{Div}(\pi) \simeq \bigoplus_{i\in I} \mathbf{Z}E_i$.

If π, π' are blowups and $\pi' = \pi \circ \mu \geq \pi$, then there are natural maps

$$\mu^* : \mathrm{Div}(\pi) \to \mathrm{Div}(\pi') \quad \text{and} \quad \mu_* : \mathrm{Div}(\pi') \to \mathrm{Div}(\pi)$$

satisfying the projection formula $\mu_*\mu^* = \mathrm{id}$. In many circumstances it is natural to identify an exceptional divisor $Z \in \mathrm{Div}(\pi)$ with its pullback $\mu^*Z \in \mathrm{Div}(\pi')$.

7.3.4. *Intersection form.* We denote by $(Z \cdot W)$ the intersection number between exceptional divisors $Z, W \in \mathrm{Div}(\pi)$. If $\pi' = \pi \circ \mu$, then $(\mu^* Z \cdot W') = (Z \cdot \mu_* W')$ and hence $(\mu^* Z \cdot \mu^* W) = (Z \cdot W)$ for $Z, W \in \mathrm{Div}(\pi)$, $Z' \in \mathrm{Div}(\pi')$.

Proposition 7.2. *The intersection form on* $\mathrm{Div}(\pi)$ *is negative definite and unimodular.*

Proof. We argue by induction on the number of blowups in π. If $\pi = \pi_0$ is the simple blowup of $0 \in \mathbf{A}^2$, then $\mathrm{Div}(\pi) = \mathbf{Z} E_0$ and $(E_0 \cdot E_0) = -1$. For the inductive step, suppose $\pi' = \pi \circ \mu$, where μ is the simple blowup of a closed point on $\pi^{-1}(0)$, resulting in an exceptional prime E. Then we have an orthogonal decomposition $\mathrm{Div}(\pi') = \mu^* \mathrm{Div}(\pi) \oplus \mathbf{Z} E$. The result follows since $(E \cdot E) = -1$.

Alternatively, we may view $\mathbf{A}^2$ as embedded in $\mathbf{P}^2$ and X_π accordingly embedded in a smooth compact surface $\bar{X}_\pi$. The proposition can then be obtained as a consequence of the Hodge Index Theorem [Har77, p.364] and Poincaré Duality applied to the smooth rational surface $\bar{X}_\pi$. □

7.3.5. *Positivity.* It follows from Proposition 7.2 that for any $i \in I$ there exists a unique divisor $\check{E}_i \in \mathrm{Div}(\pi)$ such that $(\check{E}_i \cdot E_i) = 1$ and $(\check{E}_i \cdot E_j) = 0$ for $j \neq i$.

An exceptional divisor $Z \in \mathrm{Div}(\pi)$ is *relatively nef*[19] if $(Z \cdot E_i) \geq 0$ for all exceptional primes E_i. We see that the set of relatively nef divisors is a free semigroup generated by the $\check{E}_i$, $i \in I$. Similarly, the set of *effective* divisors is a free semigroup generated by the E_i, $i \in I$.

Using the negativity of the intersection form and some elementary linear algebra, one shows that the divisors $\check{E}_i$ have strictly negative coefficients in the basis $(E_j)_{j \in I}$. Hence any relatively nef divisor is antieffective.[20]

We encourage the reader to explicitly construct the divisors $\check{E}_i$ using the procedure in the proof of Proposition 7.2. Doing this, one sees directly that $\check{E}_i$ is antieffective. See also §7.4.7.

7.3.6. *Invariants of exceptional primes.* To any exceptional prime E (or the associated divisorial valuation $\mathrm{ord}_E \in \hat{\mathcal{V}}_0^*$) we can associate two basic numerical invariants α_E and A_E. We shall not directly use them in this paper, but they seem quite fundamental and their cousins at infinity (see §9.3.3) will be of great importance.

To define α_E, pick a blowup $\pi \in \mathfrak{B}_0$ for which E is an exceptional prime. Above we defined the divisor $\check{E} = \check{E}_\pi \in \mathrm{Div}(\pi)$ by duality: $(\check{E}_\pi \cdot E) = 1$ and $(\check{E}_\pi \cdot F) = 0$ for all exceptional primes $F \neq E$ of π. Note that if $\pi' \in \mathfrak{B}_0$ dominates π, then the divisor $\check{E}_{\pi'} \in \mathrm{Div}(\pi')$ is the pullback of $\check{E}_\pi$ under the morphism $X_{\pi'} \to X_\pi$. In particular, the self-intersection number

$$\alpha_E := \alpha(\mathrm{ord}_E) := (\check{E} \cdot \check{E})$$

is an integer independent of the choice of π. Since $\check{E}$ is antieffective, $\alpha_E \leq -1$.

[19]The acronym “nef” is due to M. Reid who meant it to stand for “numerically eventually free” although many authors refer to it as “numerically effective”.

[20]A higher-dimensional version of this result is known as the “Negativity Lemma” in birational geometry: see [KM98, Lemma 3.39] and also [BdFF10, Proposition 2.11].

The second invariant is the *log discrepancy* A_E.[21] This is an important invariant in higher dimensional birational geometry, see [Kol97]. Here we shall use a definition adapted to our purposes. Let ω be a nonvanishing regular 2-form on $\mathbf{A}^2$. If $\pi \in \mathfrak{B}_0$ is a blowup, then $\pi^*\omega$ is a regular 2-form on X_π. For any exceptional prime E of π with associated divisorial valuation $\mathrm{ord}_E \in \hat{\mathcal{V}}_0^*$, we define

$$A_E := A(\mathrm{ord}_E) := 1 + \mathrm{ord}_E(\pi^*\omega). \tag{7.1}$$

Note that $\mathrm{ord}_E(\pi^*\omega)$ is simply the order of vanishing along E of the Jacobian determinant of π. The log discrepancy A_E is a positive integer whose value does not depend on the choice of π or ω. A direct calculation shows that $A(\mathrm{ord}_0) = 2$.

7.3.7. *Ideals and log resolutions.* A *log resolution* of a primary ideal $\mathfrak{a} \subseteq R$ is a blowup $\pi \in \mathfrak{B}_0$ such that the ideal sheaf $\mathfrak{a} \cdot \mathcal{O}_{X_\pi}$ on X_π is locally principal:

$$\mathfrak{a} \cdot \mathcal{O}_{X_\pi} = \mathcal{O}_{X_\pi}(Z) \tag{7.2}$$

for some exceptional divisor $Z = Z_\pi(\mathfrak{a}) \in \mathrm{Div}(\pi)$. This means that the pullback of the ideal $\mathfrak{a}$ to X_π is locally generated by a single monomial in coordinates defining the exceptional primes. It is an important basic fact that any primary ideal $\mathfrak{a} \subseteq R$ admits a log resolution.

If π is a log resolution of $\mathfrak{a}$ and $\pi' = \pi \circ \mu \geq \pi$, then π' is also a log resolution of $\mathfrak{a}$ and $Z_{\pi'}(\mathfrak{a}) = \mu^* Z_\pi(\mathfrak{a})$.

Example 7.3. The ideal $\mathfrak{a} = (z_2^2 - z_1^3, z_1^2 z_2)$ admits a log resolution that is a composition of four point blowups. Each time we blow up the base locus of the strict transform of $\mathfrak{a}$. The first blowup is at the origin. In the terminology of §7.3.2, the second and fourth blowups occur at free points whereas the third blowup is at a satellite point. See Figure 7.1.

7.3.8. *Ideals and positivity.* The line bundle $\mathcal{O}_{X_\pi}(Z)$ on X_π in (7.2) is *relatively base point free*, that is, it admits a nonvanishing section at any point of $\pi^{-1}(0)$. Conversely, if $Z \in \mathrm{Div}(\pi)$ is an exceptional divisor such that $\mathcal{O}_{X_\pi}(Z)$ is relatively base point free, then $Z = Z_\pi(\mathfrak{a})$ for $\mathfrak{a} = \pi_*\mathcal{O}_{X_\pi}(Z)$.

If a line bundle $\mathcal{O}_{X_\pi}(Z)$ is relatively base point free, then its restriction to any exceptional prime E is also base point free, implying $(Z \cdot E) = \deg(\mathcal{O}_{X_\pi}(Z)|_E) \geq 0$, so that Z is relatively nef. It is an important fact that the converse implication also holds:

Proposition 7.4. *If $Z \in \mathrm{Div}(\pi)$ is relatively nef, then the line bundle $\mathcal{O}_{X_\pi}(Z)$ is relatively base point free.*

Since $0 \in \mathbf{A}^2$ is a trivial example of a rational singularity, Proposition 7.4 is merely a special case of a result by Lipman, see [Lip69, Proposition 12.1 (ii)]. The proof in *loc. cit.* uses sheaf cohomology as well as the Zariski-Grothendieck theorem on formal functions, techniques that will not be exploited elsewhere in the paper. Here we outline a more elementary proof, taking advantage of $0 \in \mathbf{A}^2$ being a smooth point and working over an algebraically closed ground field.

[21] The log discrepancy is called *thinness* in [FJ04, FJ05a, FJ05b, FJ07].

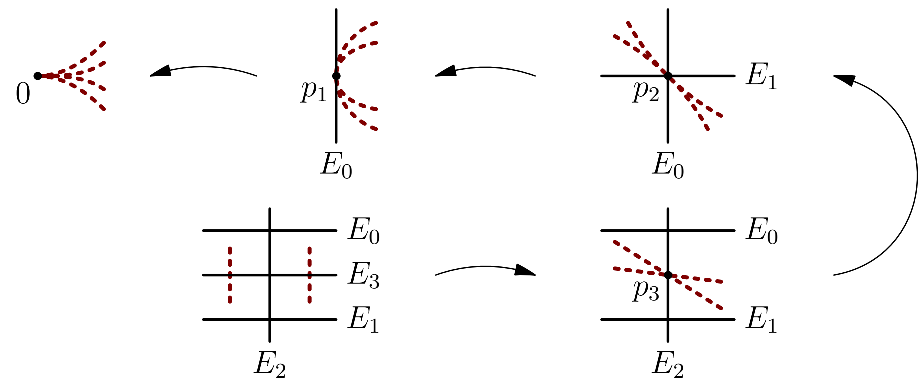


FIGURE 7.1. A log resolution of the primary ideal $\mathfrak{a} = (z_2^2 - z_1^3, z_1^2 z_2)$. The dotted curves show the strict transforms of curves of the form $C_a = \{z_2^2 - z_1^3 = a z_1^2 z_2\}$ for two different values of $a \in K^*$. The first blowup is the blowup of the origin; then we successively blow up the intersection of the exceptional divisor with the strict transform of the curves C_a. In the terminology of §7.3.2, the second and fourth blowups occur at free points whereas the third blowup is at a satellite point.

Sketch of proof of Proposition 7.4. By the structure of the semigroup of relatively nef divisors, we may assume $Z = \check{E}$ for an exceptional prime E of π. Pick two distinct free points ξ_1, ξ_2 on E and formal curves $\tilde{C}_i$ at ξ_i, $i = 1, 2$, intersecting E transversely. Then $C_i := \pi(\tilde{C}_i)$, $i = 1, 2$ are formal curves at $0 \in \mathbf{A}^2$ satisfying $\pi^* C_i = \tilde{C}_i + G_i$, where $G_i \in \mathrm{Div}(\pi)$ is an exceptional divisor. Now $(\pi^* C_i \cdot F) = 0$ for every exceptional prime F of π, so $(G_i \cdot F) = -(\tilde{C}_i \cdot F) = -\delta_{EF} = (-\check{E} \cdot F)$. Since the intersection pairing on $\mathrm{Div}(\pi)$ is nondegenerate, this implies $G_i = -\check{E}$, that is, $\pi^* C_i = \tilde{C}_i - \check{E}$ for $i = 1, 2$.

Pick $\phi_i \in \widehat{\mathcal{O}}_{\mathbf{A}^2,0}$ defining C_i. Then the ideal $\hat{\mathfrak{a}}$ generated by ϕ_1 and ϕ_2 is primary so the ideal $\mathfrak{a} := \hat{\mathfrak{a}} \cap \mathcal{O}_{\mathbf{A}^2,0}$ is also primary and satisfies $\mathfrak{a} \cdot \widehat{\mathcal{O}}_{\mathbf{A}^2,0} = \hat{\mathfrak{a}}$. Since $\mathrm{ord}_F(\mathfrak{a}) = \mathrm{ord}_F(\phi_i) = -\mathrm{ord}_F(\check{E})$, $i = 1, 2$, for any exceptional prime F and the (formal) curves $\tilde{C}_i$ are disjoint, it follows that $\mathfrak{a} \cdot \mathcal{O}_{X_\pi} = \mathcal{O}_{X_\pi}(\check{E})$ as desired. □

7.4. **Dual graphs and fans.** To a blowup $\pi \in \mathfrak{B}_0$ we can associate two basic combinatorial objects, equipped with additional structure.

7.4.1. *Dual graph.* First we have the classical notion of the *dual graph* $\Delta(\pi)$. This is an abstract simplicial complex of dimension one. Its vertices correspond to exceptional primes of π and its edges to proper intersections between exceptional primes. In the literature one often labels each vertex with the self-intersection number of the corresponding exceptional prime. We shall not do so here since this number is not an invariant of the corresponding divisorial valuation but depends also on the blowup π. From the point of view of these notes, it is more natural to use invariants such as the ones in §7.3.6.

The dual graph $\Delta(\pi)$ is connected and simply connected. This can be seen using the decomposition of π as a composition of point blowups, see §7.4.3. Alternatively, the connectedness of $\Delta(\pi)$ follows from Zariski's Main Theorem [Har77, p.280] and the simple connectedness can be deduced from sheaf cohomology considerations, see [Art66, Corollary 7].

See Figure 7.2 for an example of a dual graph.

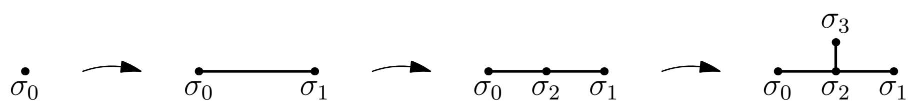


FIGURE 7.2. The dual graphs of the blowups leading up to the log resolution of the primary ideal $\mathfrak{a} = (z_2^2 - z_1^3, z_1^2 z_2)$ described in Example 7.3 and depicted in Figure 7.1. Here σ_i is the vertex corresponding to E_i.

7.4.2. *Dual fan.* While the dual graph $\Delta(\pi)$ is a natural object, the *dual fan* $\hat{\Delta}(\pi)$ is arguably more canonical. To describe it, we use basic notation and terminology from toric varieties, see [KKMS73, Ful93, Oda88].[22] Set

$$N(\pi) := \mathrm{Hom}(\mathrm{Div}(\pi), \mathbf{Z}).$$

If we label the exceptional primes E_i, $i \in I$, then we can write $N(\pi) = \bigoplus_{i \in I} \mathbf{Z} e_i \simeq \mathbf{Z}^I$ with e_i satisfying $\langle e_i, E_j \rangle = \delta_{ij}$. Note that if we identify $N(\pi)$ with $\mathrm{Div}(\pi)$ using the unimodularity of the intersection product (Proposition 7.2), then e_i corresponds to the divisor $\check{E}_i$ in §7.3.5.

Set $N_{\mathbf{R}}(\pi) := N(\pi) \otimes_{\mathbf{Z}} \mathbf{R} \simeq \mathbf{R}^I$. The one-dimensional cones in $\hat{\Delta}(\pi)$ are then of the form $\hat{\sigma}_i := \mathbf{R}_+ e_i$, $i \in I$, and the two-dimensional cones are of the form $\hat{\sigma}_{ij} := \mathbf{R}_+ e_i + \mathbf{R}_+ e_j$, where $i, j \in I$ are such that E_i and E_j intersect properly. Somewhat abusively, we will write $\hat{\Delta}(\pi)$ both for the fan and for its support (which is a subset of $N_{\mathbf{R}}(\pi)$).

Note that the dual fan $\hat{\Delta}(\pi)$ is naturally a cone over the dual graph $\Delta(\pi)$. In §7.4.6 we shall see how to embed the dual graph inside the dual fan.

A point $t \in \hat{\Delta}(\pi)$ is *irrational* if $t = t_1 e_1 + t_2 e_2$ with $t_i > 0$ and $t_1/t_2 \notin \mathbf{Q}$; otherwise t is *rational*. Note that the rational points are always dense in $\hat{\Delta}(\pi)$. The irrational points are also dense except if $\pi = \pi_0$, the simple blowup of $0 \in \mathbf{A}^2$.

7.4.3. *Free and satellite blowups.* Using the factorization of birational surface maps into simple point blowups, we can understand the structure of the dual graph and fan of a blowup $\pi \in \mathfrak{B}_0$.

First, when $\pi = \pi_0$ is a single blowup of the origin, there is a unique exceptional prime E_0, so $\hat{\Delta}(\pi_0)$ consists of a single, one-dimensional cone $\hat{\sigma}_0 = \mathbf{R}_+ e_0$ and $\Delta(\pi) = \{\sigma_0\}$ is a singleton.

Now suppose π' is obtained from π by blowing up a closed point $\xi \in \pi^{-1}(0)$. Let E_i, $i \in I$ be the exceptional primes of π. Write $I = \{1, 2, \dots, n-1\}$, where $n \geq 2$. If $E_n \subseteq X_{\pi'}$ is the preimage of ξ, then the exceptional primes of π' are E_i, $i \in I'$,

[22]We shall not, however, actually consider the toric variety defined by the fan $\hat{\Delta}(\pi)$.

where $I' = \{1, 2, \dots, n\}$. Recall that we are identifying an exceptional prime of π with its strict transform in $X_{\pi'}$.

To see what happens in detail, first suppose ξ is a free point, belonging to a unique exceptional prime of π, say E_1. In this case, the dual graph $\Delta(\pi')$ is obtained from $\Delta(\pi)$ by connecting a new vertex σ_n to σ_1. See Figure 7.3.

If instead ξ is a satellite point, belonging to two distinct exceptional primes of π, say E_1 and E_2, then we obtain $\Delta(\pi')$ from $\Delta(\pi)$ by subdividing the edge σ_{12} into two edges σ_{1n} and σ_{2n}. Again see Figure 7.3.

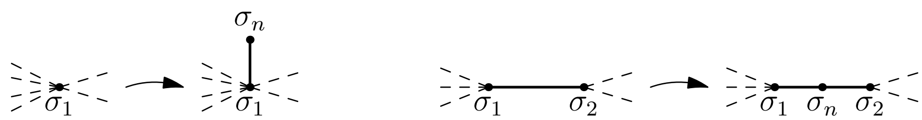


FIGURE 7.3. Behavior of the dual graph under a single blowup. The left part of the picture illustrates the blowup of a free point on E_1, creating a new vertex σ_n connected to the vertex σ_1. The right part of the picture illustrates the blowup of the satellite point $E_1 \cap E_2$, creating a new vertex σ_n and subdividing the segment σ_{12} into two segments σ_{1n} and σ_{2n}.

7.4.4. *Integral affine structure.* We define the *integral affine structure* on $\hat{\Delta}(\pi)$ to be the lattice

$$\mathrm{Aff}(\pi) := \mathrm{Hom}(N(\pi), \mathbf{Z}) \simeq \mathbf{Z}^I$$

and refer to its elements as integral affine functions. By definition, $\mathrm{Aff}(\pi)$ can be identified with the group $\mathrm{Div}(\pi)$ of exceptional divisors on X_π.

7.4.5. *Projections and embeddings.* Consider blowups $\pi, \pi' \in \mathfrak{B}_0$ with $\pi \leq \pi'$, say $\pi' = \pi \circ \mu$, with $\mu : X_{\pi'} \to X_\pi$ a birational morphism. Then μ gives rise to an injective homomorphism $\mu^* : \mathrm{Div}(\pi) \to \mathrm{Div}(\pi')$ and we let

$$r_{\pi\pi'} : N(\pi') \to N(\pi)$$

denote its transpose. It is clear that $r_{\pi\pi'} \circ r_{\pi'\pi''} = r_{\pi\pi''}$ when $\pi \leq \pi' \leq \pi''$.

Lemma 7.5. *Suppose $\pi, \pi' \in \mathfrak{B}_0$ and $\pi \leq \pi'$. Then:*

(i) $r_{\pi\pi'}(\hat{\Delta}(\pi')) = \hat{\Delta}(\pi)$*;*

(ii) *any irrational point in $\hat{\Delta}(\pi)$ has a unique preimage in $\hat{\Delta}(\pi')$;*

(iii) *if $\hat{\sigma}'$ is a 2-dimensional cone in $\hat{\Delta}(\pi)$ then either $r_{\pi\pi'}(\hat{\sigma}')$ is a one-dimensional cone in $\hat{\Delta}(\pi)$, or $r_{\pi\pi'}(\hat{\sigma}')$ is a two-dimensional cone contained in a two-dimensional cone $\hat{\sigma}$ of $\hat{\Delta}(\pi)$. In the latter case, the restriction of $r_{\pi\pi'}$ to $\hat{\sigma}'$ is unimodular in the sense that $r^*_{\pi\pi'}\,\mathrm{Aff}(\pi)|_{\hat{\sigma}'} = \mathrm{Aff}(\pi')|_{\hat{\sigma}'}$.*

We use the following notation. If e_i is a basis element of $N(\pi)$ associated to an exceptional prime E_i, then e'_i denotes the basis element of $N(\pi')$ associated to the strict transform of E_i.

Proof. It suffices to treat the case when $\pi' = \pi \circ \mu$, where μ is a single blowup of a closed point $\xi \in \pi^{-1}(0)$. As in §7.4.3 we let E_i, $i \in I$ be the exceptional primes of π. Write $I = \{1, 2, \dots, n-1\}$, where $n \geq 2$. If $E_n \subseteq X_{\pi'}$ is the preimage of ξ, then the exceptional primes of π' are E_i, $i \in I'$, where $I' = \{1, 2, \dots, n\}$.

First suppose $\xi \in E_1$ is a free point. Then $r_{\pi\pi'}(e'_i) = e_i$ for $1 \leq i < n$ and $r_{\pi\pi'}(e'_n) = e_1$. Conditions (i)-(iii) are immediately verified: $r_{\pi\pi'}$ maps the cone $\hat{\sigma}'_{1n}$ onto $\hat{\sigma}_1$ and maps all other cones $\hat{\sigma}'_{ij}$ onto the corresponding cones $\hat{\sigma}_{ij}$, preserving the integral affine structure.

Now suppose $\xi \in E_1 \cap E_2$ is a satellite point. The linear map $r_{\pi\pi'}$ is then determined by $r_{\pi\pi'}(e'_i) = e_i$ for $1 \leq i < n$ and $r_{\pi\pi'}(e'_n) = e_1 + e_2$. We see that the cones $\hat{\sigma}_{1n}$ and $\hat{\sigma}_{2n}$ in $\hat{\Delta}(\pi')$ map onto the subcones $\mathbf{R}_+ e_1 + \mathbf{R}_+(e_1 + e_2)$ and $\mathbf{R}_+ e_2 + \mathbf{R}_+(e_1 + e_2)$, respectively, of the cone $\hat{\sigma}_{12}$ in $\hat{\Delta}(\pi)$. Any other cone $\hat{\sigma}'_{ij}$ of $\hat{\Delta}(\pi')$ is mapped onto the corresponding cone $\hat{\sigma}_{ij}$ of $\hat{\Delta}(\pi)$, preserving the integral affine structure. Conditions (i)–(iii) follow. □

Using Lemma 7.5 we can show that $r_{\pi\pi'}$ admits a natural one-side inverse.

Lemma 7.6. *Let $\pi, \pi' \in \mathfrak{B}_0$ be as above. Then there exists a unique continuous, homogeneous map $\iota_{\pi'\pi} : \hat{\Delta}(\pi) \to \hat{\Delta}(\pi')$ such that:*

(i) $r_{\pi\pi'} \circ \iota_{\pi'\pi} = \mathrm{id}$ *on* $\hat{\Delta}(\pi)$;
(ii) $\iota_{\pi'\pi}(e_i) = e'_i$ *for all* i.

Further, a two-dimensional cone $\hat{\sigma}'$ in $\hat{\Delta}(\pi')$ is contained in the image of $\iota_{\pi'\pi}$ iff $r_{\pi\pi'}(\hat{\sigma}')$ is two-dimensional.

It follows easily from the uniqueness statement that $\iota_{\pi''\pi} = \iota_{\pi''\pi'} \circ \iota_{\pi'\pi}$ when $\pi \leq \pi' \leq \pi''$. We emphasize that $\iota_{\pi'\pi}$ is only *piecewise* linear and not the restriction to $\hat{\Delta}(\pi)$ of a linear map $N_{\mathbf{R}}(\pi) \to N_{\mathbf{R}}(\pi')$.

Proof of Lemma 7.6. Uniqueness is clear: when $\pi = \pi_0$ is the simple blowup of $0 \in \mathbf{A}^2$, $\iota_{\pi'\pi}$ is determined by (ii) and when $\pi \neq \pi_0$, the irrational points are dense in $\hat{\Delta}(\pi)$ and uniqueness is a consequence of Lemma 7.5 (ii).

As for existence, it suffices to treat the case when $\pi' = \pi \circ \mu$, where μ is a simple blowup of a closed point $\xi \in \pi^{-1}(0)$.

When $\xi \in E_1$ is a free point, $\iota_{\pi'\pi}$ maps e_i to e'_i for $1 \leq i < n$ and maps any cone $\hat{\sigma}_{ij}$ in $\hat{\Delta}(\pi)$ onto the corresponding cone $\hat{\sigma}'_{ij}$ in $\hat{\Delta}(\pi')$ linearly via $\iota_{\pi'\pi}(t_i e_i + t_j e_j) = (t_i e'_i + t_j e'_j)$.

If instead $\xi \in E_1 \cap E_2$ is a satellite point, then $\iota_{\pi'\pi}(e_i) = e'_i$ for $1 \leq i < n$. Further, $\iota_{\pi'\pi}$ is piecewise linear on the cone $\hat{\sigma}_{12}$:

$$\iota_{\pi'\pi}(t_1 e_1 + t_2 e_2) = \begin{cases} (t_1 - t_2)e'_1 + t_2 e'_n & \text{if } t_1 \geq t_2 \\ (t_2 - t_1)e'_2 + t_1 e'_n & \text{if } t_1 \leq t_2 \end{cases} \tag{7.3}$$

and maps any other two-dimensional cone $\hat{\sigma}_{ij}$ onto $\hat{\sigma}'_{ij}$ linearly via $\iota_{\pi'\pi}(t_i e_i + t_j e_j) = (t_i e'_i + t_j e'_j)$. □

7.4.6. *Embedding the dual graph in the dual fan.* We have noted that $\hat{\Delta}(\pi)$ can be viewed as a cone over $\Delta(\pi)$. Now we embed $\Delta(\pi)$ in $\hat{\Delta}(\pi) \subseteq N_{\mathbf{R}}$, in a way that remembers the maximal ideal $\mathfrak{m}_0$. For $i \in I$ define an integer $b_i \geq 1$ by

$$b_i := \operatorname{ord}_{E_i}(\mathfrak{m}_0),$$

where ord_{E_i} is the divisorial valuation given by order of vanishing along E_i. There exists a unique function $\varphi_0 \in \operatorname{Aff}(\pi)$ such that $\varphi_0(e_i) = b_i$. It is the integral affine function corresponding to the exceptional divisor $-Z_0 \in \operatorname{Div}(\pi)$, where $Z_0 = -\sum_{i \in I} b_i E_i$. Note that π is a log resolution of the maximal ideal $\mathfrak{m}_0$ and that $\mathfrak{m}_0 \cdot \mathcal{O}_{X_\pi} = \mathcal{O}_{X_\pi}(Z_0)$.

We now define $\Delta(\pi)$ as the subset of $\hat{\Delta}(\pi)$ given by $\varphi_0 = 1$. In other words, the vertices of $\Delta(\pi)$ are of the form

$$\sigma_i := \hat{\sigma}_i \cap \Delta(\pi) = b_i^{-1} e_i$$

and the edges of the form

$$\sigma_{ij} := \hat{\sigma}_{ij} \cap \Delta(\pi) = \{t_i e_i + t_j e_j \mid t_i, t_j \geq 0, b_i t_i + b_j t_j = 1\}.$$

If $\pi, \pi' \in \mathfrak{B}_0$ and $\pi' \geq \pi$, then $r_{\pi\pi'}(\Delta(\pi')) = \Delta(\pi)$ and $\iota_{\pi'\pi}(\Delta(\pi)) \subseteq \Delta(\pi')$.

7.4.7. *Auxiliary calculations.* For further reference let us record a few calculations involving the numerical invariants A, α and b above.

If $\pi_0 \in \mathfrak{B}_0$ is the simple blowup of the origin, then

$$A_{E_0} = 2, \quad b_{E_0} = 1, \quad \check{E}_0 = -E_0 \quad \text{and} \quad \alpha_{E_0} = -1.$$

Now suppose $\pi' = \pi \circ \mu$, where μ is the simple blowup of a closed point ξ and let us check how the numerical invariants behave. We use the notation of §7.4.3. In the case of a free blowup we have

$$A_{E_n} = A_{E_1} + 1, \quad b_{E_n} = b_{E_1} \quad \text{and} \quad \check{E}_n = \check{E}_1 - E_n, \tag{7.4}$$

where, in the right hand side, we identify the divisor $\check{E}_1 \in \operatorname{Div}(\pi)$ with its pullback in $\operatorname{Div}(\pi')$. Since $(E_n \cdot E_n) = -1$ we derive as a consequence,

$$\alpha_{E_n} := (\check{E}_n \cdot \check{E}_n) = (\check{E}_1 \cdot \check{E}_1) - 1 = \alpha_{E_1} - 1. \tag{7.5}$$

In the case of a satellite blowup,

$$A_{E_n} = A_{E_1} + A_{E_2}, \quad b_{E_n} = b_{E_1} + b_{E_2} \quad \text{and} \quad \check{E}_n = \check{E}_1 + \check{E}_2 - E_n. \tag{7.6}$$

Using $(E_n \cdot E_n) = -1$ this implies

$$\alpha_{E_n} := \alpha_{E_1} + \alpha_{E_2} + 2(\check{E}_1 \cdot \check{E}_2) - 1. \tag{7.7}$$

We also claim that if E_i, E_j are exceptional primes that intersect properly in some X_π, then

$$((b_i \check{E}_j - b_j \check{E}_i) \cdot (b_i \check{E}_j - b_j \check{E}_i)) = -b_i b_j. \tag{7.8}$$

Note that both sides of (7.8) are independent of the blowup $\pi \in \mathfrak{B}_0$ but we have to assume that E_i and E_j intersect properly in *some* blowup.

To prove (7.8), we proceed inductively. It suffices to consider the case when E_i is obtained by blowing up a closed point $\xi \in E_j$. When ξ is free, we have $b_i = b_j$, $\check{E}_i = \check{E}_j - E_i$ and (7.8) reduces to the fact that $(E_i \cdot E_i) = -1$. When instead $\xi \in E_j \cap E_k$ is a satellite point, we have $((b_i \check{E}_k - b_k \check{E}_i) \cdot (b_i \check{E}_k - b_k \check{E}_i)) = -b_i b_k$ by induction. Furthermore, $b_i = b_j + b_k$, $\check{E}_i = \check{E}_j + \check{E}_k - E_i$; we obtain (7.8) from these equations and from simple algebra.

In the dual graph depicted in Figure 7.2 we have $b_0 = b_1 = 1$, $b_2 = b_3 = 2$, $\alpha_0 = -1$, $\alpha_1 = -2$, $\alpha_2 = -6$, $\alpha_3 = -7$, $A_0 = 2$, $A_1 = 3$, $A_2 = 5$ and $A_3 = 6$.

7.4.8. *Extension of the numerical invariants.* We extend the numerical invariants A and α in §6.9 to functions on the dual fan

$$A_\pi : \hat{\Delta}(\pi) \to \mathbf{R}_+ \quad \text{and} \quad \alpha_\pi : \hat{\Delta}(\pi) \to \mathbf{R}_-$$

as follows. First we set $A_\pi(e_i) = A_{E_i}$ and extend A_π uniquely as an (integral) linear function on $\hat{\Delta}(\pi)$. Thus we set $A_\pi(t_i e_i) = t_i A_\pi(e_i)$ and

$$A_\pi(t_i e_i + t_j e_j) = t_i A_\pi(e_i) + t_j A_\pi(e_j). \tag{7.9}$$

In particular, A_π is integral affine on each simplex in the dual graph $\Delta(\pi)$.

Second, we set $\alpha_\pi(e_i) = \alpha_{E_i} = (\check{E}_i \cdot \check{E}_i)$ and extend α_π as a homogeneous function of order *two* on $\hat{\Delta}(\pi)$ which is affine on each simplex in the dual graph $\Delta(\pi)$. In other words, we set $\alpha_\pi(t_i e_i) = t_i^2 \alpha_\pi(e_i)$ for any $i \in I$ and

$$\begin{aligned}\alpha_\pi(t_i e_i + t_j e_j) &= (b_i t_i + b_j t_j)^2 \left(\frac{b_i t_i}{b_i t_i + b_j t_j} \alpha_\pi(\sigma_i) + \frac{b_j t_j}{b_i t_i + b_j t_j} \alpha_\pi(\sigma_j) \right) \\ &= (b_i t_i + b_j t_j) \left(\frac{t_i}{b_i} \alpha_\pi(e_i) + \frac{t_j}{b_j} \alpha_\pi(e_j) \right)\end{aligned} \tag{7.10}$$

whenever E_i and E_j intersect properly.

Let us check that

$$A_{\pi'} \circ \iota_{\pi'\pi} = A_\pi \quad \text{and} \quad \alpha_{\pi'} \circ \iota_{\pi'\pi} = \alpha_\pi$$

on $\hat{\Delta}(\pi)$ whenever $\pi' \geq \pi$. It suffices to do this when $\pi' = \pi \circ \mu$ and μ is the blowup of X_π at a closed point ξ. Further, the only case that requires verification is when $\xi \in E_1 \cap E_2$ is a satellite point, in which case it suffices to prove $A_\pi(e_1 + e_2) = A_{\pi'}(e'_n)$ and $\alpha_\pi(e_1 + e_2) = \alpha_{\pi'}(e'_n)$. The first of these formulas follows from (7.6) and (7.9) whereas the second results from (7.7), (7.8) and (7.10). The details are left to the reader.

In the dual graph depicted in Figure 7.2 we have $A_\pi(\sigma_0) = 2$, $A_\pi(\sigma_1) = 3$, $A_\pi(\sigma_2) = 5/2$, $A_\pi(\sigma_3) = 3$, $\alpha_\pi(\sigma_0) = -1$, $\alpha_\pi(\sigma_1) = -2$, $\alpha_\pi(\sigma_2) = -3/2$, and $\alpha_\pi(\sigma_3) = -7/4$.

7.4.9. *Multiplicity of edges in the dual graph.* We define the *multiplicity* $m(\sigma)$ of an edge σ in a dual graph $\Delta(\pi)$ as follows. Let $\sigma = \sigma_{ij}$ have endpoints $v_i = b_i^{-1} e_i$ and $v_j = b_j^{-1} e_j$. We set

$$m(\sigma_{ij}) := \gcd(b_i, b_j). \tag{7.11}$$

Let us see what happens when π' is obtained from π by blowing up a closed point $\xi \in \pi^{-1}(0)$. We use the notation above. See also Figure 7.3.

If $\xi \in E_1$ is a free point, then we have seen in (7.4) that $b_n = b_1$ and hence

$$m(\sigma_{1n}) = b_1. \tag{7.12}$$

If instead $\xi \in E_1 \cap E_2$ is a satellite point, then (7.6) gives $b_n = b_1 + b_2$ and hence

$$m(\sigma_{1n}) = m(\sigma_{2n}) = m(\sigma_{12}). \tag{7.13}$$

This shows that the multiplicity does not change when subdividing a segment.

In the dual graph depicted in Figure 7.2 we have $m_{02} = m_{12} = 1$ and $m_{23} = 2$.

7.4.10. *Metric on the dual graph.* Having embedded $\Delta(\pi)$ inside $\hat{\Delta}(\pi)$, the integral affine structure $\mathrm{Aff}(\pi)$ gives rise to an abelian group of functions on $\Delta(\pi)$ by restriction. Following [KKMS73, p.95], this further induces a volume form on each simplex in $\Delta(\pi)$. In our case, this simply means a metric on each edge σ_{ij}. The length of σ_{ij} is the largest positive number l_{ij} such that $\varphi(\sigma_i) - \varphi(\sigma_j)$ is an integer multiple of l_{ij} for all $\varphi \in \mathrm{Aff}(\pi)$. From this description it follows that $l_{ij} = \mathrm{lcm}(b_i, b_j)^{-1}$.

However, it turns out that the "correct" metric for doing potential theory is the one for which

$$d_\pi(\sigma_i, \sigma_j) = \frac{1}{b_i b_j} = \frac{1}{m_{ij}} \cdot \frac{1}{\mathrm{lcm}(b_i, b_j)}, \tag{7.14}$$

where $m_{ij} = \gcd(b_i, b_j)$ is the multiplicity of the edge σ_{ij} as in §7.4.9.

We have seen that the dual graph is connected and simply connected. It follows that $\Delta(\pi)$ is a metric tree. The above results imply that if $\pi, \pi' \in \mathfrak{B}_0$ and $\pi' \geq \pi$, then $\iota_{\pi'\pi} : \Delta(\pi) \hookrightarrow \Delta(\pi')$ is an isometric embedding.

Let us see more concretely what happens when π' is obtained from π by blowing up a closed point $\xi \in \pi^{-1}(0)$. We use the notation above.

If $\xi \in E_1$ is a free point, then $b_n = b_1$ and the dual graph $\Delta(\pi')$ is obtained from $\Delta(\pi)$ by connecting a new vertex σ_n to σ_1 using an edge of length b_1^{-2}. See Figure 7.3.

If instead $\xi \in E_1 \cap E_2$ is a satellite point, then $b_n = b_1 + b_2$ and we obtain $\Delta(\pi')$ from $\Delta(\pi)$ by subdividing the edge σ_{12}, which is of length $\frac{1}{b_1 b_2}$ into two edges σ_{1n} and σ_{2n}, of lengths $\frac{1}{b_1(b_1+b_2)}$ and $\frac{1}{b_2(b_1+b_2)}$, respectively. Note that these lengths add up to $\frac{1}{b_1 b_2}$. Again see Figure 7.3.

In the dual graph depicted in Figure 7.2 we have $d(\sigma_0, \sigma_2) = d(\sigma_1, \sigma_2) = 1/2$ and $d(\sigma_2, \sigma_3) = 1/4$.

7.4.11. *Rooted tree structure.* The dual graph $\Delta(\pi)$ is a tree in the sense of §2.1. We turn it into a rooted tree by declaring the root to be the vertex σ_0 corresponding to the strict transform of E_0, the exceptional prime of π_0, the simple blowup of 0.

When restricted to the dual graph, the functions α_π and A_π on the dual fan $\hat{\Delta}(\pi)$ described in §7.4.8 define parametrizations

$$\alpha_\pi : \Delta(\pi) \to]-\infty, -1] \quad \text{and} \quad A_\pi : \Delta(\pi) \to [2, \infty[\tag{7.15}$$

satisfying $A_{\pi'} \circ \iota_{\pi'\pi} = A_\pi$ and $\alpha_{\pi'} \circ \iota_{\pi'\pi} = \alpha_\pi$ whenever $\pi' \geq \pi$.

We claim that α_π induces the metric on the dual graph given by (7.14). For this, it suffices to show that $|\alpha_\pi(\sigma_i) - \alpha_\pi(\sigma_j)| = \frac{1}{b_i b_j}$ when E_i, E_j are exceptional primes intersecting properly. In fact, it suffices to verify this when E_i is obtained by blowing up a free point on E_j. But then $b_i = b_j$ and it follows from (7.5) that

$$\alpha_\pi(\sigma_i) - \alpha_\pi(\sigma_j) = b_i^{-2}(\alpha_{E_i} - \alpha_{E_j}) = -b_i^{-2} = -d(\sigma_i, \sigma_j).$$

In a similar way we see that the parametrization A_π of $\Delta(\pi)$ induces by the log discrepancy gives rise to the metric induced by the integral affine structure as in §7.4.10. In other words, if E_i, E_j are exceptional primes of X_π intersecting properly, then

$$A(\sigma_j) - A(\sigma_i) = -m_{ij}(\alpha(\sigma_j) - \alpha(\sigma_i)), \tag{7.16}$$

where $m_{ij} = \gcd(b_i, b_j)$ is the multiplicity of the edge σ_{ij}.

7.5. **Valuations and dual graphs.** Now we shall show how to embed the dual graph into the valuative tree.

7.5.1. *Center.* It follows from the valuative criterion of properness that any semivaluation $v \in \hat{\mathcal{V}}_0^*$ admits a *center* on X_π, for any blowup $\pi \in \mathfrak{B}_0$. The center is the unique (not necessarily closed) point $\xi = c_\pi(v) \in X_\pi$ such that $v \geq 0$ on the local ring $\mathcal{O}_{X_\pi,\xi}$ and such that $\{v > 0\} \cap \mathcal{O}_{X_\pi,\xi}$ equals the maximal ideal $\mathfrak{m}_{X_\pi,\xi}$. If $\pi' \geq \pi$, then the map $X_{\pi'} \to X_\pi$ sends $c_{\pi'}(v)$ to $c_\pi(v)$.

7.5.2. *Evaluation.* Consider a semivaluation $v \in \hat{\mathcal{V}}_0^*$ and a blowup $\pi \in \mathfrak{B}_0$. We can evaluate v on exceptional divisors $Z \in \mathrm{Div}(\pi)$. Concretely, if $Z = \sum_{i \in I} r_i E_i$, $\xi = c_\pi(v)$ is the center of v on X_π and E_j, $j \in J$ are the exceptional primes containing ξ, then $v(Z) = \sum_{j \in J} r_j v(\zeta_j)$, where $\zeta_j \in \mathcal{O}_{X_\pi,\xi}$ and $E_j = \{\zeta_j = 0\}$.

This gives rise to an *evaluation map*

$$\mathrm{ev}_\pi : \hat{\mathcal{V}}_0^* \to N_{\mathbf{R}}(\pi) \tag{7.17}$$

that is continuous, more or less by definition. The image of ev_π is contained in the dual fan $\hat{\Delta}(\pi)$. Furthermore, the embedding of the dual graph $\Delta(\pi)$ in the dual fan $\hat{\Delta}(\pi)$ was exactly designed so that $\mathrm{ev}_\pi(\mathcal{V}_0) \subseteq \Delta(\pi)$. In fact, we will see shortly that these inclusions are equalities.

It follows immediately from the definitions that

$$r_{\pi\pi'} \circ \mathrm{ev}_{\pi'} = \mathrm{ev}_\pi \tag{7.18}$$

when $\pi' \geq \pi$.

Notice that if the center of $v \in \hat{\mathcal{V}}_0^*$ on X_π is the generic point of $\bigcap_{i \in J} E_i$, then $\mathrm{ev}_\pi(v)$ lies in the relative interior of the cone $\sum_{i \in J} \mathbf{R}_+ e_i$.

7.5.3. *Embedding and quasimonomial valuations.* Next we construct a one-sided inverse to the evaluation map in (7.17).

Lemma 7.7. *Let $\pi \in \mathfrak{B}_0$ be a blowup. Then there exists a unique continuous map $\mathrm{emb}_\pi : \hat{\Delta}^*(\pi) \to \hat{\mathcal{V}}_0^*$ such that:*

(i) $\mathrm{ev}_\pi \circ \mathrm{emb}_\pi = \mathrm{id}$ *on* $\hat{\Delta}^*(\pi)$*;*
(ii) *for $t \in \hat{\Delta}^*(\pi)$, the center of* $\mathrm{emb}_\pi(t)$ *is the generic point of the intersection of all exceptional primes E_i of π such that $\langle t, E_i \rangle > 0$.*

Furthermore, condition (ii) is superfluous except in the case when $\pi = \pi_0$ is a simple blowup of $0 \in \mathbf{A}^2$ in which case the dual graph $\Delta(\pi)$ is a singleton.

As a consequence of (i), $\mathrm{emb}_\pi : \hat{\Delta}^*(\pi) \to \hat{\mathcal{V}}_0^*$ is injective and $\mathrm{ev}_\pi : \hat{\mathcal{V}}_0^* \to \hat{\Delta}^*(\pi)$ surjective.

Corollary 7.8. *If $\pi, \pi' \in \mathfrak{B}_0$ and $\pi' \geq \pi$, then* $\mathrm{emb}_{\pi'} \circ \iota_{\pi'\pi} = \mathrm{emb}_\pi$*.*

As in §6.10 we say that a valuation $v \in \hat{\mathcal{V}}_0^*$ is *quasimonomial* if it lies in the image of emb_π for some blowup $\pi \in \mathfrak{B}_0$. By Corollary 7.8, v then lies in the image of $\mathrm{emb}_{\pi'}$ for all $\pi' \geq \pi$.

Proof of Corollary 7.8. We may assume $\pi' \neq \pi$ so that π' is not the simple blowup of $0 \in \mathbf{A}^2$. The map $\mathrm{emb}'_\pi := \mathrm{emb}_{\pi'} \circ \iota_{\pi'\pi} : \Delta(\pi) \to \mathcal{V}_0$ is continuous and satisfies

$$\mathrm{ev}_\pi \circ \mathrm{emb}'_\pi = r_{\pi\pi'} \circ \mathrm{ev}_{\pi'} \circ \mathrm{emb}_{\pi'} \circ \iota_{\pi'\pi} = r_{\pi\pi'} \circ \iota_{\pi'\pi} = \mathrm{id}\,.$$

By Lemma 7.7 this implies $\mathrm{emb}'_\pi = \mathrm{emb}_\pi$. □

Proof of Lemma 7.7. We first prove existence. Consider a point $t = \sum_{i\in I} t_i e_i \in \hat{\Delta}^*(\pi)$ and let $J \subseteq I$ be the set of indices i such that $t_i > 0$. Let ξ be the generic point of $\bigcap_{i\in J} E_i$ and write $E_i = (\zeta_i = 0)$ in local algebraic coordinates ζ_i, $i \in J$ at ξ. Then we let $\mathrm{emb}_\pi(t)$ be the monomial valuation with weights t_i on ζ_i as in §6.10. More concretely, after relabeling we may assume that either $J = \{1\}$ is a singleton and $\mathrm{emb}_\pi(t) = t_1 \,\mathrm{ord}_{E_1}$ is a divisorial valuation, or $J = \{1, 2\}$ in which case v_t is defined on $R \subseteq \widehat{\mathcal{O}}_{X_\pi,\xi} \simeq K[\![\zeta_1, \zeta_2]\!]$ by

$$\mathrm{emb}_\pi(t)\Big(\sum_{\beta_1,\beta_2\geq 0} c_{\beta_1\beta_2}\zeta_1^{\beta_1}\zeta_2^{\beta_2}\Big) = \min\{t_1\beta_1 + t_2\beta_2 \mid c_\beta \neq 0\}. \tag{7.19}$$

It is clear that emb_π is continuous and that $\mathrm{ev}_\pi \circ \mathrm{emb}_\pi = \mathrm{id}$.

The uniqueness statement is clear when $\pi = \pi_0$ since the only valuation whose center on X_π is the generic point of the exceptional divisor E_0 is proportional to $\mathrm{ord}_{E_0} = \mathrm{ord}_0$.

Now suppose $\pi \neq \pi_0$ and that $\mathrm{emb}'_\pi : \hat{\Delta}^*(\pi) \to \hat{\mathcal{V}}_0^*$ is another continuous map satisfying $\mathrm{ev}_\pi \circ \iota_\pi = \mathrm{id}$. It suffices to show that $\mathrm{emb}'_\pi(t) = \mathrm{emb}_\pi(t)$ for any *irrational* $t \in \hat{\Delta}^*(\pi)$. But if t is irrational, the value of $\mathrm{emb}'_\pi(t)$ on a monomial $\zeta_1^{\beta^1}\zeta_2^{\beta_2}$ is $t_1\beta_1 + t_2\beta_2$. In particular, the values on distinct monomials are distinct, so it follows that the value of $\mathrm{emb}'_\pi(t)$ on a formal power series is given as in (7.19). Hence $\mathrm{emb}'_\pi(t) = \mathrm{emb}_\pi(t)$, which completes the proof.

In particular the divisorial valuation in $\mathcal{V}_0$ associated to the exceptional prime E_i is given by

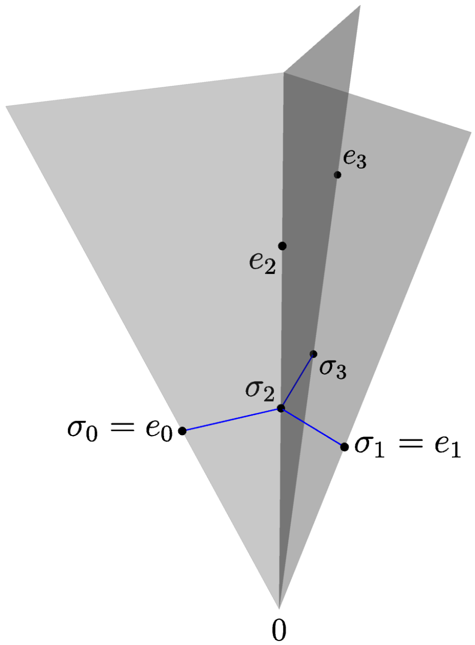
$$v_i := b_i^{-1}\,\mathrm{ord}_{E_i} \quad \text{where} \quad b_i := \mathrm{ord}_{E_i}(\mathfrak{m}_0) \in \mathbf{N}$$

□

The embedding $\mathrm{emb}_\pi : \hat{\Delta}^*(\pi) \hookrightarrow \hat{\mathcal{V}}_0^* \subseteq \mathbf{A}^2_{\mathrm{Berk}}$ extends to the full cone fan $\hat{\Delta}(\pi)$ and maps the apex $0 \in \hat{\Delta}(\pi)$ to the trivial valuation $\mathrm{triv}_{\mathbf{A}^2}$ on R. The boundary of $\mathrm{emb}_\pi : \hat{\Delta}^*(\pi)$ inside $\mathbf{A}^2_{\mathrm{Berk}}$ consists of $\mathrm{triv}_{\mathbf{A}^2}$ and the semivaluation triv_0. Thus $\mathrm{emb}_\pi(\hat{\Delta}(\pi))$ looks like a "double cone". See Figure 7.4.

7.5.4. *Structure theorem.* Because of (7.18), the evaluation maps ev_π induce a continuous map

$$\mathrm{ev} : \mathcal{V}_0 \to \varprojlim_\pi \Delta(\pi), \tag{7.20}$$



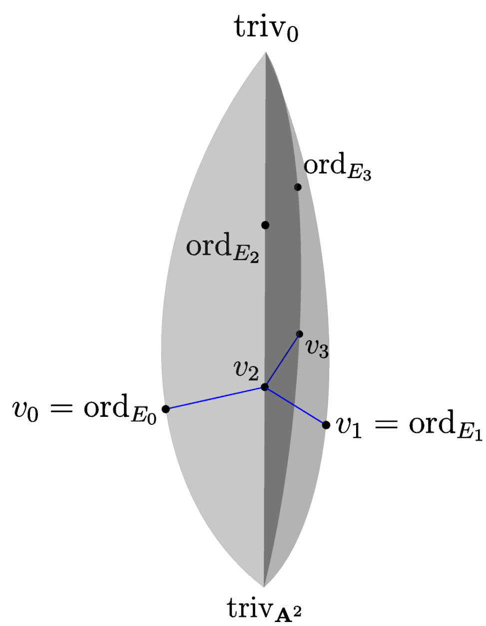


FIGURE 7.4. The dual fan of a blowup. The picture on the left illustrates the dual fan $\hat{\Delta}(\pi)$, where π is the log resolution illustrated in Figure 7.1. The picture on the left illustrates the closure of the embedding of the dual fan inside the Berkovich affine plane. The line segments illustrate the dual graph $\Delta(\pi)$ and its embedding inside the valuative tree $\mathcal{V}_0$.

where the right hand side is equipped with the inverse limit topology. Similarly, the embeddings emb_π define an embedding

$$\mathrm{emb} : \varinjlim_{\pi} \Delta(\pi) \to \mathcal{V}_0, \tag{7.21}$$

where the direct limit is defined using the maps $\iota_{\pi'\pi}$ and is equipped with the direct limit topology. The direct limit is naturally a dense subset of the inverse limit and under this identification we have $\mathrm{ev} \circ \mathrm{emb} = \mathrm{id}$.

Theorem 7.9. *The map* $\mathrm{ev} : \mathcal{V}_0 \to \varprojlim \Delta(\pi)$ *is a homeomorphism.*

By homogeneity, we also obtain a homeomorphism $\mathrm{ev} : \hat{\mathcal{V}}_0^* \to \varprojlim \hat{\Delta}^*(\pi)$.

Proof. Since r is continuous and both sides of (7.20) are compact, it suffices to show that r is bijective. The image of r contains the dense subset $\varinjlim \Delta(\pi)$ so surjectivity is clear.

To prove injectivity, pick $v, w \in \mathcal{V}_0$ with $v \neq w$. Then there exists a primary ideal $\mathfrak{a} \subseteq R$ such that $v(\mathfrak{a}) \neq w(\mathfrak{a})$. Let $\pi \in \mathfrak{B}_0$ be a log resolution of $\mathfrak{a}$ and write $\mathfrak{a} \cdot \mathcal{O}_{X_\pi} = \mathcal{O}_{X_\pi}(Z)$, where $Z \in \mathrm{Div}(\pi)$. Then

$$\langle \mathrm{ev}_\pi(v), Z \rangle = -v(\mathfrak{a}) \neq -w(\mathfrak{a}) = \mathrm{ev}_\pi(Z) = \langle \mathrm{ev}_\pi(w), Z \rangle,$$

so that $\mathrm{ev}_\pi(v) \neq \mathrm{ev}_\pi(w)$ and hence $\mathrm{ev}(v) \neq \mathrm{ev}(w)$. □

7.5.5. *Integral affine structure.* We set

$$\mathrm{Aff}(\hat{\mathcal{V}}_0^*) = \varinjlim_{\pi} \mathrm{ev}_\pi^* \mathrm{Aff}(\pi).$$

Thus a function $\varphi : \hat{\mathcal{V}}_0^* \to \mathbf{R}$ is integral affine iff it is of the form $\varphi = \varphi_\pi \circ \mathrm{ev}_\pi$, with $\varphi_\pi \in \mathrm{Aff}(\pi)$. In other words, φ is defined by an exceptional divisor in some blowup.

7.6. **Tree structure on $\mathcal{V}_0$.** Next we use Theorem 7.9 to equip $\mathcal{V}_0$ with a tree structure.

7.6.1. *Metric tree structure.* The metric on a dual graph $\Delta(\pi)$ defined in §7.4.10 turns this space into a finite metric tree in the sense of §2.2. Further, if $\pi' \geq \pi$, then the embedding $\iota_{\pi'\pi} : \Delta(\pi) \hookrightarrow \Delta(\pi')$ is an isometry. It then follows from the discussion in §2.2.2 that $\mathcal{V}_0 \simeq \varprojlim \Delta(\pi)$ is a metric tree.

Lemma 7.10. *The ends of $\mathcal{V}_0$ are exactly the valuations that are not quasimonomial.*

Proof. The assertion in the lemma amounts to the ends of the tree $\varprojlim \Delta(\pi)$ being exactly the points that do not belong to any single dual graph. It is clear that all points of the latter type are ends. On the other hand, if $t \in \Delta(\pi)$ for some blowup π, then there exists a blowup $\pi' \in \mathfrak{B}_0$ dominating π such that $\iota_{\pi'\pi}(t)$ is not an end of $\Delta(\pi')$. When t is already not an endpoint of $\Delta(\pi)$, this is clear. Otherwise $t = b_i^{-1} e_i$, in which case π' can be chosen as the blowup of a free point on the associated exceptional prime E_i. □

The hyperbolic space $\mathbf{H} \subseteq \mathcal{V}_0$ induced by the generalized metric on $\mathcal{V}_0$ contains all quasimonomial valuations but also some non-quasimonomial ones, see §7.7.5.

7.6.2. *Rooted tree structure.* We choose the valuation ord_0 as the root of the tree $\mathcal{V}_0$ and write $\leq$ for the corresponding partial ordering.

The two parametrizations α_π and A_π on the dual graph $\Delta(\pi)$ in §7.4.11 give rise to parametrizations[23]

$$\alpha : \mathcal{V}_0 \to [-\infty, -1] \quad \text{and} \quad A : \mathcal{V}_0 \to [2, \infty]. \tag{7.22}$$

The parametrization α gives rise to the generalized metric on $\mathcal{V}_0$ and we have

$$\alpha(v) = -(1 + d(v, \mathrm{ord}_0)). \tag{7.23}$$

The choice of parametrization will be justified in §7.8.1. Note that hyperbolic space $\mathbf{H} \subseteq \mathcal{V}_0$ is given by $\mathbf{H} = \{\alpha > -\infty\}$.

There is also a unique, lower semicontinuous *multiplicity* function

$$m : \mathcal{V}_0 \to \mathbf{N} \cup \{\infty\}$$

on $\mathcal{V}_0$ induced by the multiplicity on dual graphs. It has the property that $m(w) divides m(v)$ if $w \leq v$. The two parametrizations α and A are related through the multiplicity by

$$A(v) = 2 + \int_{\mathrm{ord}_0}^{v} m(w)\, d\alpha(w);$$

this follows from (7.16).

[23]The increasing parametrization $-\alpha$ is denoted by α and called *skewness* in [FJ04]. The increasing parametrization A is called *thinness* in *loc. cit.* .

There is also a generalized metric induced by A, but we shall not use it.

7.6.3. *Retraction.* It will be convenient to regard the dual graph and fan as subsets of the valuation spaces $\mathcal{V}_0$ and $\hat{\mathcal{V}}_0$, respectively. To this end, we introduce

$$|\Delta(\pi)| := \mathrm{emb}_\pi(\Delta(\pi)) \quad \text{and} \quad |\hat{\Delta}^*(\pi)| := \mathrm{emb}_\pi(\hat{\Delta}^*(\pi)).$$

Note that if $\pi' \geq \pi$, then $|\hat{\Delta}^*(\pi)| \subseteq |\hat{\Delta}^*(\pi')|$.

The evaluation maps now give rise to *retractions*

$$r_\pi := \mathrm{emb}_\pi \circ \mathrm{ev}_\pi$$

of $\hat{\mathcal{V}}_0^*$ and $\mathcal{V}_0$ onto $|\Delta_0^*|$ and $|\Delta(\pi)|$, respectively. It is not hard to see that $r'_\pi \circ r_\pi = r_\pi$ when $\pi' \geq \pi$.

Let us describe the retraction in more detail. Let $\xi = c_\pi(v)$ be the center of v on X_π and let E_i, $i \in J$ be the exceptional primes containing ξ. Write $E_i = (\zeta_i = 0)$ in local algebraic coordinates ζ_i at ξ and set $t_i = v(\zeta_i) > 0$. Then $w := r_\pi(v) \in |\hat{\Delta}^*(\pi)|$ is the monomial valuation such that $w(\zeta_i) = t_i$, $i \in J$.

It follows from Theorem 7.9 that

$$r_\pi \to \mathrm{id} \quad \text{as } \pi \to \infty.$$

In fact, we have the following more precise result.

Lemma 7.11. *If $v \in \hat{\mathcal{V}}_0^*$ and $\pi \in \mathfrak{B}_0$ is a blowup, then*

$$(r_\pi v)(\mathfrak{a}) \leq v(\mathfrak{a})$$

for every ideal $\mathfrak{a} \subseteq R$, with equality if the strict transform of $\mathfrak{a}$ to X_π does not vanish at the center of v on X_π. In particular, equality holds if $\mathfrak{a}$ is primary and π is a log resolution of $\mathfrak{a}$.

Proof. Pick $v \in \hat{\mathcal{V}}_0^*$ and set $w = r_\pi(v)$. Let ξ be the center of v on X_π and $E_i = (\zeta_i = 0)$, $i \in J$, the exceptional primes of π containing ξ. By construction, w is the smallest valuation on $\widehat{\mathcal{O}}_{X_\pi,\xi}$ taking the same values as v on the ζ_i. Thus $w \leq v$ on $\widehat{\mathcal{O}}_{X_\pi,\xi} \supseteq R$, which implies $w(\mathfrak{a}) \leq v(\mathfrak{a})$ for all ideals $\mathfrak{a} \subseteq R$.

Moreover, if the strict transform of $\mathfrak{a}$ to X_π does not vanish at ξ, then $\mathfrak{a} \cdot \widehat{\mathcal{O}}_{X_\pi,\xi}$ is generated by a single monomial in the ζ_i, and then it is clear that $v(\mathfrak{a}) = w(\mathfrak{a})$. □

7.7. **Classification of valuations.** Similarly to points in the Berkovich affine line, we can classify semivaluations in the valuative tree into four classes. The classification is discussed in detail in [FJ04] but already appears in a slightly different form in the work of Spivakovsky [Spi90]. One can show that the set of semivaluations of each of the four types below is dense in $\mathcal{V}_0$, see [FJ04, Proposition 5.3].

Recall that any semivaluation $v \in \hat{\mathcal{V}}_0^*$ extends to the fraction field F of R. In particular, it extends to the local ring $\mathcal{O}_0 := \mathcal{O}_{\mathbf{A}^2,0}$. Since $v(\mathfrak{m}_0) > 0$, v also defines a semivaluation on the completion $\widehat{\mathcal{O}}_0$.

7.7.1. *Curve semivaluations.* The subset $\mathfrak{p} := \{v = \infty\} \subsetneq \widehat{\mathcal{O}}_0$ is a prime ideal and v defines a valuation on the quotient ring $\widehat{\mathcal{O}}_0/\mathfrak{p}$. If $\mathfrak{p} \neq 0$, then $\widehat{\mathcal{O}}_0/\mathfrak{p}$ is principal and we say that v is a *curve semivaluation* as $v(\phi)$ is proportional to the order of vanishing at 0 of the restriction of ϕ to the formal curve defined by $\mathfrak{p}$. A curve semivaluation $v \in \mathcal{V}_0$ is always an endpoint in the valuative tree. One can check that they satisfy $\alpha(v) = -\infty$ and $A(v) = \infty$.

7.7.2. *Numerical invariants.* Now suppose v defines a *valuation* on $\widehat{\mathcal{O}}_0$, that is, $\mathfrak{p} = (0)$. As in §6.9 we associate to v two basic numerical invariants: the rational rank and the transcendence degree. It does not make a difference whether we compute these in R, $\mathcal{O}_0$ or $\widehat{\mathcal{O}}_0$. The Abhyankar inequality says that

$$\operatorname{tr.deg} v + \operatorname{rat.rk} v \le 2$$

and equality holds iff v is a quasimonomial valuation.

7.7.3. *Divisorial valuations.* A valuation $v \in \hat{\mathcal{V}}_0^*$ is *divisorial* if it has the numerical invariants $\operatorname{tr.deg} v = \operatorname{rat.rk} v = 1$. In this situation there exists a blowup $\pi \in \mathfrak{B}_0$ such that the center of v on X_π is the generic point of an exceptional prime E_i of π. In other words, v belongs to the one-dimensional cone $\hat{\sigma}_i$ of the dual fan $|\hat{\Delta}^*(\pi)|$ and $v = t \operatorname{ord}_{E_i}$ for some $t > 0$. We then set $b(v) := b_i = \operatorname{ord}_{E_i}(\mathfrak{m}_0)$.

More generally, suppose $v \in \hat{\mathcal{V}}_0^*$ is divisorial and $\pi \in \mathfrak{B}_0$ is a blowup such that the center of v on X_π is a closed point ξ. Then there exists a blowup $\pi' \in \mathfrak{B}_0$ dominating π in which the (closure of the) center of v is an exceptional prime of π'. Moreover, by a result of Zariski (cf. [Kol97, Theorem 3.17]), the birational morphism $X_{\pi'} \to X_\pi$ is an isomorphism above $X_\pi \setminus \{\xi\}$ and can be constructed by successively blowing up the center of v.

We will need the following result in §8.4.

Lemma 7.12. *Let $\pi \in \mathfrak{B}_0$ be a blowup and $v \in \hat{\mathcal{V}}_0^*$ a semivaluation. Set $w := r_\pi(v)$.*

(i) *if $v \notin |\hat{\Delta}^*(\pi)|$, then w is necessarily divisorial;*
(ii) *if $v \notin |\hat{\Delta}^*(\pi)|$ and v is divisorial, then $b(w)$ divides $b(v)$;*
(iii) *if $v \in |\hat{\Delta}^*(\pi)|$, then v is divisorial iff it is a rational point in the given integral affine structure; in this case, there exists a blowup $\pi' \ge \pi$ such that $|\hat{\Delta}^*(\pi')| = |\hat{\Delta}^*(\pi)|$ as subsets of $\hat{\mathcal{V}}_0^*$ and such that v belongs to a one-dimensional cone of $|\hat{\Delta}^*(\pi')|$;*
(iv) *if $v \in |\hat{\Delta}^*(\pi)|$ is divisorial and lies in the interior of a two-dimensional cone, say $\hat{\sigma}_{12}$ of $|\hat{\Delta}(\pi)|$, then $b(v) \ge b_1 + b_2$.*

Sketch of proof. For (i), let ξ be the common center of v and w on X_π. If there is a unique exceptional prime E_1 containing ξ, then it is clear that w is proportional to ord_{E_1} and hence divisorial. Now suppose ξ is the intersection point between two distinct exceptional primes E_1 and E_2. Pick coordinates ζ_1, ζ_2 at ξ such that $E_i = (\zeta_i = 0)$ for $i = 1, 2$. If $v(\zeta_1)$ and $v(\zeta_2)$ are rationally independent, then v gives different values to all monomials $\zeta_1^{\beta_1}\zeta_2^{\beta_2}$, so we must have $v = w$, contradicting $v \notin |\hat{\Delta}^*(\pi)|$. Hence $w(\zeta_1) = v(\zeta_1)$ and $w(\zeta_2) = v(\zeta_2)$ are rationally dependent, so $\operatorname{rat.rk} w = 1$. Since w is quasimonomial, it must be divisorial.

For (iii), we may assume that the center of v on X_π is the intersection point between two distinct exceptional primes $E_1 = (\zeta_1 = 0)$ and $E_2 = (\zeta_2 = 0)$ as above. Then v is monomial in coordinates (ζ_1, ζ_2) and it is clear that $\operatorname{rat.rk} v = 1$ if $v(\zeta_1)/v(\zeta_2) \in \mathbf{Q}$ and $\operatorname{rat.rk} v = 2$ otherwise. This proves the first statement. Now suppose v is divisorial. We can construct π' in (iii) by successively blowing up the center of v using the result of Zariski referred to above. Since v is monomial, the center is always a satellite point and blowing it up does not change the dual fan, viewed as a subset of $\hat{\mathcal{V}}_0^*$.

When proving (ii) we may by (iii) assume that w belongs to a one-dimensional cone $\hat{\sigma}_1$ of $|\hat{\Delta}(\pi)|$. Then $b(w) = b_1$. We now successively blow up the center of v. This leads to a sequence of divisorial valuations $w_0 = w, w_1, \dots, w_m = v$. Since the first blowup is at a free point, we have $b(w_1) = b_1$ in view of (7.12). Using (7.12) and (7.13) one now shows by induction that b_1 divides $b(w_j)$ for $j \le m$, concluding the proof of (ii).

Finally, in (iv) we obtain v after a finitely many satellite blowups, so the result follows from (7.13). □

7.7.4. *Irrational valuations.* A valuation $v \in \hat{\mathcal{V}}_0^*$ is *irrational* if $\operatorname{tr.deg} v = 0, \operatorname{rat.rk} v = 2$. In this case v is not divisorial but still quasimonomial; it belongs to a dual fan $|\hat{\Delta}^*(\pi)|$ for some blowup $\pi \in \mathfrak{B}_0$ and for any such π, v belongs to the interior of a two-dimensional cone.

7.7.5. *Infinitely singular valuations.* A valuation $v \in \hat{\mathcal{V}}_0^*$ is *infinitely singular* if it has the numerical invariants $\operatorname{rat.rk} v = 1$, $\operatorname{tr.deg} v = 0$. Every infinitely singular valuation in the valuative tree $\mathcal{V}_0$ is an end. However, some of these ends still belong to hyperbolic space $\mathbf{H}$,

Example 7.13. Consider a sequence $(v_j)_{j=0}^\infty$ defined as follows. First, $v_0 = \operatorname{ord}_0 = \operatorname{ord}_{E_0}$. Then $v_j = b_j^{-1} \operatorname{ord}_{E_j}$ is defined inductively as follows: for j odd, E_j is obtained by blowing up a free point on E_{j-1} and for j even, E_j is obtained by blowing up the satellite point $E_{j-1} \cap E_{j-2}$. The sequence $(v_{2j})_{j=0}^\infty$ is increasing and converges to an infinitely singular valuation v, see Figure 7.5. We have $b_{2n} = b_{2n+1} = 2^{-n}$, $A(v_{2n}) = 3 - 2^{-n}$ and $\alpha(v_{2n}) = -\frac{1}{3}(5 - 2^{1-2n})$. Thus $\alpha(v) = -5/3$ and $A(v) = 3$. In particular, $v \in \mathbf{H}$.

For more information on infinitely singular valuations, see [FJ04, Appendix A]. We shall not describe them further here, but they do play a role in dynamics.

7.8. **Potential theory.** In §2.5 we outlined the first elements of a potential theory on a general metric tree and in §4.9 we applied this to the Berkovich projective line.

However, the general theory applied literally to the valuative tree $\mathcal{V}_0$ does not quite lead to a satisfactory notion. The reason is that one should really view a function on $\mathcal{V}_0$ as the restriction of a homogeneous function on the cone $\hat{\mathcal{V}}_0$. In analogy with the situation over the complex numbers, one expects that for any ideal $\mathfrak{a} \subseteq R$, the function $\log|\mathfrak{a}|$ defined by[24]

$$\log|\mathfrak{a}|(v) := -v(\mathfrak{a})$$

[24]The notation reflects the fact that $|\cdot| := e^{-v}$ is a seminorm on R, see (6.1).

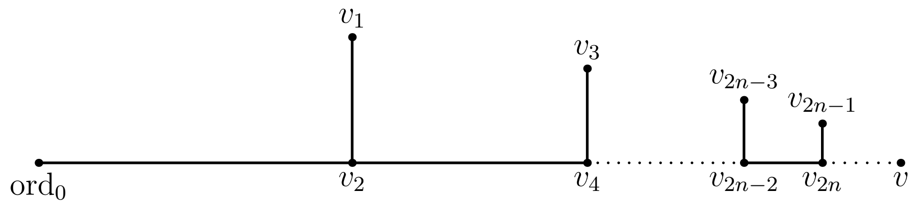


FIGURE 7.5. An infinitely singular valuation. The divisorial valuation v_j is obtained by performing a sequence of $j+1$ blowups, every other free, and every other a satellite blowup. The picture is not to scale: we have $d(v_{2n}, v_{2n+2}) = d(v_{2n+1}, v_{2n+2}) = 2^{-(2n+1)}$ for $n \geq 0$. Further, $\alpha(v) = -5/3$, $A(v) = -3$ and $d(\mathrm{ord}_0, v) = 2/3$. In particular, v belongs to hyperbolic space $\mathbf{H}$.

should be plurisubharmonic on $\hat{\mathcal{V}}_0$. Indeed, $\log|\mathfrak{a}|$ is a maximum of finitely many functions of the form $\log|\phi|$, where $\phi \in R$ is a polynomial. As a special case, the function $\log|\mathfrak{m}_0|$ should be plurisubharmonic on $\hat{\mathcal{V}}_0$. This function has a pole (with value $-\infty$) at the point triv_0 and so should definitely not be pluriharmonic on $\hat{\mathcal{V}}_0$. However, it is constantly equal to -1 on $\mathcal{V}_0$, and so would be harmonic there with the usual definition of the Laplacian.

7.8.1. *Subharmonic functions and Laplacian on $\mathcal{V}_0$.* An *ad hoc* solution to the problem above is to extend the valuative tree $\mathcal{V}_0$ to a slightly larger tree $\tilde{\mathcal{V}}_0$ by connecting the root ord_0 to a "ground" point $G \in \tilde{\mathcal{V}}_0$ using an interval of length one. See Figure 7.6.

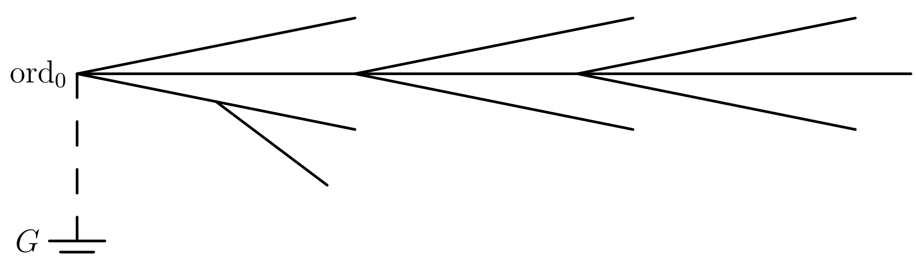


FIGURE 7.6. Connecting the valuative tree $\mathcal{V}_0$ to "ground" gives rise to the auxiliary tree $\tilde{\mathcal{V}}_0$.

Denote the Laplacian on $\tilde{\mathcal{V}}_0$ by $\tilde{\Delta}$. We define the class $\mathrm{SH}(\mathcal{V}_0)$ of *subharmonic functions*[25] on $\mathcal{V}_0$ as the set of restrictions to $\mathcal{V}_0$ of functions $\varphi \in \mathrm{QSH}(\tilde{\mathcal{V}}_0)$ with

$$\varphi(G) = 0 \quad \text{and} \quad \tilde{\Delta}\varphi = \rho - a\delta_G,$$

where ρ is a positive measure supported on $\mathcal{V}_0$ and $a = \rho(\mathcal{V}_0) \geq 0$. In particular, φ is affine of slope $\varphi(\mathrm{ord}_0)$ on the segment $[G, \mathrm{ord}_0[= \tilde{\mathcal{V}}_0 \setminus \mathcal{V}_0$. We then define

$$\Delta\varphi := \rho = (\tilde{\Delta}\varphi)|_{\mathcal{V}_0}.$$

For example, if $\varphi \equiv -1$ on $\mathcal{V}_0$, then $\tilde{\Delta}\varphi = \delta_{\mathrm{ord}_0} - \delta_G$ and $\Delta\varphi = \delta_{\mathrm{ord}_0}$.

[25] If $\varphi \in \mathrm{SH}(\mathcal{V}_0)$, then $-\varphi$ is a *positive tree potential* in the sense of [FJ04].

From this definition and the analysis in §2.5 one deduces:

Proposition 7.14. *Let $\varphi \in \mathrm{SH}(\mathcal{V}_0)$ and write $\rho = \Delta\varphi$. Then:*

(i) *φ is decreasing in the partial ordering of $\mathcal{V}_0$ rooted in* ord_0*;*
(ii) *$\varphi(\mathrm{ord}_0) = -\rho(\mathcal{V}_0)$;*
(iii) *$|D_{\vec{v}}\varphi| \le \rho(\mathcal{V}_0)$ for all tangent directions $\vec{v}$ in $\mathcal{V}_0$.*

As a consequence we have the estimate

$$-\alpha(v)\varphi(\mathrm{ord}_0) \le \varphi(v) \le \varphi(\mathrm{ord}_0) \le 0 \tag{7.24}$$

for all $v \in \mathcal{V}_0$, where $\alpha : \mathcal{V}_0 \to [-\infty, -1]$ is the parametrization given by (7.23). The exact sequence in (2.8) shows that

$$\Delta : \mathrm{SH}(\mathcal{V}_0) \to \mathcal{M}^+(\mathcal{V}_0), \tag{7.25}$$

is a homeomorphism whose inverse is given by

$$\varphi(v) = \int_{\mathcal{V}_0} \alpha(w \wedge_{\mathrm{ord}_0} v)\, d\rho(w). \tag{7.26}$$

In particular, for any $C > 0$, the set $\{\varphi \in \mathrm{SH}(\mathcal{V}_0) \mid \varphi(\mathrm{ord}_0) \ge -C\}$ is compact. Further, if $(\varphi_i)_i$ is a decreasing net in $\mathrm{SH}(\mathcal{V}_0)$, and $\varphi := \lim \varphi_i$, then either $\varphi_i \equiv -\infty$ on $\mathcal{V}_0$ or $\varphi \in \mathrm{SH}(\mathcal{V}_0)$. Moreover, if $(\varphi_i)_i$ is a family in $\mathrm{SH}(\mathcal{V}_0)$ with $\sup_i \varphi(\mathrm{ord}_\infty) < \infty$, then the upper semicontinuous regularization of $\varphi := \sup_i \varphi_i$ belongs to $\mathrm{SH}(\mathcal{V}_0)$.

7.8.2. *Subharmonic functions from ideals.* The definitions above may seem arbitrary, but the next result justifies them. It shows that the Laplacian is intimately connected to intersection numbers and shows that the generalized metric on $\mathcal{V}_0$ is the correct one.

Proposition 7.15. *If $\mathfrak{a} \subseteq R$ is a primary ideal, then the function* $\log|\mathfrak{a}|$ *on $\mathcal{V}_0$ is subharmonic. Moreover, if $\pi \in \mathfrak{B}_0$ is a log resolution of $\mathfrak{a}$, with exceptional primes E_i, $i \in I$, and if we write $\mathfrak{a} \cdot \mathcal{O}_{X_\pi} = \mathcal{O}_{X_\pi}(Z)$, then*

$$\Delta \log|\mathfrak{a}| = \sum_{i \in I} b_i (Z \cdot E_i) \delta_{v_i},$$

where $b_i = \mathrm{ord}_{E_i}(\mathfrak{m}_0)$ and $v_i = b_i^{-1} \mathrm{ord}_{E_i} \in \mathcal{V}_0$.

Proof. Write $\varphi = \log|\mathfrak{a}|$. It follows from Lemma 7.11 that $\varphi = \varphi \circ r_\pi$, so $\Delta\varphi$ is supported on the dual graph $|\Delta(\pi)| \subseteq \mathcal{V}_0$. Moreover, the proof of the same lemma shows that φ is affine on the interior of each 1-dimensional simplex so $\Delta\varphi$ is zero there. Hence it suffices to compute the mass of $\Delta\varphi$ at each v_i.

Note that π dominates π_0, the simple blowup of 0. Let E_0 be the strict transform of the exceptional divisor of π_0. Write $\mathfrak{m}_0 \cdot \mathcal{O}_{X_\pi} = \mathcal{O}_{X_\pi}(Z_0)$, where $Z_0 = -\sum_i b_i E_i$. Since π_0 already is a log resolution of $\mathfrak{m}_0$ we have

$$(Z_0 \cdot E_0) = 1 \quad \text{and} \quad (Z_0 \cdot E_j) = 0,\ j \ne 0. \tag{7.27}$$

Fix $i \in I$ and let E_j, $j \in J$ be the exceptional primes that intersect E_i properly. First assume $i \neq 0$. Using (7.27) and $(E_i \cdot E_j) = 1$ for $j \in J$ we get

$$\Delta\varphi\{v_i\} = \sum_{j\in J} \frac{\varphi(v_j) - \varphi(v_i)}{d(v_i, v_j)} = \sum_{j\in J} b_i b_j(\varphi(v_j) - \varphi(v_i)) =$$
$$= \sum_{j\in J} (b_i \operatorname{ord}_{E_j}(Z) - b_j \operatorname{ord}_{E_i}(Z))(E_i \cdot E_j) =$$
$$= b_i(Z \cdot E_i) - \operatorname{ord}_{E_i}(Z)(Z_0 \cdot E_i) = b_i(Z \cdot E_i).$$

If instead $i = 0$, then, by the definition of the Laplacian on $\mathcal{V}_0 \subseteq \tilde{\mathcal{V}}_0$, we get

$$\Delta\varphi\{v_0\} = \sum_{j\in J} \frac{\varphi(v_j) - \varphi(v_0)}{d(v_0, v_j)} + \varphi(\operatorname{ord}_0) = \sum_{j\in J} b_j(\varphi(v_j) - \varphi(v_i)) + \varphi(\operatorname{ord}_0) =$$
$$= \sum_{j\in J} (\operatorname{ord}_{E_j}(Z) - b_j\varphi(\operatorname{ord}_0))(E_j \cdot E_0) + \varphi(\operatorname{ord}_0) =$$
$$= (Z \cdot E_0) - \varphi(\operatorname{ord}_0)(Z_0 \cdot E_0) + \varphi(\operatorname{ord}_0) = (Z \cdot E_0),$$

which completes the proof. (Note that $b_0 = 1$.) □

Corollary 7.16. *If $v = v_E = b_E^{-1} \operatorname{ord}_E \in \mathcal{V}_0$ is a divisorial valuation, then there exists a primary ideal $\mathfrak{a} \subseteq R$ such that $\Delta \log |\mathfrak{a}| = b_E \delta_{v_E}$.*

Proof. Let $\pi \in \mathfrak{B}_0$ be a blowup such that E is among the exceptional primes E_i, $i \in I$. As in §7.3.5 above, define $\check{E} \in \operatorname{Div}(\pi)$ by $(\check{E} \cdot F) = \delta_{EF}$. Thus $\check{E}$ is relatively nef, so by Proposition 7.4 there exists a primary ideal $\mathfrak{a} \subseteq R$ such that $\mathfrak{a} \cdot \mathcal{O}_{X_\pi} = \mathcal{O}_{X_\pi}(\check{E})$. The result now follows from Proposition 7.15. □

Remark 7.17. One can show that the function $\log |\mathfrak{a}|$ determines a primary ideal $\mathfrak{a}$ up to integral closure. (This fact is true in any dimension.) Furthermore, the product of two integrally closed ideals is integrally closed. Corollary 7.16 therefore shows that the assignment $\mathfrak{a} \mapsto \Delta \log |\mathfrak{a}|$ is a semigroup isomorphism between integrally closed primary ideals of R and finite atomic measures on $\mathcal{V}_0$ whose mass at a divisorial valuation v_E is an integer divisible by b_E.

Corollary 7.18. *If $\phi \in R \setminus \{0\}$ is a nonzero polynomial, then the function $\log |\phi|$ on $\mathcal{V}_0$ is subharmonic. More generally, the function $\log |\mathfrak{a}|$ is subharmonic for any nonzero ideal $\mathfrak{a} \subseteq R$.*

Proof. For $n \geq 1$, the ideal $\mathfrak{a}_n := \mathfrak{a} + \mathfrak{m}_0^n$ is primary. Set $\varphi_n = \log |\mathfrak{a}_n|$. Then φ_n decreases pointwise on $\mathcal{V}_0$ to $\varphi := \log |\mathfrak{a}|$. Since the φ_n are subharmonic, so is φ. □

Exercise 7.19. If $\phi \in \mathfrak{m}_0$ is a nonzero irreducible polynomial, show that

$$\Delta \log |\phi| = \sum_{j=1}^{n} m_j \delta_{v_j}$$

where v_j, $1 \leq j \leq n$ are the curve valuations associated to the local branches C_j of $\{\phi = 0\}$ at 0 and where m_j is the multiplicity of C_j at 0, that is, $m_j = \operatorname{ord}_0(\phi_j)$, where $\phi_j \in \hat{\mathcal{O}}_0$ is a local equation of C_j. *Hint* Let $\pi \in \mathfrak{B}_0$ be an embedded resolution of singularities of the curve $C = \{\phi = 0\}$.

This exercise confirms that the generalized metric on $\mathcal{V}_0$ is the correct one.

While we shall not use it, we have the following regularization result.

Theorem 7.20. *Any subharmonic function on $\mathcal{V}_0$ is a decreasing limit of a sequence $(\varphi_n)_{n\geq 1}$, where $\varphi_n = c_n \log|\mathfrak{a}_n|$, with c_n a positive rational number and $\mathfrak{a}_n \subseteq R$ a primary ideal.*

Proof. By Theorem 2.10 (applied to the tree $\tilde{\mathcal{V}}_0$) any given function $\varphi \in \mathrm{SH}(\mathcal{V}_0)$ is the limit of a decreasing sequence $(\varphi_n)_n$ of functions in $\mathrm{SH}(\mathcal{V}_0)$ such that $\Delta\varphi_n$ is a finite atomic measure supported on quasimonomial valuations. Let $\pi_n \in \mathfrak{B}_0$ be a blowup such that $\Delta\varphi_n$ is supported on the dual graph $|\Delta(\pi_n)|$. Since the divisorial valuations are dense in $|\Delta(\pi_n)|$, we may pick $\psi_n \in \mathrm{SH}(\mathcal{V}_0)$ such that $\Delta\psi_n$ is a finite atomic measure supported on divisorial valuations in $|\Delta(\pi_n)|$, with rational weights, such that $|\psi_n - \varphi_n| \leq 2^{-n}$ on $\mathcal{V}_0$. The sequence $(\psi_n + 3\cdot 2^{-n})_{n\geq 1}$ is then decreasing and $\Delta(\psi_n + 3\cdot 2^{-n}) = \Delta\psi_n + 3\cdot 2^{-n}\delta_{\mathrm{ord}_0}$ is a finite atomic measure supported on divisorial valuations in $|\Delta(\pi_n)|$, with rational weights. The result now follows from Corollary 7.16. □

Regularization results such as Theorem 7.20 play an important role in higher dimensions, but the above proof, which uses tree arguments together with Lipman's result in Proposition 7.4, does not generalize. Instead, one can construct the ideals $\mathfrak{a}_n$ as *valuative multiplier ideals.* This is done in [FJ05b] in dimension two, and in [BFJ08b] in higher dimensions.

7.9. Intrinsic description of the tree structure on $\mathcal{V}_0$.

As explained in §7.6, the valuative tree inherits a partial ordering and a (generalized) metric from the dual graphs. We now describe these two structures intrinsically, using the definition of elements in $\mathcal{V}_0$ as functions on R. The potential theory in §7.8 is quite useful for this purpose.

7.9.1. *Partial ordering.* The following result gives an intrinsic description of the partial ordering on $\mathcal{V}_0$.

Proposition 7.21. *If $w, v \in \mathcal{V}_0$, then the following are equivalent:*

(i) *$v \leq w$ in the partial ordering induced by $\mathcal{V}_0 \simeq \varprojlim \Delta(\pi)$;*
(ii) *$v(\phi) \leq w(\phi)$ for all polynomials $\phi \in R$;*
(iii) *$v(\mathfrak{a}) \leq w(\mathfrak{a})$ for all primary ideals $\mathfrak{a} \subseteq R$.*

Proof. The implication (i) $\Longrightarrow$ (ii) is a consequence of Proposition 7.14 and the fact that $\log|\phi|$ is subharmonic. That (ii) implies (iii) is obvious. It remains to prove that (iii) implies (i). Suppose that $v \not\leq w$ in the sense of (i). After replacing v and w by $r_\pi(v)$ and $r_\pi(w)$, respectively, for a sufficiently large π, we may assume that $v, w \in |\Delta(\pi)|$. Set $v' := v \wedge w$. Then $v' < v$, $v' \leq w$ and $]v', v] \cap [v', w] = \emptyset$. Replacing v by a divisorial valuation in $]v', v]$ we may assume that v is divisorial. By Corollary 7.16 we can find an ideal $\mathfrak{a} \subseteq R$ such that $\Delta \log|\mathfrak{a}|$ is supported at v. Then $w(\mathfrak{a}) = v'(\mathfrak{a}) < v(\mathfrak{a})$, so (iii) does not hold. □

7.9.2. *Integral affine structure.* Next we give an intrinsic description of the integral affine structure.

Proposition 7.22. *If $\pi \in \mathfrak{B}_0$ is a blowup, then a function $\varphi : \hat{\mathcal{V}}_0 \to \mathbf{R}$ belongs to* $\mathrm{Aff}(\pi)$ *iff it is of the form $\varphi = \log|\mathfrak{a}| - \log|\mathfrak{b}|$, where $\mathfrak{a}$ and $\mathfrak{b}$ are primary ideals of R for which π is a common log resolution.*

Sketch of proof. After unwinding definitions this boils down to the fact that any exceptional divisor can be written as the difference of two relatively nef divisors. Indeed, by Proposition 7.4, if Z is relatively nef, then there exists a primary ideal $\mathfrak{a} \subseteq R$ such that $\mathfrak{a} \cdot \mathcal{O}_{X_\pi} = \mathcal{O}_{X_\pi}(Z)$. □

Corollary 7.23. *A function $\varphi : \hat{\mathcal{V}}_0^* \to \mathbf{R}$ is integral affine iff it is of the form $\varphi = \log|\mathfrak{a}| - \log|\mathfrak{b}|$, where $\mathfrak{a}$ and $\mathfrak{b}$ are primary ideals in R.*

7.9.3. *Metric.* Recall the parametrization α of $\mathcal{V}_0 \simeq \varprojlim \Delta(\pi)$ given by (7.23).

Proposition 7.24. *For any $v \in \mathcal{V}_0$ we have*

$$\alpha(v) = -\sup\left\{\frac{v(\phi)}{\mathrm{ord}_0(\phi)} \;\middle|\; p \in \mathfrak{m}_0\right\} = -\sup\left\{\frac{v(\mathfrak{a})}{\mathrm{ord}_0(\mathfrak{a})} \;\middle|\; \mathfrak{a} \subseteq R \ \mathfrak{m}_0\text{-primary}\right\}$$

and the suprema are attained when v is quasimonomial.

In fact, one can show that supremum in the second equality is attained *only* if v is quasimonomial. Further, the supremum in the first equality is never attained when v is infinitely singular, but is attained if v is a curve semivaluation (in which case $\alpha(v) = -\infty$), and we allow $\phi \in \mathfrak{m}_0 \cdot \widehat{\mathcal{O}}_0$.

Proof. Since the functions $\log|\mathfrak{a}|$ and $\log|\phi|$ are subharmonic, (7.24) shows that $v(\mathfrak{a}) \le -\alpha(v)\,\mathrm{ord}_0(\mathfrak{a})$ and $v(\phi) \le -\alpha(v)\,\mathrm{ord}_0(\phi)$ for all $\mathfrak{a}$ and all ϕ.

Let us prove that equality can be achieved when v is quasimonomial. Pick a blowup $\pi \in \mathfrak{B}_0$ such that $v \in |\Delta(\pi)|$ and pick $w \in |\Delta(\pi)|$ divisorial with $w \ge v$. By Corollary 7.16 there exists a primary ideal $\mathfrak{a}$ such that $\Delta \log|\mathfrak{a}|$ is supported at w. This implies that the function $\log|\mathfrak{a}|$ is affine with slope $-\mathrm{ord}_0(\mathfrak{a})$ on the segment $[\mathrm{ord}_0, w]$. In particular, $v(\mathfrak{a}) = -\alpha(v)\,\mathrm{ord}_0(\mathfrak{a})$. By picking ϕ as a general element in $\mathfrak{a}$ we also get $v(\phi) = -\alpha(v)\,\mathrm{ord}_0(\phi)$.

The case of a general $v \in \mathcal{V}_0$ follows from what precedes, given that $r_\pi v(\phi)$, $r_\pi v(\mathfrak{a})$ and $\alpha(r_\pi(v))$ converge to $v(\phi)$, $v(\mathfrak{a})$ and $\alpha(v)$, respectively, as $\pi \to \infty$. □

Notice that Proposition 7.24 gives a very precise version of the Izumi-Tougeron inequality (6.7). Indeed, $\alpha(v) > -\infty$ for all quasimonomial valuations $v \in \mathcal{V}_0$.

7.9.4. *Multiplicity.* The multiplicity function $m : \mathcal{V}_0 \to \mathbf{N} \cup \{\infty\}$ can also be characterized intrinsically. For this, one first notes that if $v = v_C$ is a curve semivaluation, defined by a formal curve C, then $m(v) = \mathrm{ord}_0(C)$. More generally, one can show that

$$m(v) = \min\{m(C) \mid v \le v_C\}.$$

In particular, $m(v) = \infty$ iff v cannot be dominated by a curve semivaluation, which in turn is the case iff v is infinitely singular.

7.9.5. *Topology.* Theorem 7.9 shows that the topology on $\mathcal{V}_0$ induced from $\mathbf{A}^2_{\mathrm{Berk}}$ coincides with the tree topology on $\mathcal{V}_0 \simeq \varprojlim \Delta(\pi)$. It is also possible to give a more geometric description of the topology.

For this, consider a blowup $\pi \in \mathfrak{B}_0$ and a *closed* point $\xi \in \pi^{-1}(0)$. Define $U(\xi) \subseteq \mathcal{V}_0$ as the set of semivaluations having center ξ on X_π. This means precisely that $v(\mathfrak{m}_\xi) > 0$, where $\mathfrak{m}_\xi$ is the maximal ideal of the local ring $\mathcal{O}_{X_\pi,\xi}$. Thus $U(\xi)$ is open in $\mathcal{V}_0$. One can in fact show that these sets $U(\xi)$ generate the topology on $\mathcal{V}_0$.

If ξ is a *free* point, belonging to a unique exceptional prime E of X_π, then we have $U(\xi) = U(\vec{v})$ for a tangent direction $\vec{v}$ at v_E in $\mathcal{V}_0$, namely, the tangent direction for which $\mathrm{ord}_\xi \in U(\vec{v})$. As a consequence, the open set $U(\xi)$ is connected and its boundary is a single point: $\partial U(\xi) = \{v_E\}$.

7.10. **Relationship to the Berkovich unit disc.** Let us briefly sketch how to relate the valuative tree with the Berkovich unit disc. Fix global coordinates (z_1, z_2) on $\mathbf{A}^2$ vanishing at 0 and let $L = K((z_1))$ be the field of Laurent series in z_1. There is a unique extension of the trivial valuation on K to a valuation v_L on L for which $v_L(z_1) = 1$. The Berkovich open unit disc over L is the set of semivaluations $v : L[z_2] \to \mathbf{R}_+$, extending v_L, for which $v(z_2) > 0$. If v is such a semivaluation, then $v/\min\{1, v(z_2)\}$ is an element in the valuative tree $\mathcal{V}_0$. Conversely, if $v \in \mathcal{V}_0$ is not equal to the curve semivaluation v_C associated to the curve $(z_1 = 0)$, then $v/v(z_1)$ defines an element in the Berkovich open unit disc over L.

Even though L is not algebraically closed, the classification of the points in the Berkovich affine line into Type 1-4 points still carries over, see §3.9.1. Curve valuations become Type 1 points, divisorial valuations become Type 2 points and irrational valuations become Type 3 points. An infinitely singular valuation $v \in \mathcal{V}_0$ is of Type 4 or Type 1, depending on whether the log discrepancy $A(v)$ is finite or infinite. The parametrization and partial orderings on $\mathcal{V}_0$ and the Berkovich unit disc are related, but different. See [FJ04, §3.9, §4.5] for more details.

Note that the identification of the valuative tree with the Berkovich unit disc depends on a choice of coordinates. In the study of polynomial dynamics in §8, it would usually not be natural to fix coordinates. The one exception to this is when studying the dynamics of a skew product

$$f(z_1, z_2) = (\phi(z_1), \psi(z_1, z_2)),$$

with $\phi(0) = 0$, in a neighborhood of the invariant line $z_1 = 0$. However, it will be more efficient to study general polynomial mappings in two variables using the Berkovich affine plane over the trivially valued field K.

As noted in §6.7, the Berkovich unit disc over the field $K((z_1))$ of Laurent series is in fact more naturally identified with the space $\mathcal{V}_C$, where $C = \{z_1 = 0\}$.

7.11. **Other ground fields.** Let us briefly comment on the case when the field K is not algebraically closed.

Let K^a denote the algebraic closure and $G = \mathrm{Gal}(K^a/K)$ the Galois group. Using general theory we have an identification $\mathbf{A}^2_{\mathrm{Berk}}(K) \simeq \mathbf{A}^2_{\mathrm{Berk}}(K^a)/G$.

First suppose that the closed point $0 \in \mathbf{A}^2(K)$ is K-rational, that is, $\mathcal{O}_0/\mathfrak{m}_0 \simeq K$. Then 0 has a unique preimage $0 \in \mathbf{A}^2(K^a)$. Let $\mathcal{V}_0(K^a) \subseteq \mathbf{A}^2_{\mathrm{Berk}}(K^a)$ denote the valuative tree at $0 \in \mathbf{A}^2(K^a)$. Every $g \in G$ induces an automorphism of $\mathbf{A}^2_{\mathrm{Berk}}(K^a)$

that leaves $\mathcal{V}_0(K^a)$ invariant. In fact, one checks that g preserves the partial ordering as well as the parametrizations α and A and the multiplicity m. Therefore, the quotient $\mathcal{V}_0(K) \simeq \mathcal{V}_0(K^a)$ also is naturally a tree. As in §3.9.1 we define a parametrization α of $\mathcal{V}_0(K)$ using the corresponding parametrization of $\mathcal{V}_0(K^a)$ and the degree of the map $\mathcal{V}_0(K^a) \to \mathcal{V}_0(K)$. This parametrization gives rise to the correct generalized metric in the sense that the analogue of Exercise 7.19 holds.

When the closed point 0 is not K-rational, it has finitely many preimages $0_j \in \mathbf{A}^2(K^a)$. At each 0_j we have a valuative tree $\mathcal{V}_{0_j} \subseteq \mathbf{A}^2_{\mathrm{Berk}}(K^a)$ and $\mathcal{V}_0$, which is now the quotient of the disjoint union of the $\mathcal{V}_{0_j}$ by G, still has a natural metric tree structure.

In fact, even when K is not algebraically closed, we can analyze the valuative tree using blowups and dual graphs much as we have done above. One thing to watch out for, however, is that the intersection form on $\mathrm{Div}(\pi)$ is no longer unimodular. Further, when E_i, E_j are exceptional primes intersecting properly, it is no longer true that $(E_i \cdot E_j) = 1$. In order to get the correct metric on the valuative tree, so that Proposition 7.15 holds for instance, we must take into account the degree over K of the residue field whenever we blow up a closed point ξ. The resulting metric is the same as the one obtained above using the Galois action.

7.12. **Notes and further references.** The valuative tree was introduced and studied extensively in the monograph [FJ04] by Favre and myself. One of our original motivations was in fact to study superattracting fixed points, but it turned out that while valuations on surfaces had been classified by Spivakovsky, the structure of this valuation space had not been explored.

It was not remarked in [FJ04] that the valuative tree can be viewed as a subset of the Berkovich affine plane over a trivially valued field. The connection that was made was with the Berkovich unit disc over the field of Laurent series.

In [FJ04], several approaches to the valuative tree are pursued. The first approach is algebraic, using *key polynomials* as developed by MacLane [Mac36]. While beautiful, this method is coordinate dependent and involves some quite delicate combinatorics. In addition, even though there is a notion of key polynomials in higher dimensions [Vaq07], these seem hard to use for our purposes.

The geometric approach, using blowups and dual graphs is also considered in [FJ04] but perhaps not emphasized as much as here. As already mentioned, this approach can be partially generalized to higher dimensions, see [BFJ08b], where it is still true that the valuation space $\mathcal{V}_0$ is an inverse limit of dual graphs. The analogue of the Laplace operator on $\mathcal{V}_0$ is then a nonlinear Monge-Ampère operator, but this operator is defined geometrically, using intersection theory, rather than through the simplicial structure of the space. In higher dimensions, the relation between the different positivity notions on exceptional divisors is much more subtle than in dimension two. Specifically, Proposition 7.4 is no longer true.

Granja [Gra07] has generalized the construction of the valuative tree to a general two-dimensional regular local ring.

The valuative tree gives an efficient way to encode singularities in two dimensions. For example, it can be used to study the singularities of planar plurisubharmonic functions, see [FJ05a, FJ05b]. It is also related to many other constructions in singularity theory. We shall not discuss this further here, but refer to the paper [Pop11] by Popescu-Pampu for further references. In this paper, the author, defines an interesting object, the *kite* (cerf-volant), which also encodes the combinatorics of the exceptional primes of a blowup.

In order to keep notes reasonably coherent, and in order to reflect changing trends, I have taken the freedom to change some of the notation and terminology from [FJ04]. Notably, in [FJ04], the valuative tree is simply denoted $\mathcal{V}$ and its elements are called valuations. Here we wanted to be more precise, so we call them semivaluations. What is called subharmonic functions here correspond to positive tree potentials in [FJ04]. The valuation ord_0 is called $\nu_{\mathfrak{m}}$ in [FJ04].

8. Local plane polynomial dynamics

Next we will see how the valuative tree can be used to study superattracting fixed points for polynomial maps of $\mathbf{A}^2$.

8.1. **Setup.** Let K be an algebraically closed field, equipped with the trivial valuation. (See §8.8 for the case of other ground fields.) Further, R and F are the coordinate ring and function field of the affine plane $\mathbf{A}^2$ over K. Recall that the Berkovich affine plane $\mathbf{A}^2_{\mathrm{Berk}}$ is the set of semivaluations on R that restrict to the trivial valuation on K.

8.2. **Definitions and results.** We briefly recall the setup from §1.2 of the introduction. Let K be an algebraically closed field of characteristic zero. Consider a polynomial mapping $f : \mathbf{A}^2 \to \mathbf{A}^2$ over K. We assume that f is *dominant*, since otherwise the image of f is contained in a curve. Consider a (closed) fixed point $0 = f(0) \in \mathbf{A}^2$ and define

$$c(f) := \mathrm{ord}_0(f^*\mathfrak{m}_0),$$

where $\mathfrak{m}_0$ denotes the maximal ideal at 0. We say that f is *superattracting* if $c(f^n) > 1$ for some $n \geq 1$.

Exercise 8.1. Show that if f is superattracting, then in fact $c(f^2) > 1$. On the other hand, find an example of a superattracting f for which $c(f) = 1$.

Exercise 8.2. Show that if f is superattracting and $K = \mathbf{C}$, then there exists a neighborhood $0 \in U \subseteq \mathbf{A}^2$ (in the usual Euclidean topology) such that $f(U) \subseteq U$, and $f^n(z) \to 0$ as $n \to \infty$ for any $z \in U$.

As mentioned in the introduction, the sequence $(c(f^n))_{n\geq 1}$ is supermultiplicative, so the limit

$$c_\infty(f) := \lim_{n\to\infty} c(f^n)^{1/n} = \sup_{n\to\infty} c(f^n)^{1/n}$$

exists.

Exercise 8.3. Verify these statements! Also show that f is superattracting iff $c_\infty(f) > 1$ iff df_0 is nilpotent.

Exercise 8.4. In coordinates (z_2, z_2) on $\mathbf{A}^2$, let f_c be the homogeneous part of f of degree $c = c(f)$. Show that if $f_c^2 \not\equiv 0$, then in fact $f_c^n \not\equiv 0$ for all $n \geq 1$, so that $c(f^n) = c^n$ and $c_\infty = c = c(f)$ is an integer.

Example 8.5. If $f(z_1, z_2) = (z_2, z_1 z_2)$, then $c(f^n)$ is the $(n+2)$th Fibonacci number and $c_\infty = \frac{1}{2}(\sqrt{5} + 1)$ is the golden mean.

For the convenience of the reader, we recall the result that we are aiming for:

Theorem B. *The number $c_\infty = c_\infty(f)$ is a quadratic integer: there exists $a, b \in \mathbf{Z}$ such that $c_\infty^2 = ac_\infty + b$. Moreover, there exists a constant $\delta > 0$ such that*

$$\delta c_\infty^n \leq c(f^n) \leq c_\infty^n$$

for all $n \geq 1$.

Here it is the left-hand inequality that is nontrivial.

8.3. **Induced map on the Berkovich affine plane.** As outlined in §1.2, we approach Theorem B by studying the induced map

$$f : \mathbf{A}^2_{\mathrm{Berk}} \to \mathbf{A}^2_{\mathrm{Berk}}$$

on the Berkovich affine plane $\mathbf{A}^2_{\mathrm{Berk}}$. Recall the subspaces

$$\mathcal{V}_0 \subseteq \hat{\mathcal{V}}_0^* \subseteq \hat{\mathcal{V}}_0 \subseteq \mathbf{A}^2_{\mathrm{Berk}}$$

introduced in §7: $\hat{\mathcal{V}}_0$ is the set of semivaluations whose center on $\mathbf{A}^2$ is the point 0. It has the structure of a cone over the valuative tree $\mathcal{V}_0$, with apex at triv_0. It is clear that

$$f(\hat{\mathcal{V}}_0) \subseteq \hat{\mathcal{V}}_0 \quad \text{and} \quad f(\mathrm{triv}_0) = \mathrm{triv}_0\,.$$

In general, f does not map the pointed cone $\hat{\mathcal{V}}_0^*$ into itself. Indeed, suppose there exists an algebraic curve $C = \{\phi = 0\} \subseteq \mathbf{A}^2$ passing through 0 and contracted to 0 by f. Then any semivaluation $v \in \hat{\mathcal{V}}_0^*$ such that $v(\phi) = \infty$ satisfies $f(v) = \mathrm{triv}_0$. To rule out this behavior, we introduce

Assumption 8.6. From now on, and until §8.6 we assume that the germ f is *finite*.

This assumption means that the ideal $f^*\mathfrak{m}_0 \subseteq \mathcal{O}_0$ is primary, that is, $\mathfrak{m}_0^s \subseteq f^*\mathfrak{m}_0$ for some $s \geq 1$, so it exactly rules out the existence of contracted curves. Certain modifications are required to handle the more general case when f is merely dominant. See §8.6 for some of this.

The finiteness assumption implies that $f^{-1}\{\mathrm{triv}_0\} = \{\mathrm{triv}_0\}$. Thus we obtain a well-defined map

$$f : \hat{\mathcal{V}}_0^* \to \hat{\mathcal{V}}_0^*,$$

which is clearly continuous and homogeneous.

While f preserves $\hat{\mathcal{V}}_0^*$, it does not preserve the "section" $\mathcal{V}_0 \subseteq \hat{\mathcal{V}}_0^*$ given by the condition $v(\mathfrak{m}_0) = 1$. Indeed, if $v(\mathfrak{m}_0) = 1$, there is no reason why $f(v)(\mathfrak{m}_0) = 1$. Rather, we define

$$c(f, v) := v(f^*\mathfrak{m}_0) \quad \text{and} \quad f_\bullet v := \frac{f(v)}{c(f,v)}.$$

The assumption that f is finite at 0 is equivalent to the existence of a constant $C > 0$ such that $1 \leq c(f, v) \leq C$ for all $v \in \mathcal{V}_0$. Indeed, we can pick C as any integer s such that $f^*\mathfrak{m}_0 \supseteq \mathfrak{m}_0^s$. Also note that

$$c(f) = c(f, \mathrm{ord}_0).$$

The normalization factors $c(f, v)$ naturally define a dynamical *cocycle*. Namely, we can look at $c(f^n, v)$ for every $n \geq 0$ and $v \in \mathcal{V}_0$ and we then have

$$c(f^n, v) = \prod_{i=0}^{n-1} c(f, v_i),$$

where $v_i = f_\bullet^i v$ for $0 \leq i < n$.

Apply this equality to $v = \mathrm{ord}_0$. By definition, we have $v_i = f_\bullet^i \mathrm{ord}_0 \geq \mathrm{ord}_0$ for all i. This gives $c(f, v_i) \geq c(f, \mathrm{ord}_0) = c(f)$, and hence $c(f^n) \geq c(f)^n$, as we already knew. More importantly, we shall use the *multiplicative cocycle* $c(f^n, v)$ in order to study the *supermultiplicative sequence* $(c(f^n))_{n\geq 0}$.

8.4. **Fixed points on dual graphs.** Consider a blowup $\pi \in \mathfrak{B}_0$. We have seen that the dual graph of π embeds as a subspace $|\Delta(\pi)| \subseteq \mathcal{V}_0$ of the valuative tree, and that there is a retraction $r_\pi : \mathcal{V}_0 \to |\Delta(\pi)|$. We shall study the selfmap

$$r_\pi f_\bullet : |\Delta(\pi)| \to |\Delta(\pi)|.$$

Notice that this map is continuous since r_π and $f_\bullet$ are. Despite appearances, it does not really define an induced dynamical system on $|\Delta(\pi)|$, as, in general, we may have $(r_\pi f_\bullet)^2 \neq r_\pi f_\bullet^2$. However, the fixed points of $r_\pi f_\bullet$ will play an important role.

It is easy to see that a continuous selfmap of a finite simplicial tree always has a fixed point. (See also Proposition 2.17.) Hence we can find $v_0 \in |\Delta(\pi)|$ such that $r_\pi f_\bullet v_0 = v_0$. There are then three possibilities:

(1) v_0 is divisorial and $f_\bullet v_0 = v_0$;
(2) v_0 is divisorial and $f_\bullet v_0 \neq v_0$;
(3) v_0 is irrational and $f_\bullet v_0 = v_0$.

Indeed, if $v \in \mathcal{V}_0 \setminus |\Delta(\pi)|$ is any valuation, then $r_\pi(v)$ is divisorial, see Lemma 7.12. The same lemma also allows us to assume, in cases (1) and (2), that the center of v_0 on X_π is an exceptional prime $E \subseteq X_\pi$.

In case (2), this means that the center of $f_\bullet v_0$ on X_π is a *free* point $\xi \in E$, that is, a point that does not belong to any other exceptional prime of π.

8.5. **Proof of Theorem B.** Using the fixed point v_0 that we just constructed, and still assuming f finite, we can now prove Theorem B.

The proof that c_∞ is a quadratic integer relies on a calculation using value groups. Recall that the value group of a valuation v is defined as $\Gamma_v = v(F)$, where F is the fraction field of R.

Lemma 8.7. *In the notation above, we have $c(f, v_0)\Gamma_{v_0} \subseteq \Gamma_{v_0}$. As a consequence, $c(f, v_0)$ is a quadratic integer.*

We shall see that under suitable assumptions on the blowup π we have $c(f, v_0) = c_\infty(f)$. This will show that $c_\infty(f)$ is a quadratic integer.

Proof. In general, $\Gamma_{f(v)} \subseteq \Gamma_v$ and $\Gamma_{r_\pi(v)} \subseteq \Gamma_v$ for $v \in \hat{\mathcal{V}}_0^*$. If we write $c_0 = c(f, v_0)$, then this leads to

$$c_0\Gamma_{v_0} = c_0\Gamma_{r_\pi f_\bullet v_0} \subseteq c_0\Gamma_{f_\bullet v_0} = \Gamma_{c_0 f_\bullet v_0} = \Gamma_{f(v_0)} \subseteq \Gamma_{v_0},$$

which proves the first part of the lemma.

Now v_0 is quasimonomial, so the structure of its value group is given by (4.5). When v_0 is divisorial, $\Gamma_{v_0} \simeq \mathbf{Z}$ and the inclusion $c_0\Gamma_{v_0} \subseteq \Gamma_{v_0}$ immediately implies that c_0 is an integer. If instead v_0 is irrational, $\Gamma_{v_0} \simeq \mathbf{Z} \oplus \mathbf{Z}$ and c_0 is a quadratic integer. Indeed, if we write $\Gamma_{v_0} = t_1\mathbf{Z} \oplus t_2\mathbf{Z}$, then there exist integers a_{ij} such that $c_0 t_i = \sum_{j=1}^2 a_{ij} t_j$ for $i = 1, 2$. But then c_0 is an eigenvalue of the matrix (a_{ij}), hence a quadratic integer. □

It remains to be seen that $c(f, v_0) = c_\infty(f)$ and that the estimates in Theorem B hold. We first consider cases (1) and (3) above, so that $f_\bullet v_0 = v_0$. It follows from (7.24) that the valuations v_0 and ord_0 are comparable. More precisely, $\mathrm{ord}_0 \leq$

$v_0 \le -\alpha_0 \operatorname{ord}_0$, where $\alpha_0 = \alpha(v_0)$. The condition $f_\bullet v_0 = v_0$ means that $f(v_0) = cv_0$, where $c = c(f, v_0)$. This leads to

$$c(f^n) = \operatorname{ord}_0(f^{n*}\mathfrak{m}_0) \le v_0(f^{n*}\mathfrak{m}_0) = (f^n_* v_0)(\mathfrak{m}_0) = c^n v_0(\mathfrak{m}_0) = c^n$$

and, similarly, $c^n \le -\alpha_0 c(f^n)$. In view of the definition of c_∞, this implies that $c_\infty = c$, so that

$$f(v_0) = c_\infty v_0 \quad \text{and} \quad -\alpha_0^{-1} c_\infty^n \le c(f^n) \le c_\infty^n,$$

proving Theorem B in this case.

Case (2) is more delicate and is in some sense the typical case. Indeed, note that we have not made any restriction on the modification π. For instance, π could be a simple blowup of the origin. In this case $|\Delta(\pi)| = \{\operatorname{ord}_0\}$ is a singleton, so $v_0 = \operatorname{ord}_0$ but there is no reason why $f_\bullet \operatorname{ord}_0 = \operatorname{ord}_0$. To avoid this problem, we make

Assumption 8.8. The map $\pi : X_\pi \to \mathbf{A}^2$ defines a log resolution of the ideal $f^*\mathfrak{m}$. In other words, the ideal sheaf $f^*\mathfrak{m} \cdot \mathcal{O}_{X_\pi}$ is locally principal.

Such a π exists by resolution of singularities. Indeed our current assumption that f be a *finite* germ implies that $f^*\mathfrak{m}$ is an $\mathfrak{m}$-primary ideal.

For us, the main consequence of π being a log resolution of $f^*\mathfrak{m}$ is that

$$c(v) = v(f^*\mathfrak{m}_0) = (r_\pi v)(f^*\mathfrak{m}_0) = c(r_\pi v)$$

for all $v \in \mathcal{V}_0$, see Lemma 7.11.

As noted above, we may assume that the center of v_0 on X_π is an exceptional prime E. Similarly, the center of $f_\bullet v_0$ on X_π is a free point $\xi \in E$. Let $U(\xi)$ be the set of all valuations $v \in \mathcal{V}_0$ whose center on X_π is the point ξ. By §7.9.5, this is a connected open set and its closure is given by $\overline{U(\xi)} = U(\xi) \cup \{v_0\}$. We have $r_\pi U(\xi) = \{v_0\}$, so $c(f, v) = c(f, v_0)$ for all $v \in U(\xi)$ by Lemma 7.11.

We claim that $f_\bullet(\overline{U(\xi)}) \subseteq U(\xi)$. To see this, we could use §2.6 but let us give a direct argument. Note that $v \ge v_0$, and hence $f(v) \ge f(v_0)$ for all $v \in \overline{U(\xi)}$. Since $c(f, v) = c(f, v_0)$, this implies $f_\bullet v \ge f_\bullet v_0 > v_0$ for all $v \in \overline{U(\xi)}$. In particular, $f_\bullet v \ne v_0$ for all $v \in \overline{U(\xi)}$, so that

$$\overline{U(\xi)} \cap f_\bullet^{-1} U(\xi) = \overline{U(\xi)} \cap f_\bullet^{-1} \overline{U(\xi)}.$$

It follows that $\overline{U(\xi)} \cap f_\bullet^{-1} U(\xi)$ is a subset of $\overline{U(\xi)}$ that is both open and closed. It is also nonempty, as it contains v_0. By connectedness of $\overline{U(\xi)}$, we conclude that $f_\bullet(\overline{U(\xi)}) \subseteq U(\xi)$.

The proof of Theorem B can now be concluded in the same way as in cases (1) and (3). Set $v_n := f_\bullet^n v_0$ for $n \ge 0$. Then we have $v_n \in \overline{U(\xi)}$ and hence $c(f, v_n) = c(f, v_0) =: c$ for all $n \ge 0$. This implies $c(f^n, v_0) = \prod_{i=0}^{n-1} c(f, v_i) = c^n$ for all $n \ge 1$. As before, this implies that $c = c_\infty$ and $-\alpha_0^{-1} c_\infty^n \le c(f^n) \le c_\infty^n$, where $\alpha_0 = \alpha(v_0) < \infty$.

8.6. **The case of a non-finite germ.** Let us briefly discuss the situation when $f : \mathbf{A}^2 \to \mathbf{A}^2$ is dominant but not finite at a fixed point $0 = f(0)$. In other words, the ideal $f^*\mathfrak{m}_0 \subseteq \mathfrak{m}_0$ is not primary. In this case, the subset $I_f \subseteq \mathcal{V}_0$ given by $c(f, \cdot) = +\infty$ is nonempty but finite. Each element of I_f is a curve valuation associated to an irreducible germ of a curve C at 0 such that $f(C) = 0$. In particular,

I_f does not contain any quasimonomial valuations. Write $\hat{I}_f = \mathbf{R}_+^* I_f$, $\hat{D}_f := \hat{\mathcal{V}}_0^* \setminus \hat{I}_f = \{c(f,\cdot) < +\infty\}$ and $D_f := \mathcal{V}_0 \setminus I_f = \hat{D}_f \cap \mathcal{V}_0$. For $v \in \hat{I}_f$ we have $f(v) = \mathrm{triv}_0$. We can view $f : \hat{\mathcal{V}}_0^* \dashrightarrow \hat{\mathcal{V}}_0^*$ as a partially defined map having domain of definition $\hat{D}_f$. On D_f we define $f_\bullet$ as before, namely $f_\bullet v = f(v)/c(f,v)$. One can show that $f_\bullet$ extends continuously through I_f to a map $f_\bullet : \mathcal{V}_0 \to \mathcal{V}_0$. More precisely, any $v \in I_f$ is associated to an analytically irreducible branch of an algebraic curve $D \subseteq \mathbf{A}^2$ for which $f(D) = 0$. The valuation $f(\mathrm{ord}_D)$ is divisorial and has 0 as its center on $\mathbf{A}^2$, hence $f(\mathrm{ord}_D) = rv_E$, where $r \in \mathbf{N}$ and $v_E \in \mathcal{V}_0$ is divisorial. The continuous extension of $f_\bullet$ across v is then given by $f_\bullet v = v_E$. In particular, $f_\bullet I_f \cap I_f = \emptyset$.

Now we can find a log resolution $\pi : X_\pi \to \mathbf{A}^2$ of the ideal $f^*\mathfrak{m}_0$. By this we mean that the ideal sheaf $f^*\mathfrak{m}_0 \cdot \mathcal{O}_{X_\pi}$ on X_π is locally principal and given by a normal crossings divisor in a neighborhood of $\pi^{-1}(0)$. We can embed the dual graph of this divisor as a finite subtree $|\Delta| \subseteq \mathcal{V}_0$. Note that $|\Delta|$ contains all elements of I_f. There is a continuous retraction map $r : \mathcal{V}_0 \to |\Delta|$. Thus we get a continuous selfmap $rf_\bullet : |\Delta| \to |\Delta|$, which admits a fixed point $v \in |\Delta|$. Note that $v \notin I_f$ since $f_\bullet I_f \cap I_f = \emptyset$ and $r^{-1} I_f = I_f$. Therefore v is quasimonomial. The proof now goes through exactly as in the finite case.

8.7. **Further properties.** Let us outline some further results from [FJ07] that one can obtain by continuing the analysis.

First, one can construct an *eigenvaluation*, by which we mean a semivaluation $v \in \mathcal{V}_0$ such that $f(v) = c_\infty v$. Indeed, suppose f is finite for simplicity and look at the three cases (1)–(3) in §8.4. In cases (1) and (3) the valuation v_0 is an eigenvaluation. In case (2) one can show that the sequence $(f_\bullet^n v_0)_{n=0}^\infty$ increases to an eigenvaluation.

Second, we can obtain local normal forms for the dynamics. For example, in Case (2) in §8.4 we showed that $f_\bullet$ mapped the open set $U(\xi)$ into itself, where $U(\xi)$ is the set of semivaluations whose center of X_π is equal to ξ, the center of $f_\bullet v_0$ on X_π. This is equivalent to the the lift $f : X_\pi \dashrightarrow X_\pi$ being regular at ξ and $f(\xi) = \xi$. By choosing X_π judiciously one can even guarantee that $f : (X_\pi, \xi) \to (X_\pi, \xi)$ is a *rigid* germ, a dynamical version of simple normal crossings singularities. Such a rigidification result was proved in [FJ07] for superattracting germs and later extended by Matteo Ruggiero [Rug09] to more general germs.

When f is finite, $f_\bullet : \mathcal{V}_0 \to \mathcal{V}_0$ is a tree map in the sense of §2.6, so the results in that section apply, but in our approach here we did not need them. In contrast, the approach in [FJ07] consists of first using the tree analysis in §2.6 to construct an eigenvaluation.

Using numerical invariants one can show that f preserves the type of a valuation in the sense of §7.7. There is also a rough analogue of the ramification locus for selfmaps of the Berkovich projective line as in §4.7. At least in the case of a finite map, the ramification locus is a finite subtree given by the convex hull of the preimages of the root ord_0.

While this is not pursued in [FJ07], the induced dynamics on the valuative tree is somewhat similar to the dynamics of a selfmap of the unit disc over $\mathbf{C}$. Indeed, recall from §7.10 that we can embed the valuative tree inside the Berkovich unit disc over the field of Laurent series (although this does not seem very useful from a dynamical point of view). In particular, the dynamics is (essentially) globally attracting. This is in sharp contrast with selfmaps of the Berkovich projective line that are nonrepelling on hyperbolic space $\mathbf{H}$.

For simplicity we only studied the dynamics of polynomial maps, but the analysis goes through also for formal fixed point germs. In particular, it applies to fixed point germs defined by rational maps of a projective surface and to holomorphic (perhaps transcendental) fixed point germs. In the latter case, one can really interpret $c_\infty(f)$ as a speed at which typical orbits tend to 0, see [FJ07, Theorem B].

8.8. **Other ground fields.** Let us briefly comment on the case when the field K is not algebraically closed. Specifically, let us argue why Theorem B continues to hold in this case.

Let K^a be the algebraic closure of K and $G = \mathrm{Gal}(K^a/K)$ the Galois group. Then $\mathbf{A}^2(K) \simeq \mathbf{A}^2(K^a)/G$ and any polynomial mapping $f : \mathbf{A}^2(K) \to \mathbf{A}^2(K)$ induces a equivariant polynomial mapping $f : \mathbf{A}^2(K^a) \to \mathbf{A}^2(K^a)$.

If the point $0 \in \mathbf{A}^2(K)$ is K-rational, then it has a unique preimage in $0 \in K^a$ and the value of $\mathrm{ord}_0(\phi)$, for $\phi \in R$, is the same when calculated over K or over K^a. The same therefore holds for $c(f^n)$, so since Theorem B holds over K^a, it also holds over K.

In general, $0 \in \mathbf{A}^2$ has finitely many preimages $0_j \in \mathbf{A}^2(K^a)$ but if $\phi \in R$ is a polynomial with coefficients in K, then $\mathrm{ord}_0(\phi) = \mathrm{ord}_{0_j}(\varphi)$ for all j. Again we can deduce Theorem B over K from its counterpart over K^a, although some care needs to be taken to prove that c_∞ is a quadratic integer in this case.

Alternatively, we can consider the action of f directly on $\mathbf{A}^2_{\mathrm{Berk}}(K)$. As noted in §7.11, the subset of semivaluations centered at 0 is still the cone over a tree and we can consider the induced dynamics. The argument for proving that c_∞ is a quadratic integer, using value groups, carries over to this setting.

8.9. **Notes and further references.** In [FJ07] and [FJ11] we used the notation f_*v instead of $f(v)$ as the action of f on the valuative tree is given as a pushforward. However, one usually does not denote induced maps on Berkovich spaces as pushforwards, so I decided to deviate from *loc. cit.* in order to keep the notation uniform across these notes.

In analogy with the degree growth of polynomial maps (see 10.7) I would expect the sequence $(c(f^n))_{n=0}^\infty$ to satisfy an integral linear recursion relation, but this has not yet been established.

My own path to Berkovich spaces came through joint work with Charles Favre. Theorem B, in a version for holomorphic selfmaps of $\mathbf{P}^2$, has ramifications for problem of equidistribution to the Green current. See [FJ03] and also [DS08, Par11] for higher dimensions.

9. The valuative tree at infinity

In order to study the dynamics at infinity of polynomial maps of $\mathbf{A}^2$ we will use the subspace of the Berkovich affine plane $\mathbf{A}^2_{\mathrm{Berk}}$ consisting of semivaluations centered at infinity. As in the case of semivaluations centered at a point, this is a cone over a tree that we call the *valuative tree at infinity*.[26] Its structure is superficially similar to that of the valuative tree at a point, which we will refer to as the *local case*, but, as we will see, there are some significant differences.

9.1. **Setup.** Let K be an algebraically closed field of characteristic zero, equipped with the trivial valuation. (See §9.8 for the case of other ground fields.) Further, R and F are the coordinate ring and function field of the affine plane $\mathbf{A}^2$ over K. Recall that the Berkovich affine plane $\mathbf{A}^2_{\mathrm{Berk}}$ is the set of semivaluations on R that restrict to the trivial valuation on K.

A *linear system* $|\mathfrak{M}|$ of curves on $\mathbf{A}^2$ is the projective space associated to a nonzero, finite-dimensional vector space $\mathfrak{M} \subseteq R$. The system is *free* if its base locus is empty, that is, for every point $\xi \in \mathbf{A}^2$ there exists a polynomial $\phi \in \mathfrak{M}$ with $\phi(\xi) \neq 0$. For any linear system $|\mathfrak{M}|$ and any $v \in \mathbf{A}^2_{\mathrm{Berk}}$ we write $v(|\mathfrak{M}|) = \min\{v(\phi) \mid \phi \in \mathfrak{M}\}$.

9.2. **Valuations centered at infinity.** We let $\hat{\mathcal{V}}_\infty \subseteq \mathbf{A}^2_{\mathrm{Berk}}$ denote the set of semivaluations v having center at infinity, that is, such that $v(\phi) < 0$ for some polynomial $\phi \in R$. Note that $\hat{\mathcal{V}}_\infty$ is naturally a pointed cone: in contrast to $\hat{\mathcal{V}}_0$ there is no element 'triv_∞'.

The valuative tree at infinity is the base of this cone and we want to realize it as a "section'. In the local case, the valuative tree at a closed point $0 \in \mathbf{A}^2$ was defined using the maximal ideal $\mathfrak{m}_0$. In order to do something similar at infinity, we fix an embedding $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$. This allows us to define the *degree* of a polynomial in R and in particular defines the free linear system $|\mathfrak{L}|$ of *lines*, associated to the subspace $\mathfrak{L} \subseteq R$ of *affine functions* on $\mathbf{A}^2$, that is, polynomials of degree at most one. Note that $v \in \mathbf{A}^2_{\mathrm{Berk}}$ has center at infinity iff $v(|\mathfrak{L}|) < 0$.

We say that two polynomials z_1, z_2 are affine coordinates on $\mathbf{A}^2$ if $\deg z_i = 1$ and $R = K[z_1, z_2]$. In this case, $F = K(z_1, z_2)$ and $v(|\mathfrak{L}|) = \min\{v(z_1), v(z_2)\}$.

Definition 9.1. The *valuative tree at infinity* $\mathcal{V}_\infty$ is the set of semivaluations $v \in \mathbf{A}^2_{\mathrm{Berk}}$ such that $v(|\mathfrak{L}|) = -1$.

The role of $\mathrm{ord}_0 \in \mathcal{V}_0$ is played by the valuation $\mathrm{ord}_\infty \in \mathcal{V}_\infty$, defined by

$$\mathrm{ord}_\infty(\phi) = -\deg(\phi). \tag{9.1}$$

In particular, $v(\phi) \geq \mathrm{ord}_\infty(\phi)$ for every $\phi \in R$ and every $v \in \mathcal{V}_\infty$. We emphasize that both $\mathcal{V}_\infty$ and ord_∞ depend on a choice of embedding $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$.

We equip $\mathcal{V}_\infty$ and $\hat{\mathcal{V}}_\infty$ with the subspace topology from $\mathbf{A}^2_{\mathrm{Berk}}$. It follows from Tychonoff's theorem that $\mathcal{V}_\infty$ is a compact Hausdorff space. The space $\hat{\mathcal{V}}_\infty$ is open in $\mathbf{A}^2_{\mathrm{Berk}}$ and its boundary consists of the trivial valuation $\mathrm{triv}_{\mathbf{A}^2}$ and the set of semivaluations centered at a curve in $\mathbf{A}^2$.

[26]The notation in these notes differs from [FJ07, FJ11] where the valuative tree at infinity is denoted by $\mathcal{V}_0$. In *loc. cit.* the valuation ord_∞ defined in (9.1) is denoted by $-\deg$.

As in the local case, we can classify the elements of $\hat{\mathcal{V}}_\infty$ into curve semivaluations, divisorial valuations, irrational valuations and infinitely singular valuations. We do this by considering v as a semivaluation on the ring $\widehat{\mathcal{O}}_{\mathbf{P}^2,\xi}$, where ξ is the center of ξ on $\mathbf{P}^2$.

9.3. **Admissible compactifications.** The role of a blowup of $\mathbf{A}^2$ above a closed point is played here by a *compactification* of $\mathbf{A}^2$, by which we mean a projective surface containing $\mathbf{A}^2$ as Zariski open subset. To make the analogy even stronger, recall that we have fixed an embedding $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$. We will use

Definition 9.2. An *admissible compactification* of $\mathbf{A}^2$ is a smooth projective surface X containing $\mathbf{A}^2$ as a Zariski open subset, such that the induced birational map $X \dashrightarrow \mathbf{P}^2$ induced by the identity on $\mathbf{A}^2$, is regular.

By the structure theorem of birational surface maps, this means that the morphism $X \to \mathbf{P}^2$ is a finite composition of point blowups above infinity. The set of admissible compactifications is naturally partially ordered and in fact a directed set: any two admissible compactifications are dominated by a third.

Many of the notions below will in fact not depend on the choice of embedding $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$ but would be slightly more complicated to state without it.

Remark 9.3. Some common compactifications of $\mathbf{A}^2$, for instance $\mathbf{P}^1 \times \mathbf{P}^1$, are not admissible in our sense. However, the set of admissible compactifications is cofinal among compactifications of $\mathbf{A}^2$: If Y is an irreducible, normal projective surface containing $\mathbf{A}^2$ as a Zariski open subset, then there exists an admissible compactification X of $\mathbf{A}^2$ such that the birational map $X \dashrightarrow Y$ induced by the identity on $\mathbf{A}^2$ is regular. Indeed, X is obtained by resolving the indeterminacy points of the similarly defined birational map $\mathbf{P}^2 \dashrightarrow Y$. See [Mor73, Kis02] for a classification of smooth compactifications of $\mathbf{A}^2$.

9.3.1. *Primes and divisors at infinity.* Let X be an admissible compactification of $\mathbf{A}^2$. A *prime at infinity* of X is an irreducible component of $X \setminus \mathbf{A}^2$. We often identify a prime of X at infinity with its strict transform in any compactification X' dominating X. In this way we can identify a prime at infinity E (of some admissible compactification) with the corresponding divisorial valuation ord_E.

Any admissible compactification contains a special prime L_∞, the strict transform of $\mathbf{P}^2 \setminus \mathbf{A}^2$. The corresponding divisorial valuation is $\mathrm{ord}_{L_\infty} = \mathrm{ord}_\infty$.

We say that a point in $X \setminus \mathbf{A}^2$ is a *free point* if it belongs to a unique prime at infinity; otherwise it is a *satellite point.*

A *divisor at infinity* on X is a divisor supported on $X \setminus \mathbf{A}^2$. We write $\mathrm{Div}_\infty(X)$ for the abelian group of divisors at infinity. If E_i, $i \in I$ are the primes of X at infinity, then $\mathrm{Div}_\infty(X) \simeq \bigoplus_i \mathbf{Z}E_i$.

9.3.2. *Intersection form and linear equivalence.* We have the following basic facts.

Proposition 9.4. *Let X be an admissible compactification of $\mathbf{A}^2$. Then*

(i) *Every divisor on X is linearly equivalent to a unique divisor at infinity, so* $\mathrm{Div}_\infty(X) \simeq \mathrm{Pic}(X)$.

(ii) *The intersection form on* $\mathrm{Div}_\infty(X)$ *is nondegenerate and unimodular. It has signature* $(1, \rho(X) - 1)$.

Proof. We argue by induction on the number of blowups needed to obtain X from $\mathbf{P}^2$. If $X = \mathbf{P}^2$, then the statement is clear: $\mathrm{Div}_\infty(X) = \mathrm{Pic}(X) = \mathbf{Z}L_\infty$ and $(L_\infty \cdot L_\infty) = 1$. For the inductive step, suppose $\pi' = \pi \circ \mu$, where μ is the simple blowup of a closed point on $X \setminus \mathbf{A}^2$, resulting in an exceptional prime E. Then we have an orthogonal decomposition $\mathrm{Div}_\infty(X') = \mu^* \mathrm{Div}_\infty(X) \oplus \mathbf{Z}E$, $\mathrm{Pic}(X') = \mu^* \mathrm{Pic}(X) \oplus \mathbf{Z}E$ and $(E \cdot E) = -1$.

Statement (ii) about the intersection form is also a consequence of the Hodge Index Theorem and Poincaré Duality. □

Concretely, the isomorphism $\mathrm{Pic}(X) \simeq \mathrm{Div}_\infty(X)$ can be understood as follows. Any irreducible curve C in X that is not contained in $X \setminus \mathbf{A}^2$ is the closure in X of an affine curve $\{\phi = 0\}$ for some polynomial $\phi \in R$. Then C is linearly equivalent to the element in $\mathrm{Div}_\infty(X)$ defined as the divisor of poles of ϕ, where the latter is viewed as a rational function on X.

Let E_i, $i \in I$ be the primes of X at infinity. It follows from Proposition 9.4 that for each $i \in I$ there exists a divisor $\check{E}_i \in \mathrm{Div}_\infty(X)$ such that $(\check{E}_i \cdot E_i) = 1$ and $(\check{E}_i \cdot E_j) = 0$ for all $j \neq i$.

9.3.3. *Invariants of primes at infinity.* Analogously to the local case (see 7.3.6) we associate two basic numerical invariants α_E and A_E to any prime E at infinity (or, equivalently, to the associated divisorial valuation $\mathrm{ord}_E \in \hat{\mathcal{V}}_\infty$.

To define α_E, pick an admissible compactification X of $\mathbf{A}^2$ in which E is a prime at infinity. Above we defined the divisor $\check{E} = \check{E}_X \in \mathrm{Div}_\infty(X)$ by duality: $(\check{E}_X \cdot E) = 1$ and $(\check{E}_X \cdot F) = 0$ for all primes $F \neq E$ of X at infinity. Note that if X' is an admissible compactification dominating X, then the divisor $\check{E}_{X'}$ on X' is the pullback of $\check{E}_X$ under the morphism $X' \to X$. In particular, the self-intersection number

$$\alpha_E := \alpha(\mathrm{ord}_E) := (\check{E} \cdot \check{E})$$

is an integer independent of the choice of X.

The second invariant is the *log discrepancy* A_E. Let ω be a nonvanishing regular 2-form on $\mathbf{A}^2$. If X is an admissible compactification of $\mathbf{A}^2$, then ω extends as a rational form on X. For any prime E of X at infinity, with associated divisorial valuation $\mathrm{ord}_E \in \hat{\mathcal{V}}_\infty$, we define

$$A_E := A(\mathrm{ord}_E) := 1 + \mathrm{ord}_E(\omega). \tag{9.2}$$

This is an integer whose value does not depend on the choice of X or ω. Note that $A_{L_\infty} = -2$ since ω has a pole of order 3 along L_∞. In general, A_E can be positive or negative.

We shall later need the analogues of (7.4) and (7.5). Thus let X be an admissible compactification of $\mathbf{A}^2$ and X' the blowup of X at a free point $\xi \in X \setminus \mathbf{A}^2$. Let E' be the "new" prime of X', that is, the inverse image of ξ in X'. Then

$$A_{E'} = A_E + 1,\ b_{E'} = b_E \quad \text{and} \quad \check{E}' = \check{E} - E', \tag{9.3}$$

where, in the right hand side, we identify the divisor $\check{E} \in \mathrm{Div}_\infty(X)$ with its pullback to X'. As a consequence,

$$\alpha_{E'} := (\check{E}' \cdot \check{E}') = (\check{E} \cdot \check{E}) - 1 = \alpha_E - 1. \tag{9.4}$$

Generalizing both §7.3.6 and §9.3.3, the invariants α_E and A_E can in fact be defined for any divisorial valuation ord_E in the Berkovich affine plane.

9.3.4. *Positivity.* Recall that in the local case, the notion of relative positivity was very well behaved and easy to understand, see §7.3.5. Here the situation is much more subtle, and this will account for several difficulties.

As usual, we say that a divisor $Z \in \mathrm{Div}(X)$ is *effective* if it is a positive linear combination of prime divisors on X. We also say that $Z \in \mathrm{Div}(X)$ is *nef* if $(Z \cdot W) \geq 0$ for all effective divisors W. These notions make sense also for $\mathbf{Q}$-divisors. It is a general fact that if $Z \in \mathrm{Div}(X)$ is nef, then $(Z \cdot Z) \geq 0$.

Clearly, the semigroup of effective divisors in $\mathrm{Div}_\infty(X)$ is freely generated by the primes E_i, $i \in I$ at infinity. A divisor $Z \in \mathrm{Div}_\infty(X)$ is *nef at infinity* if $(Z \cdot W) \geq 0$ for every effective divisor $W \in \mathrm{Div}_\infty(X)$. This simply means that $(Z \cdot E_i) \geq 0$ for all $i \in I$. It follows easily that the subset of $\mathrm{Div}_\infty(X)$ consisting of divisors that are nef at infinity is a free semigroup generated by the $\check{E}_i$, $i \in I$.

We see that a divisor $Z \in \mathrm{Div}_\infty(X)$ is nef iff it is nef at infinity and, in addition, $(Z \cdot C) \geq 0$ whenever C is the closure in X of an irreducible curve in $\mathbf{A}^2$. In general, a divisor that is nef at infinity may not be nef.

Example 9.5. Consider the surface X obtained by first blowing up any closed point at infinity, creating the prime E_1, then blowing up a free point on E_1, creating the prime E_2. Then the divisor $Z := \check{E}_2 = L_\infty - E_2$ is nef at infinity but Z is not nef since $(Z \cdot Z) = -1 < 0$.

However, a divisor $Z \in \mathrm{Div}_\infty(X)$ that is nef at infinity and effective is always nef: as above it suffices to show that $(Z \cdot C) \geq 0$ whenever C is the closure in X of a curve in $\mathbf{A}^2$. But $(E_i \cdot C) \geq 0$ for all $i \in I$, so since Z has nonnegative coefficients in the basis E_i, $i \in I$, we must have $(Z \cdot C) \geq 0$.

On the other hand, it is possible for a divisor to be nef but not effective. The following example was communicated by Adrien Dubouloz [Dub11].

Example 9.6. Pick two distinct points ξ_1, ξ_2 on the line at infinity L_∞ in $\mathbf{P}^2$ and let C be a conic passing through ξ_1 and ξ_2. Blow up ξ_1 and let D be the exceptional divisor. Now blow up ξ_2, creating E_1, blow up $C \cap E_1$, creating E_2 and finally blow up $C \cap E_2$ creating F. We claim that the non-effective divisor $Z = 2D + 5L_\infty + 3E_1 + E_2 - F$ on the resulting surface X is nef.

To see this, we successively contract the primes L_∞, E_1 and E_2. A direct computation shows that each of these is a (-1)-curve at the time we contract it, so by Castelnuovo's criterion we obtain a birational morphism $\mu : X \to Y$, with Y a smooth rational surface. Now Y is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. Indeed, one checks that $(F \cdot F) = (C \cdot C) = 0$ and $(F \cdot C) = 1$ on Y and it is easy to see in coordinates that each of F and C is part of a fibration on Y. Now Z is the pullback of the divisor $W = 2D - F$ on Y, Further, $\mathrm{Pic}(Y) \simeq \mathbf{Z}C \oplus \mathbf{Z}F$ and $(W \cdot C) = 1 > 0$ and $(W \cdot F) = 2 > 0$, so W is ample on Y and hence $Z = \mu^* W$ is nef on X.

Finally, in contrast to the local case (see Proposition 7.4) it can happen that a divisor $Z \in \mathrm{Div}_\infty(X)$ is nef but that the line bundle $\mathcal{O}_X(Z)$ has base points, that is, it is not generated by its global sections.

Example 9.7. Consider the surface X obtained from blowing $\mathbf{P}^2$ nine times, as follows. First blow up at three distinct points on L_∞, creating primes E_{1j}, $j = 1, 2, 3$. On each E_{1j} blow up a free point, creating a new prime E_{2j}. Finally blow up a free point on each E_{2j}, creating a new prime E_{3j}. Set $Z = 3L_\infty + \sum_{j=1}^{3}(2E_{2j} + E_{1j})$. Then $Z = \sum_{j=1}^{3} \check{E}_{3j}$, so Z is nef at infinity. Since Z is also effective, it must be nef.

However, we claim that if the points at which we blow up are generically chosen, then the line bundle $\mathcal{O}_X(Z)$ is not generated by its global sections. To see this, consider a global section of $\mathcal{O}_X(Z)$ that does not vanish identically along L_∞. Such a section is given by a polynomial $\phi \in R$ of degree 3 satisfying $\mathrm{ord}_{E_{ij}}(\phi) = 3 - i$, $1 \le i, j \le 3$. This gives nine conditions on ϕ. Note that if ϕ is such a section, then so is $\phi - c$ for any constant c, so we may assume that ϕ has zero constant coefficient. Thus ϕ is given by eight coefficients. For a generic choice of points blown up, no such polynomial ϕ will exist. This argument is of course not rigorous, but can be made so by an explicit computation in coordinates that we invite the reader to carry out.

9.4. **Valuations and dual fans and graphs.** Analogously to §7.5 we can realize $\hat{\mathcal{V}}_\infty$ and $\mathcal{V}_\infty$ as inverse limits of dual fans and graphs, respectively.

To an admissible compactification X of $\mathbf{A}^2$ we associate a dual fan $\hat{\Delta}(X)$ with integral affine structure $\mathrm{Aff}(X) \simeq \mathrm{Div}_\infty(X)$. This is done exactly as in the local case, replacing exceptional primes with primes at infinity. Inside the dual fan we embed the dual graph $\Delta(X)$ using the integral affine function associated to the divisor $\pi^* L_\infty = \sum_i b_i E_i \in \mathrm{Div}_\infty(X)$. The dual graph is a tree.

The numerical invariants A_E and α_E uniquely to homogeneous functions A and α on the dual fan $\hat{\Delta}(X)$ of degree one and two, respectively and such that these functions are affine on the dual graph. Then A and α give parametrizations of the dual graph rooted in the vertex corresponding to L_∞. We equip the dual graph with the metric associated to the parametrization α: the length of a simplex σ_{ij} is equal to $1/(b_i b_j)$. We could also (but will not) use A to define a metric on the dual graph. This metric is the same as the one induced by the integral affine structure: the length of the simplex σ_{ij} is $m_{ij}/(b_i b_j)$, where $m_{ij} = \gcd\{b_i, b_j\}$ is the multiplicity of the segment.

Using monomial valuations we embed the dual fan as a subset $|\hat{\Delta}(X)|$ of the Berkovich affine plane. The image $|\hat{\Delta}^*(X)|$ of the punctured dual fan lies in $\hat{\mathcal{V}}_\infty$. The preimage of $\mathcal{V}_\infty \subseteq \hat{\mathcal{V}}_\infty$ under the embedding $|\hat{\Delta}^*(X)| \subseteq \hat{\mathcal{V}}_\infty$ is exactly $|\Delta(X)|$. In particular, a vertex σ_E of the dual graph is identified with the corresponding normalized valuation $v_E \in \mathcal{V}_\infty$, defined by

$$v_E = b_E^{-1} \,\mathrm{ord}_E \quad \text{where } b_E := -\mathrm{ord}_E(|\mathfrak{L}|). \tag{9.5}$$

Note that $v_{L_\infty} = \mathrm{ord}_{L_\infty} = \mathrm{ord}_\infty$.

We have a retraction $r_X : \hat{\mathcal{V}}_\infty \to |\hat{\Delta}^*(X)|$ that maps $\mathcal{V}_\infty$ onto $|\Delta(X)|$. The induced maps

$$r : \mathcal{V}_\infty \to \varprojlim_X |\Delta(X)| \quad \text{and} \quad r : \hat{\mathcal{V}}_\infty \to \varprojlim_X |\hat{\Delta}^*(X)| \tag{9.6}$$

are homeomorphisms. The analogue of Lemma 7.12 remains true and we have the following analogue of Lemma 7.11.

Lemma 9.8. *If $v \in \hat{\mathcal{V}}_\infty$ and X is an admissible compactification of $\mathbf{A}^2$, then*

$$(r_X v)(\phi) \le v(\phi)$$

for every polynomial $\phi \in R$, with equality if the closure in X of the curve $(\phi = 0) \subseteq \mathbf{A}^2$ does not pass through the center of v on X.

The second homeomorphism in (9.6) equips $\hat{\mathcal{V}}_\infty$ with an integral affine structure: a function φ on $\hat{\mathcal{V}}_\infty$ is integral affine if it is of the form $\varphi = \varphi_X \circ r_X$, where $\varphi_X \in \mathrm{Aff}(X)$.

The first homeomorphism in (9.6) induces a metric tree structure on $\mathcal{V}_\infty$ as well as two parametrizations[27]

$$\alpha : \mathcal{V}_\infty \to [-\infty, 1] \quad \text{and} \quad A : \mathcal{V}_\infty \to [2, \infty] \tag{9.7}$$

of $\mathcal{V}_\infty$, viewed as a tree rooted in ord_∞. We extend A and α as homogeneous functions on $\hat{\mathcal{V}}_\infty$ of degrees one and two, respectively.

9.5. **Potential theory.** Since $\mathcal{V}_\infty$ is a metric tree, we can do potential theory on it, but just as in the case of the valuative tree at a closed point, we need to tweak the general approach in §2.5. The reason is again that one should view a function on $\mathcal{V}_\infty$ as the restriction of a homogeneous function on $\hat{\mathcal{V}}_\infty$.

A first guideline is that functions of the form $\log|\mathfrak{M}|$, defined by[28]

$$\log|\mathfrak{M}|(v) = -v(|\mathfrak{M}|) \tag{9.8}$$

should be subharmonic on $\mathcal{V}_\infty$, for any linear system $|\mathfrak{M}|$ on $\mathbf{A}^2$. In particular, the function $\log|\mathfrak{L}| \equiv 1$ should be subharmonic (but not harmonic). A second guideline is that the Laplacian should be closely related to the intersection product on divisors at infinity.

9.5.1. *Subharmonic functions and Laplacian on $\mathcal{V}_\infty$.* As in §7.8.1 we extend the valuative tree $\mathcal{V}_\infty$ to a slightly larger tree $\tilde{\mathcal{V}}_\infty$ by connecting the root ord_∞ to a point G using an interval of length one. Let $\tilde{\Delta}$ denote the Laplacian on $\tilde{\mathcal{V}}_\infty$.

We define the class $\mathrm{SH}(\mathcal{V}_\infty)$ of *subharmonic functions* on $\mathcal{V}_\infty$ as the set of restrictions to $\mathcal{V}_\infty$ of functions $\varphi \in \mathrm{QSH}(\tilde{\mathcal{V}}_\infty)$ such that

$$\varphi(G) = 2\varphi(\mathrm{ord}_\infty) \quad \text{and} \quad \tilde{\Delta}\varphi = \rho - a\delta_G,$$

where ρ is a positive measure supported on $\mathcal{V}_\infty$ and $a = \rho(\mathcal{V}_\infty) \ge 0$. In particular, φ is affine of slope $-\varphi(\mathrm{ord}_\infty)$ on the segment $[G, \mathrm{ord}_\infty[= \tilde{\mathcal{V}}_\infty \setminus \mathcal{V}_\infty$. We then define $\Delta\varphi := \rho = (\tilde{\Delta}\varphi)|_{\mathcal{V}_\infty}$. For example, if $\varphi \equiv 1$ on $\mathcal{V}_\infty$, then $\varphi(G) = 2$, $\tilde{\Delta}\varphi = \delta_{\mathrm{ord}_\infty} - \delta_G$ and $\Delta\varphi = \delta_{\mathrm{ord}_\infty}$.

From this definition and the analysis in §2.5 one deduces:

[27] In [FJ04] the parametrization A is called *thinness* whereas $-\alpha$ is called *skewness*.

[28] As in §7.8 the notation reflects the fact that $|\cdot| := e^{-v}$ is a seminorm on R.

Proposition 9.9. *Let $\varphi \in \mathrm{SH}(\mathcal{V}_\infty)$ and write $\rho = \Delta\varphi$. Then:*

(i) *φ is decreasing in the partial ordering of $\mathcal{V}_\infty$ rooted in ord_∞;*
(ii) *$\varphi(\mathrm{ord}_\infty) = \rho(\mathcal{V}_\infty)$;*
(iii) *$|D_{\vec{v}}\varphi| \le \rho(\mathcal{V}_\infty)$ for all tangent directions $\vec{v}$ in $\mathcal{V}_\infty$.*

As a consequence we have the estimate

$$\alpha(v)\varphi(\mathrm{ord}_\infty) \le \varphi(v) \le \varphi(\mathrm{ord}_\infty) \tag{9.9}$$

for all $v \in \mathcal{V}_\infty$. Here $\alpha : \mathcal{V}_\infty \to [-\infty, +1]$ is the parametrization in (9.7). It is important to remark that a subharmonic function can take both positive and negative values. In particular, (9.9) is not so useful when $\alpha(v) < 0$.

The exact sequence in (2.8) shows that

$$\Delta : \mathrm{SH}(\mathcal{V}_\infty) \to \mathcal{M}^+(\mathcal{V}_\infty), \tag{9.10}$$

is a homeomorphism whose inverse is given by

$$\varphi(v) = \int_{\mathcal{V}_\infty} \alpha(w \wedge_{\mathrm{ord}_\infty} v)\, d\rho(w). \tag{9.11}$$

The compactness properties in §2.5 carry over to the space $\mathrm{SH}(\mathcal{V}_\infty)$. In particular, for any $C > 0$, the set $\{\varphi \in \mathrm{SH}(\mathcal{V}_\infty) \mid \varphi(\mathrm{ord}_\infty) \le C\}$ is compact. Further, if $(\varphi_i)_i$ is a decreasing net in $\mathrm{SH}(\mathcal{V}_\infty)$, and $\varphi := \lim \varphi_i$, then $\varphi \in \mathrm{SH}(\mathcal{V}_\infty)$. Moreover, if $(\varphi_i)_i$ is a family in $\mathrm{SH}(\mathcal{V}_\infty)$ with $\sup_i \varphi(\mathrm{ord}_\infty) < \infty$, then the upper semicontinuous regularization of $\varphi := \sup_i \varphi_i$ belongs to $\mathrm{SH}(\mathcal{V}_\infty)$.

While the function -1 on $\mathcal{V}_\infty$ is not subharmonic, it is true that $\max\{\varphi, r\}$ is subharmonic whenever $\varphi \in \mathrm{SH}(\mathcal{V}_\infty)$ and $r \in \mathbf{R}$.

9.5.2. *Laplacian of integral affine functions.* Any integral affine function φ on $\hat{\mathcal{V}}_\infty$ is associated to a divisor at infinity $Z \in \mathrm{Div}_\infty(X)$ for some admissible compactification X of $\mathbf{A}^2$: the value of φ at a divisorial valuation ord_{E_i} is the coefficient $\mathrm{ord}_{E_i}(Z)$ of E_i in Z. Using the same computations as in the proof of Proposition 7.15 we show that

$$\Delta\varphi = \sum_{i \in I} b_i (Z \cdot E_i)\delta_{v_i},$$

where $b_i = -\mathrm{ord}_{E_i}(|\mathfrak{L}|) \ge 1$ and $v_i = b_i^{-1}\mathrm{ord}_{E_i}$. In particular, φ is subharmonic iff Z is nef at infinity.

Recall that we have defined divisors $\check{E}_i \in \mathrm{Div}_\infty(X)$ such that $(\check{E}_i \cdot E_i) = 1$ and $(\check{E}_i \cdot E_j) = 0$ for all $j \ne i$. The integral affine function φ_i on $\mathcal{V}_\infty$ associated to $\check{E}_i$ is subharmonic and satisfies $\Delta\varphi_i = b_i\delta_{v_i}$. In view of (9.11), this shows that $\min_{\mathcal{V}_\infty} \varphi_i = \varphi_i(v_i) = b_i\alpha(v_i)$. This implies

$$\alpha_{E_i} = (\check{E}_i \cdot \check{E}_i) = \mathrm{ord}_{E_i}(\check{E}_i) = b_i^2\alpha(v_i) = \alpha(\mathrm{ord}_{E_i}). \tag{9.12}$$

Proposition 9.10. *Let E be a divisor at infinity on some admissible compactification X of $\mathbf{A}^2$. Let $\check{E} \in \mathrm{Div}_\infty(X)$ be the associated element of the dual basis and $v_E = b_E^{-1}\mathrm{ord}_E \in \mathcal{V}_\infty$ the associated normalized divisorial valuation. Then $\check{E}$ is nef at infinity and the following statements are equivalent:*

(i) *$\check{E}$ is nef;*
(ii) *$(\check{E} \cdot \check{E}) \ge 0$;*

(iii) $\alpha(v_E) \geq 0$.

Proof. That $\check{E}$ is nef at infinity is clear from the definition and has already been observed. That (ii) is equivalent to (iii) is an immediate consequence of (9.12). If $\check{E}$ is nef, then $(\check{E} \cdot \check{E}) \geq 0$, showing that (i) implies (ii). On the other hand, if $\alpha(v_E) \geq 0$, then we have seen above that the minimum on $\mathcal{V}_\infty$ of the integral affine function φ associated to $\check{E}$ is attained at v_E and is nonnegative. Thus $\check{E}$ is effective. Being nef at infinity and effective, $\check{E}$ must be nef, proving that (ii) implies (i). □

9.5.3. *Subharmonic functions from linear systems.* Let $|\mathfrak{M}|$ be a nonempty linear system of affine curves. We claim that the function $\log|\mathfrak{M}|$, defined by (9.8) is subharmonic on $\mathcal{V}_\infty$. To see this, note that $\log|\mathfrak{M}| = \max \log|\phi|$, where ϕ ranges over polynomials defining the curves in $|\mathfrak{M}|$. The claim therefore follows from

Exercise 9.11. If $\phi \in R$ is an irreducible polynomial, show that $\log|\phi|$ is subharmonic on $\mathcal{V}_\infty$ and that

$$\Delta \log|\phi| = \sum_{j=1}^{n} m_j \delta_{v_j}$$

where v_j, $1 \leq j \leq n$ are the curve valuations associated to the all the local branches C_j of $\{\phi = 0\}$ at infinity and where $m_j = (C_j \cdot L_\infty)$ is the local intersection number of C_j with the line at infinity in $\mathbf{P}^2$.

Example 9.12. Fix affine coordinates (z_1, z_2) on $\mathbf{A}^2$ and let $\mathfrak{M} \subseteq R$ be the vector space spanned by z_1 Then $\log|\mathfrak{M}|(v) = \max\{-v(z_1), 0\}$ and $\Delta \log|\mathfrak{M}|$ is a Dirac mass at the monomial valuation with $v(z_1) = 0$, $v(z_2) = -1$.

Example 9.13. Fix affine coordinates (z_1, z_2) on $\mathbf{A}^2$ and let $\mathfrak{M} \subseteq R$ be the vector space spanned by $z_1 z_2$ and the constant function 1. Then $\log|\mathfrak{M}|(v) = \max\{-(v(z_1) + v(z_2), 0\}$ and $\Delta \log|\mathfrak{M}| = \delta_{v_{-1,1}} + \delta_{v_{1,-1}}$, where v_{t_1,t_2} is the monomial valuation with weights $v_{t_1,t_2}(z_i) = t_i$, $i = 1, 2$.

Proposition 9.14. *Let $|\mathfrak{M}|$ be a linear system of affine curves on $\mathbf{A}^2$. Then the following conditions are equivalent:*

(i) *the base locus of $|\mathfrak{M}|$ on $\mathbf{A}^2$ contains no curves;*
(ii) *the function $\log|\mathfrak{M}|$ is bounded on $\mathcal{V}_\infty$;*
(iii) *the measure $\Delta \log|\mathfrak{M}|$ on $\mathcal{V}_\infty$ is supported at divisorial valuations.*

Linear systems $|\mathfrak{M}|$ satisfying these equivalent conditions are natural analogs of primary ideals $\mathfrak{a} \subseteq R$ in the local setting.

Sketch of proof. That (iii) implies (ii) follows from (9.11). If the base locus of $|\mathfrak{M}|$ contains an affine curve C, let $v \in \mathcal{V}_\infty$ be a curve valuation associated to one of the branches at infinity of C. Then $\log|\mathfrak{M}|(v) = -v(\varphi) = -\infty$ so (ii) implies (i).

Finally, let us prove that (i) implies (iii). Suppose the base locus on $|\mathfrak{M}|$ on $\mathbf{A}^2$ contains no curves. Then we can pick an admissible compactification of $\mathbf{A}^2$ such that the strict transform of $|\mathfrak{M}|$ to X has no base points at infinity. In this case one shows that $\Delta \log|\mathfrak{M}|$ is an atomic measure supported on the divisorial valuations associated to some of the primes of X at infinity. □

In general, it seems very hard to characterize the measures on $\mathcal{V}_\infty$ appearing in (iii). Notice that if $\Delta\log|\mathfrak{M}|$ is a Dirac mass at a divisorial valuation v then $\alpha(v)\ge 0$, as follows from (9.11). There are also sufficient conditions: using the techniques in the proof of Theorem 9.18 one can show that if ρ is an atomic measure with rational coefficients supported on divisorial valuations in the tight tree $\mathcal{V}'_\infty$ (see §9.7) then there exists a linear system $|\mathfrak{M}|$ such that $\log|\mathfrak{M}|\ge 0$ and $\Delta\log|\mathfrak{M}| = n\rho$ for some integer $n\ge 1$.

9.6. **Intrinsic description of tree structure on $\mathcal{V}_\infty$.** We can try to describe the tree structure on $\mathcal{V}_\infty \simeq \varprojlim|\Delta(X)|$ intrinsically, viewing the elements of $\mathcal{V}_\infty$ purely as semivaluations on the ring R. This is more complicated than in the case of the valuative tree at a closed point (see §7.9). However, the partial ordering can be characterized essentially as expected:

Proposition 9.15. *If $w, v\in\mathcal{V}_\infty$, then the following are equivalent:*
- (i) *$v\le w$ in the partial ordering induced by $\mathcal{V}_\infty\simeq\varprojlim|\Delta(X)|$;*
- (ii) *$v(\phi)\le w(\phi)$ for all polynomials $\phi\in R$;*
- (iii) *$v(|\mathfrak{M}|)\le w(|\mathfrak{M}|)$ for all free linear systems $|\mathfrak{M}|$ on $\mathbf{A}^2$.*

Proof. The implication (i) $\Longrightarrow$ (ii) follows from the subharmonicity of $\log|\phi|$ together with Proposition 9.9 (i). The implication (ii) $\Longrightarrow$ (iii) is obvious. It remains to prove (iii) $\Longrightarrow$ (i).

Suppose $v\not\le w$ in the partial ordering on $\mathcal{V}_\infty\simeq\varprojlim|\Delta(X)|$. We need to find a free linear system $|\mathfrak{M}|$ on $\mathbf{A}^2$ such that $v(|\mathfrak{M}|) > w(|\mathfrak{M}|)$. First assume that v and w are quasimonomial and pick an admissible compactification X of $\mathbf{A}^2$ such that $v, w\in|\Delta(X)|$. Let E_i, $i\in I$, be the primes of X at infinity. One of these primes is L_∞ and there exists another prime (not necessarily unique) E_i such that $v_i\ge v$. Fix integers r, s with $1\ll r\ll s$ and define the divisor $Z\in\operatorname{Div}_\infty(X)$ by

$$Z := \sum_{j\in I}\check{E}_j + r\check{E}_i + s\check{L}_\infty.$$

We claim that Z is an *ample* divisor on X. To prove this, it suffices, by the Nakai-Moishezon criterion, to show that $(Z\cdot Z)>0$, $(Z\cdot E_j)>0$ for all $j\in I$ and $(Z\cdot C)>0$ whenever C is the closure in X of a curve $\{\phi=0\}\subseteq\mathbf{A}^2$.

First, by the definition of $\hat{E}_j$ it follows that $(Z\cdot E_j)\ge 1$ for all j. Second, we have $(\check{L}_\infty\cdot C)=\deg\phi$ and $(\check{E}_j\cdot C) = -\operatorname{ord}_{E_i}(\phi)\ge\alpha(v_i)\deg\phi$ for all $j\in I$ in view of (9.9), so that $(Z\cdot C)>0$ for $1\le r\ll s$. Third, since $(\check{L}_\infty\cdot\check{L}_\infty)=1$, a similar argument shows that $(Z\cdot Z)>0$ for $1\le r\ll s$.

Since Z is ample, there exists an integer $n\ge 1$ such that the line bundle $\mathcal{O}_X(nZ)$ is base point free. In particular, the corresponding linear system $|\mathfrak{M}| := |\mathcal{O}_X(nZ)|$ is free on $\mathbf{A}^2$. Now, the integral affine function on $|\Delta(X)|$ induced by $\check{L}_\infty$ is the constant function $+1$. Moreover, the integral affine function on $|\Delta(X)|$ induced by $\check{E}_i$ is the function $\varphi_i = b_i\alpha(\cdot\wedge_{\operatorname{ord}_\infty} v_i)$. Since $v_i\ge v$ and $v\not\le w$, this implies $\varphi_i(v)<\varphi_i(w)$. For $r\gg 1$ this translates into $v(|\mathfrak{M}|) > w(|\mathfrak{M}|)$ as desired.

Finally, if v and w are general semivaluations in $\mathcal{V}_\infty$ with $v\not\le w$, then we can pick an admissible compactification X of $\mathbf{A}^2$ such that $r_X(v)\not\le r_X(w)$. By the previous construction there exists a free linear system $|\mathfrak{M}|$ on $\mathbf{A}^2$ such that $r_X(v)(|\mathfrak{M}|) >$

$r_X(w)(|\mathfrak{M}|)$. But since the linear system $|\mathfrak{M}|$ was free also on X, it follows that $v(|\mathfrak{M}|) = r_X(v)(|\mathfrak{M}|)$ and $w(|\mathfrak{M}|) = r_X(w)(|\mathfrak{M}|)$. This concludes the proof. □

The following result is a partial analogue of Corollary 7.23 and characterizes integral affine functions on $\hat{\mathcal{V}}_\infty$.

Proposition 9.16. *For any integral affine function φ on $\hat{\mathcal{V}}_\infty$ there exist free linear systems $|\mathfrak{M}_1|$ and $|\mathfrak{M}_2|$ on $\mathbf{A}^2$ and an integer $n \geq 1$ such that $\varphi = \frac{1}{n}(\log|\mathfrak{M}_1| - \log|\mathfrak{M}_2|)$.*

Proof. Pick an admissible compactification X of $\mathbf{A}^2$ such that φ is associated to divisor $Z \in \mathrm{Div}_\infty(X)$. We may write $Z = Z_1 - Z_2$, where $Z_i \in \mathrm{Div}_\infty(X)$ is ample. For a suitable $n \geq 1$, nZ_1 and nZ_2 are very ample, and in particular base point free. We can then take $|\mathfrak{M}_i| = |\mathcal{O}_X(nZ_i)|$, $i = 1, 2$. □

It seems harder to describe the parametrization α. While (9.9) implies

$$\alpha(v) \geq \sup_{\phi \in R\setminus 0} \frac{v(\phi)}{\mathrm{ord}_\infty(\phi)}$$

for any v, it is doubtful that equality holds in general. One can show that equality does hold when v is a quasimonomial valuation in the tight tree $\mathcal{V}'_\infty$, to be defined shortly.

9.7. **The tight tree at infinity.**

For the study of polynomial dynamics in §10, the full valuative tree at infinity is too large. Here we will introduce a very interesting and useful subtree.

Definition 9.17. The *tight tree at infinity* is the subset $\mathcal{V}'_\infty \subseteq \mathcal{V}_\infty$ consisting of semivaluations v for which $A(v) \leq 0 \leq \alpha(v)$.

Since α is decreasing and A is increasing in the partial ordering on $\mathcal{V}_\infty$, it is clear that $\mathcal{V}'_\infty$ is a subtree of $\mathcal{V}_\infty$. Similarly, α (resp. A) is lower semicontinuous (resp. upper semicontinuous) on $\mathcal{V}_\infty$, which implies that $\mathcal{V}'_\infty$ is a closed subset of $\mathcal{V}_\infty$. It is then easy to see that $\mathcal{V}'_\infty$ is a metric tree in the sense of §2.2.

Similarly, we define $\hat{\mathcal{V}}'_\infty$ as the set of semivaluations $v \in \hat{\mathcal{V}}_\infty$ satisfying $A(v) \leq 0 \leq \alpha(v)$. Thus $\hat{\mathcal{V}}'_\infty = \mathbf{R}_+^* \mathcal{V}_\infty$ The subset $\hat{\mathcal{V}}'_\infty \subset \mathbf{A}^2_{\mathrm{Berk}}$ does not depend on the choice of embedding $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$. In particular, it is invariant under polynomial automorphisms of $\mathbf{A}^2$. Further, $\hat{\mathcal{V}}'_\infty$ is nowhere dense as it contains no curve semivaluations. Its closure is the union of itself and the trivial valuation $\mathrm{triv}_{\mathbf{A}^2}$.

9.7.1. *Monomialization.*

The next, very important result characterizes some of the ends of the tree $\mathcal{V}'_\infty$.

Theorem 9.18. *Let ord_E be a divisorial valuation centered at infinity such that $A(\mathrm{ord}_E) \leq 0 = (\check{E} \cdot \check{E})$. Then $A(\mathrm{ord}_E) = -1$ and there exist coordinates (z_1, z_2) on $\mathbf{A}^2$ in which ord_E is monomial with $\mathrm{ord}_E(z_1) = -1$ and $\mathrm{ord}_E(z_2) = 0$.*

This is proved in [FJ07, Theorem A.7]. Here we provide an alternative, more geometric proof. This proof uses the Line Embedding Theorem and is the reason why we work in characteristic zero throughout §9. (It is quite possible, however, that Theorem 9.18 is true also over an algebraically closed field of positive characteristic).

Proof. Let X be an admissible compactification of $\mathbf{A}^2$ on which E is a prime at infinity. The divisor $\check{E} \in \mathrm{Div}_\infty(X)$ is nef at infinity. It is also effective, and hence nef, since $(\check{E}\cdot\check{E}) \geq 0$; see Proposition 9.10.

Let K_X be the canonical class of X. We have $(\check{E}\cdot K_X) = A(\mathrm{ord}_E) - 1 < 0$. By the Hirzebruch-Riemann-Roch Theorem we have

$$\chi(\mathcal{O}_X(\check{E})) = \chi(\mathcal{O}_X) + \frac{1}{2}((\check{E}\cdot\check{E}) - (\check{E}\cdot K_X)) > \chi(\mathcal{O}_X) = 1.$$

Serre duality yields $h^2(\mathcal{O}_X) = h^0(\mathcal{O}_X(K_X - \check{E})) = 0$, so since $h^1(\mathcal{O}_X(\check{E})) \geq 0$ we conclude that $h^0(\mathcal{O}_X(\check{E})) \geq 2$. Thus there exists a nonconstant polynomial $\phi \in R$ that defines a global section of $\mathcal{O}_X(\check{E})$. Since $\check{E}$ is effective, $\phi + t$ is also a global section for any $t \in K$.

Let C_t be the closure in X of the affine curve $(\phi + t = 0) \subset \mathbf{A}^2$. For any t we have $C_t = \check{E}$ in $\mathrm{Pic}(X)$, so $(C_t \cdot E) = 1$ and $(C_t \cdot F) = 0$ for all primes F at infinity different from E. This implies that C_t intersects $X \setminus \mathbf{A}^2$ at a unique point $\xi_t \in E$; this point is furthermore free on E, C_t is smooth at ξ_t, and the intersection is transverse. Since $\mathrm{ord}_E(\phi) = (\check{E}\cdot\check{E}) = 0$, the image of the map $t \mapsto \xi_t$ is Zariski dense in E.

For generic t, the affine curve $C_t \cap \mathbf{A}^2 = (\phi + t = 0)$ is smooth, hence C_t is smooth for these t. By adjunction, C_t is rational. In particular, $C_t \cap \mathbf{A}^2$ is a smooth curve with one place at infinity.

The Line Embedding Theorem by Abhyankar-Moh and Suzuki [AM73, Suz74] now shows that there exist coordinates (z_1, z_2) on $\mathbf{A}^2$ such that $\phi + t = z_2$. We use these coordinates to define a compactification $Y \simeq \mathbf{P}^1 \times \mathbf{P}^1$ of $\mathbf{A}^2$. Let F be the irreducible compactification of $Y \setminus \mathbf{A}^2$ that intersects the strict transform of each curve $z_2 = \text{const}$. Then the birational map $Y \dashrightarrow X$ induced by the identity on $\mathbf{A}^2$ must map F onto E. It follows that $\mathrm{ord}_E = \mathrm{ord}_F$. Now ord_F is monomial in (z_1, z_2) with $\mathrm{ord}_F(z_1) = -1$ and $\mathrm{ord}_F(z_2) = 0$. Furthermore, the 2-form $dz_1 \wedge dz_2$ has a pole of order 2 along F on Y so $A(\mathrm{ord}_F) = -1$. This completes the proof. □

9.7.2. *Tight compactifications.* We say that an admissible compactification X of $\mathbf{A}^2$ is *tight* if $|\Delta(X)| \subseteq \mathcal{V}'_\infty$. Let E_i, $i \in I$ be the primes of X at infinity. Since the parametrization α and the log discrepancy A are both affine on the simplices of $|\Delta(X)|$, X is tight iff $A(v_i) \leq 0 \leq \alpha(v_i)$ for all $i \in I$. In particular, this implies $(\check{E}_i \cdot \check{E}_i) \geq 0$, so the the divisor $\check{E}_i \in \mathrm{Div}_\infty(X)$ is nef for all $i \in I$. Since every divisor in $\mathrm{Div}_\infty(X)$ that is nef at infinity is a positive linear combination of the $\check{E}_i$, we conclude

Proposition 9.19. *If X is a tight compactification of $\mathbf{A}^2$, then the nef cone of X is simplicial.*

See [CPR02, CPR05, GM04, GM05, Mon07] for other cases when the nef cone is known to be simplicial. For a general admissible compactification of $\mathbf{A}^2$ one would, however, expect the nef cone to be rather complicated.

Lemma 9.20. *Let X be a tight compactification of $\mathbf{A}^2$ and ξ a closed point of $X \setminus \mathbf{A}^2$. Let X' be the admissible compactification of $\mathbf{A}^2$ obtained by blowing up ξ. Then X' is tight unless ξ is a free point on a prime E for which $\alpha_E = 0$ or $A_E = 0$.*

Proof. If ξ is a satellite point, then X' is tight since $|\Delta(X')| = |\Delta(X)|$.

Now suppose ξ is a free point, belonging to a unique prime on E. Let E' be the prime of X' resulting from blowing up ξ. Then X' is tight iff $\alpha_{E'} := (\check{E}' \cdot \check{E}') \geq 0 \geq A_{E'}$. But it follows from (9.3) that $A_{E'} = A_E + 1$ and $\alpha_{E'} = \alpha_E - 1$. Hence $\alpha_{E'} \geq 0 \geq A_{E'}$ unless $\alpha_E = 0$ or $A_E = 0$. The proof is complete. □

Corollary 9.21. *If X is a tight compactification of $\mathbf{A}^2$ and $v \in \hat{\mathcal{V}}'_\infty$ is a divisorial valuation, then there exists a tight compactification X' dominating X such that $v \in |\hat{\Delta}^*(X')|$.*

Proof. In the proof we shall repeatedly use the analogues at infinity of the results in §7.7.3, in particular Lemma 7.12.

We may assume $v = \mathrm{ord}_E$ for some prime E at infinity. By Lemma 7.12, the valuation $w := r_X(v)$ is divisorial and $b(w)$ divides $b(v)$. We argue by induction on the integer $b(v)/b(w)$.

By the same lemma we can find an admissible compactification X_0 dominating X such that $|\hat{\Delta}^*(X_0)| = |\hat{\Delta}^*(X)|$, and w is contained in a one-dimensional cone in $|\hat{\Delta}^*(X_0)|$. Then the center of v on X_0 is a free point ξ_0. Let X_1 be the blowup of X_0 in ξ_0. Note that since $v \neq w$ we have $\alpha(w) > \alpha(v) \geq 0 \geq A(v) > A(w)$, so by Lemma 9.20 the compactification X_1 is tight.

If $v \in |\hat{\Delta}^*(X_1)|$ then we are done. Otherwise, set $v_1 = r_{X_1}(v)$. If the center ξ_1 of v on X_1 is a satellite point, then it follows from Lemma 7.12 that $b(v_1) > b(v_0)$. If $b(w) = b(v)$, this is impossible and if $b(w) < b(v)$, we are done by the inductive hypothesis.

The remaining case is when ξ_1 is a free point on E_1, the preimage of ξ_0 under the blowup map. We continue this procedure: assuming that the center of v on X_j is a free point ξ_j, we let X_{j+1} be the blowup of X_j in ξ_j. By (9.3) we have $A_{E_n} = A_{E_0} + n$. But $A_{E_n} \leq 0$ so the procedure must stop after finitely many steps. When it stops, we either have $v \in |\hat{\Delta}^*(X_n)|$ or the center of v on X_n is a satellite point. In both cases the proof is complete in view of what precedes. □

Corollary 9.22. *If X is a tight compactification of $\mathbf{A}^2$ and $f : \mathbf{A}^2 \to \mathbf{A}^2$ is a polynomial automorphism, then there exists a tight compactification X' such that the birational map $X' \dashrightarrow X$ induced by f is* regular*.*

Proof. Let E_i, $i \in I$ be the primes of X at infinity. Now f^{-1} maps the divisorial valuations $v_i := \mathrm{ord}_{E_i}$ to divisorial valuations $v'_i = \mathrm{ord}_{E'_i}$. We have $v'_i \in \hat{\mathcal{V}}'_\infty$, so after a repeated application of Corollary 9.21 we find an admissible compactification X' of $\mathbf{A}^2$ such that $v'_i \in |\hat{\Delta}^*(X')|$ for all $i \in I$. But then it is easy to check that $f : X' \to X$ is regular. □

Corollary 9.23. *Any two tight compactifications can be dominated by a third, so the set of tight compactifications is a directed set. Furthermore, the retraction maps $r_X : \hat{\mathcal{V}}_\infty \to |\hat{\Delta}^*(X)|$ give rise to homeomorphisms*

$$\hat{\mathcal{V}}'_\infty \xrightarrow{\sim} \varprojlim_X |\hat{\Delta}^*(X)| \quad \textit{and} \quad \mathcal{V}'_\infty \xrightarrow{\sim} \varprojlim_X |\Delta(X)|,$$

where X ranges over all tight compactifications of $\mathbf{A}^2$.

9.8. **Other ground fields.** Throughout the section we assumed that the ground field was algebraically closed and of characteristic zero. Let us briefly discuss what happens when one or more of these assumptions are not satisfied.

First suppose K is algebraically closed but of characteristic $p > 0$. Everything in §9 goes through, except for the proof of the monomialization theorem, Theorem 9.18, which relies on the Line Embedding Theorem. On the other hand, it is quite possible that the proof of Theorem 9.18 can be modified to work also in characteristic $p > 0$.

Now suppose K is not algebraically closed. There are two possibilities for studying the set of semivaluations in $\mathbf{A}^2_{\mathrm{Berk}}$ centered at infinity. One way is to pass to the algebraic closure K^a. Let $G = \mathrm{Gal}(K^a/K)$ be the Galois group. Using general theory we have an identification $\mathbf{A}^2_{\mathrm{Berk}}(K) \simeq \mathbf{A}^2_{\mathrm{Berk}}(K^a)/G$ and G preserves the open subset $\hat{\mathcal{V}}_\infty(K^a)$ of semivaluations centered at infinity. Any embedding $\mathbf{A}^2(K) \hookrightarrow \mathbf{P}^2(K)$ induces an embedding $\mathbf{A}^2(K^a) \hookrightarrow \mathbf{P}^2(K^a)$ and allows us to define subsets $\mathcal{V}_\infty(K) \subseteq \hat{\mathcal{V}}_\infty(K)$ and $\mathcal{V}_\infty(K^a) \subseteq \hat{\mathcal{V}}_\infty(K^a)$. Each $g \in G$ maps $\mathcal{V}_\infty(K^a)$ into itself and preserves the partial ordering parametrizations as well as the parametrizations α and A and the multiplicity m. Therefore, the quotient $\mathcal{V}_\infty(K) \simeq \mathcal{V}_\infty(K^a)/G$ also is naturally a tree that we equip with a metric that takes into account the degree of the map $\mathcal{V}_\infty(K^a) \to \mathcal{V}_\infty(K)$.

Alternatively, we can obtain the metric tree structure directly from the dual graphs of the admissible compactifications by keeping track of the residue fields of the closed points being blown up.

9.9. **Notes and further references.** The valuative tree at infinity was introduced in [FJ07] for the purposes of studying the dynamics at infinity of polynomial mappings of $\mathbf{C}^2$ (see the next section). It was not explicitly identified as a subset of the Berkovich affine plane over a trivially valued field.

In [FJ07], the tree structure of $\mathcal{V}_\infty$ was deduced by looking at the center on $\mathbf{P}^2$ of a semivaluation in $\mathcal{V}_\infty$. Given a closed point $\xi \in \mathbf{P}^2$, the semivaluations having center at ξ form a tree (essentially the valuative tree at ξ but normalized by $v(L_\infty) = 1$). By gluing these trees together along ord_∞ we see that $\mathcal{V}_\infty$ itself is a tree. The geometric approach here, using admissible compactifications, seems more canonical and amenable to generalization to higher dimensions.

Just as with the valuative tree at a point, I have allowed myself to change the notation from [FJ07]. Specifically, the valuative tree at infinity is (regrettably) denoted $\mathcal{V}_0$ and the tight tree at infinity is denoted $\mathcal{V}_1$. The notation $\mathcal{V}_\infty$ and $\mathcal{V}'_\infty$ seems more natural. Further, the valuation ord_∞ is denoted $-\deg$ in [FJ07].

The tight tree at infinity $\mathcal{V}'_\infty$ was introduced in [FJ07] and tight compactifications in [FJ11]. They are both very interesting notions. The tight tree was studied in [FJ07] using key polynomials, more or less in the spirit of Abhyankar and Moh [AM73]. While key polynomials are interesting, they are notationally cumbersome as they contain a lot of combinatorial information and they depend on a choice of coordinates, something that I have striven to avoid here.

As indicated in the proof of Theorem 9.18, it is possible to study the tight tree at infinity using the basic theory for compact surfaces. In particular, while the proof of the structure result for $\mathcal{V}'_\infty$ in [FJ07] used the Line Embedding Theorem in a crucial way (just as in Theorem 9.18) one can use the framework of tight compactifications together with surface theory to give a proof of the Line Embedding Theorem. (It should be mentioned, however, that by now there are quite a few proofs of the line embedding theorem.)

One can also prove Jung's theorem, on the structure $\mathrm{Aut}(\mathbf{C}^2)$ using the tight tree at infinity. It would be interesting to see if there is a higher-dimensional version of the tight tree at infinity, and if this space could be used to shine some light on the wild automorphisms of $\mathbf{C}^3$, the existence of which was proved by Shestakov and Umirbaev in [SU04].

The log discrepancy used here is a slight variation of the standard notion in algebraic geometry (see [JM10]) but has the advantage of not depending on the choice of compactification. If we fix an embedding $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$ and $A_{\mathbf{P}^2}$ denotes the usual log discrepancy on $\mathbf{P}^2$, then we have $A(v) = A_{\mathbf{P}^2}(v) - 3v(|\mathfrak{L}|)$.

10. Plane polynomial dynamics at infinity

We now come to the third type of dynamics on Berkovich spaces: the dynamics at infinity of polynomial mappings of $\mathbf{A}^2$. The study will be modeled on the dynamics near a (closed) fixed point as described in §8. We will refer to the latter situation as the local case.

10.1. **Setup.** Let K is an algebraically closed field of characteristic zero, equipped with the trivial valuation. (See §10.8 for the case of other ground fields.) Further, R and F are the coordinate ring and function field of the affine plane $\mathbf{A}^2$ over K. Recall that the Berkovich affine plane $\mathbf{A}^2_{\mathrm{Berk}}$ is the set of semivaluations on R that restrict to the trivial valuation on K.

10.2. **Definitions and results.** We keep the notation from §9 and consider a polynomial mapping $f : \mathbf{A}^2 \to \mathbf{A}^2$, which we assume to be *dominant* to avoid degenerate cases. Given an embedding $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$, the degree $\deg f$ is defined as the degree of the curve $\deg f^*\ell$ for a general line $\ell \in |\mathfrak{L}|$.

The degree growth sequence $(\deg f^n)_{n\geq 0}$ is submultiplicative,

$$\deg f^{n+m} \leq \deg f^n \cdot \deg f^m,$$

and so the limit

$$d_\infty = \lim_{n\to\infty} (\deg f^n)^{1/n}$$

is well defined. Since f is assumed dominant, $\deg f^n \geq 1$ for all n, hence $d_\infty \geq 1$.

Exercise 10.1. Verify these statements!

Example 10.2. If $f(z_1, z_2) = (z_2, z_1 z_2)$, then $\deg f^n$ is the $(n+1)$th Fibonacci number and $d_\infty = \frac{1}{2}(\sqrt{5} + 1)$ is the golden mean.

Example 10.3. For $f(z_1, z_2) = (z_1^2, z_1 z_2^2)$, $\deg f^n = (n+2)2^{n-1}$ and $d_\infty = 2$.

Exercise 10.4. Compute d_∞ for a skew product $f(z_1, z_2) = (\phi(z_1), \psi(z_1, z_2))$.

Here is the result that we are aiming for.

Theorem C. *The number $d_\infty = d_\infty(f)$ is a quadratic integer: there exist $a, b \in \mathbf{Z}$ such that $d_\infty^2 = a d_\infty + b$. Moreover, we are in exactly one of the following two cases:*

(a) *there exists $C > 0$ such that $d_\infty^n \leq \deg f^n \leq C d_\infty^n$ for all n;*

(b) $\deg f^n \sim n d_\infty^n$ *as $n \to \infty$.*

Moreover, case (b) occurs iff f, after conjugation by a suitable polynomial automorphism of $\mathbf{A}^2$, is a skew product of the form

$$f(z_1, z_2) = (\phi(z_1), \psi(z_1) z_2^{d_\infty} + O_{z_1}(z_2^{d_\infty - 1})),$$

where $\deg \phi = d_\infty$ *and* $\deg \psi > 0$.

The behavior of the degree growth sequence does not depend in an essential way on our choice of embedding $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$. To see this, fix such an embedding, let $g : \mathbf{A}^2 \to \mathbf{A}^2$ be a polynomial automorphism and set $\tilde{f} := g^{-1} f g$. Then $\tilde{f}^n = g^{-1} f^n g$, $f^n = g \tilde{f}^n g^{-1}$ and so

$$\frac{1}{\deg g \deg g^{-1}} \leq \frac{\deg \tilde{f}^n}{\deg f^n} \leq \deg g \deg g^{-1}$$

for all $n \geq 1$. As a consequence, when proving Theorem C, we may conjugate by polynomial automorphisms of $\mathbf{A}^2$, if necessary.

10.3. **Induced action.** The strategy for proving Theorems C is superficially very similar to the local case explored in §8. Recall that f extends to a map

$$f : \mathbf{A}^2_{\mathrm{Berk}} \to \mathbf{A}^2_{\mathrm{Berk}},$$

given by $f(v)(\phi) := v(f^*\phi)$.

We would like to study the dynamics of f at infinity. For any admissible compactification X of $\mathbf{A}^2$, f extends to a rational map $f : X \dashrightarrow \mathbf{P}^2$. Using resolution of singularities we can find X such that $f : X \to \mathbf{P}^2$ is a morphism. There are then two cases: either $f(E) \subseteq L_\infty$ for every prime E of X at infinity, or there exists a prime E such that $f(E) \cap \mathbf{A}^2 \neq \emptyset$. The first case happens iff f is *proper*.

Recall that $\hat{\mathcal{V}}_\infty$ denotes the set of semivaluations in $\mathbf{A}^2_{\mathrm{Berk}}$ having center at infinity. It easily follows that f is proper iff $f(\hat{\mathcal{V}}_\infty) \subseteq \hat{\mathcal{V}}_\infty$. Properness is the analogue of finiteness in the local case.

10.3.1. *The proper case.* When f is proper, if induces a selfmap

$$f : \hat{\mathcal{V}}_\infty \to \hat{\mathcal{V}}_\infty.$$

Now $\hat{\mathcal{V}}_\infty$ is the pointed cone over the valuative tree at infinity $\mathcal{V}_\infty$, whose elements are normalized by the condition $v(|\mathfrak{L}|) = -1$. As in the local case, we can break the action of f on $\hat{\mathcal{V}}_\infty$ into two parts: the induced dynamics

$$f_\bullet : \mathcal{V}_\infty \to \mathcal{V}_\infty,$$

and a multiplier $d(f, \cdot) : \mathcal{V}_\infty \to \mathbf{R}_+$. Here

$$d(f, v) = -v(f^*|\mathfrak{L}|),$$

Further, $f_\bullet$ is defined by

$$f_\bullet v = \frac{f(v)}{d(f, v)}.$$

The break-up of the action is compatible with the dynamics in the sense that $(f^n)_\bullet = (f_\bullet)^n$ and

$$d(f^n, v) = \prod_{i=0}^{n-1} d(f, v_i), \quad \text{where } v_i = f_\bullet^i v.$$

Recall that $\mathrm{ord}_\infty \in \mathcal{V}_\infty$ is the valuation given by $\mathrm{ord}_\infty(\phi) = -\deg(\phi)$ for any polynomial $\phi \in R$. We then have

$$\deg f^n = d(f^n, \mathrm{ord}_\infty) = \prod_{i=0}^{n-1} d(f, v_i), \quad \text{where } v_i = f_\bullet^i \,\mathrm{ord}_\infty.$$

Now $v_i \geq \mathrm{ord}_\infty$ on R, so it follows that $\deg f^n \leq (\deg f)^n$ as we already knew. The multiplicative cocycle $d(f, \cdot)$ is the main tool for studying the submultiplicative sequence $(\deg f^n)_{n \geq 0}$.

10.3.2. *The non-proper case.* When $f : \mathbf{A}^2 \to \mathbf{A}^2$ is dominant but not necessarily proper, there exists at least one divisorial valuation $v \in \hat{\mathcal{V}}_\infty \subseteq \mathbf{A}^2_{\mathrm{Berk}}$ for which $f(v) \in \mathbf{A}^2_{\mathrm{Berk}} \setminus \hat{\mathcal{V}}_\infty$. We can view $f : \hat{\mathcal{V}}_\infty \dashrightarrow \hat{\mathcal{V}}_\infty$ as a partially defined map. Its domain of definition is the open set $\hat{D}_f \subseteq \hat{\mathcal{V}}_\infty$ consisting of semivaluations for which there exists an affine function L with $v(f^*L) < 0$. Equivalently, if we as before define $d(f, v) = -v(f^*|\mathfrak{L}|)$, then $\hat{D}_f = \{d(f, \cdot) > 0\}$. On $D_f := \hat{D}_f \cap \mathcal{V}_\infty$ we define $f_\bullet$ as before, namely $f_\bullet v = f(v)/d(f, v)$.

Notice that $D_{f^n} = \bigcap_{i=0}^{n-1} f_\bullet^{-i} D_f$, so the domain of definition of $f_\bullet^n$ decreases as $n \to \infty$. One may even wonder whether the intersection $\bigcap_n D_{f^n}$ is empty. However, a moment's reflection reveals that ord_∞ belongs to this intersection. More generally, it is not hard to see that the set of valuations $v \in \mathcal{V}_\infty$ for which $v(\phi) < 0$ for all nonconstant polynomials ϕ, is a subtree of $\mathcal{V}_\infty$ contained in D_f and invariant under f, for any dominant polynomial mapping f.

For reasons that will become apparent later, we will in fact study the dynamics on the even smaller subtree, namely the tight subtree $\mathcal{V}'_\infty \subseteq \mathcal{V}_\infty$ defined in §9.7. We shall see shortly that $f_\bullet \mathcal{V}'_\infty \subseteq \mathcal{V}'_\infty$, so we have a natural induced dynamical system on $\mathcal{V}'_\infty$ for any dominant polynomial mapping f.

10.4. **Invariance of the tight tree $\mathcal{V}'_\infty$.** Theorem B, the local counterpart to Theorem C, follows easily under the additional assumption (not always satisfied) that there exists a quasimonomial valuation $v \in \mathcal{V}_0$ such that $f_\bullet v = v$. Indeed, such a valuation satisfies

$$\mathrm{ord}_0 \le v \le \alpha v,$$

where $\alpha = \alpha(v) < \infty$. If $f(v) = cv$, then this gives $c = c_\infty$ and $\alpha^{-1} c_\infty \le c(f^n) \le c_\infty^n$. Moreover, the inclusion $c_\infty \Gamma_v = \Gamma_{f(v)} \subseteq \Gamma_v$ implies that c_∞ is a quadratic integer. See §8.5.

In the affine case, the situation is more complicated. We cannot just take *any* quasimonomial fixed point v for $f_\bullet$. For a concrete example, consider the product map $f(z_1, z_2) = (z_1^3, z_2^2)$ and let v be the monomial valuation with weights $v(z_1) = 0$, $v(z_2) = -1$. Then $f(v) = 2v$, whereas $d_\infty = 3$. The problem here is that while $v \ge \mathrm{ord}_\infty$, the reverse inequality $v \le C\,\mathrm{ord}_\infty$ does not hold for any constant $C > 0$.

The way around this problem is to use the tight tree $\mathcal{V}'_\infty$ introduced in §9.7. Indeed, if $v \in \mathcal{V}'_\infty$ is quasimonomial, then either there exists $\alpha = \alpha(v) > 0$ such that $\alpha^{-1}\,\mathrm{ord}_\infty \le v \le \mathrm{ord}_\infty$ on R, or v is monomial in suitable coordinates on $\mathbf{A}^2$, see Theorem 9.18. As the example above shows, the latter case still has to be treated with some care.

We start by showing that the tight tree is invariant.

Proposition 10.5. *For any dominant polynomial mapping $f : \mathbf{A}^2 \to \mathbf{A}^2$ we have $f(\hat{\mathcal{V}}'_\infty) \subseteq \hat{\mathcal{V}}'_\infty$. In particular, $\mathcal{V}'_\infty \subseteq D_f$ and $f_\bullet \mathcal{V}'_\infty \subseteq \mathcal{V}'_\infty$.*

Sketch of proof. It suffices to prove that if $v \in \hat{\mathcal{V}}'_\infty$ is divisorial, then $f(v) \in \hat{\mathcal{V}}'_\infty$. After rescaling, we may assume $v = \mathrm{ord}_E$. Arguing using numerical invariants as in §4.4, we show that $f(v)$ is divisorial, of the form $f(v) = r\,\mathrm{ord}_{E'}$ for some prime divisor E' on $\mathbf{A}^2$ (a priori not necessarily at infinity).

We claim that the formula

$$A(f(v)) = A(v) + v(Jf) \tag{10.1}$$

holds, where Jf denotes the Jacobian determinant of f. Note that the assumption $\alpha(v) \geq 0$ implies $v(Jf) \leq 0$ by (9.9). Together with the assumption $A(v) \leq 0$, we thus see that $A(f(v)) \leq 0$. In particular, the 2-form ω on $\mathbf{A}^2$ has a pole along E', which implies that E' must be a prime at infinity.

Hence $f(v) \in \hat{\mathcal{V}}_\infty$ and $A(f(v)) \leq 0$. It remains to prove that $\alpha(f(v)) \geq 0$. Let X' be an admissible compactification of $\mathbf{A}^2$ in which E' is a prime at infinity and pick another compactification X of $\mathbf{A}^2$ such that the induced map $f : X \to X'$ is regular. The divisors $\check{E} \in \mathrm{Div}_\infty(X)$ and $\check{E}' \in \mathrm{Div}_\infty(X')$ are both nef at infinity and satisfies $f_*\check{E} = r\check{E}'$. Since $(\check{E} \cdot \check{E}) = \alpha(v) \geq 0$, $\check{E}$ is effective (and hence nef). As a consequence, $\check{E}' = r^{-1} f_*\check{E}$ is effective and hence nef. In particular, $\alpha(f(v)) = r^2(\check{E}' \cdot \check{E}') \geq 0$, which completes the proof.

Finally we prove (10.1). Write $A_E = A(\mathrm{ord}_E)$ and $A_{E'} = A(\mathrm{ord}_{E'})$. Recall that ω is a nonvanishing 2-form on $\mathbf{A}^2$. Near E' it has a zero of order $A_{E'} - 1$. From the chain rule, and the fact that $f(\mathrm{ord}_E) = r\,\mathrm{ord}_{E'}$, it follows that $f^*\omega$ has a zero of order $r - 1 + r(A_{E'} - 1) = rA_{E'} - 1$ along E. On the other hand we have $f^*\omega = Jf \cdot \omega$ in $\mathbf{A}^2$ and the right hand side vanishes to order $\mathrm{ord}_E(Jf) + A_E - 1$ along E. This concludes the proof. □

10.5. **Some lemmas.** Before embarking on the proof of Theorem C, let us record some useful auxiliary results.

Lemma 10.6. *Let $\phi \in R$ be a polynomial, X an admissible compactification of $\mathbf{A}^2$ and E a prime of X at infinity. Let C_X be the closure in X of the curve $\{\phi = 0\}$ in $\mathbf{A}^2$ and assume that C_X intersects E. Then $\deg p \geq b_E$, where $b_E := -\mathrm{ord}_E(|\mathfrak{L}|)$.*

Proof. This follows from elementary intersection theory. Let $\pi : X \to \mathbf{P}^2$ be the birational morphism induced by the identity on $\mathbf{A}^2$ and let $C_{\mathbf{P}^2}$ be the closure in $\mathbf{P}^2$ of the curve $\{\phi = 0\} \subseteq \mathbf{A}^2$. Then $\mathrm{ord}_E(\pi^* L_\infty) = b_E$. Assuming that C_X intersects E, we get

$$b_E \leq b_E(C_X \cdot E) \leq (C_X \cdot \pi^* L_\infty) = (C_{\mathbf{P}^2} \cdot L_\infty) = \deg p,$$

where the first equality follows from the projection formula and the second from Bézout's Theorem. □

Applying Lemma 10.6 and Lemma 9.8 to $\phi = f^*L$, for L a general affine function, we obtain

Corollary 10.7. *Let $f : \mathbf{A}^2 \to \mathbf{A}^2$ be a dominant polynomial mapping, X an admissible compactification of $\mathbf{A}^2$ and E a prime of X at infinity. Assume that $\deg(f) < b_E$. Then $d(f, v) = d(f, v_E)$ for all $v \in \mathcal{V}_\infty$ such that $r_X(v) = v_E$.*

10.6. **Proof of Theorem C.** If we were to follow the proof in the local case, we would pick a log resolution at infinity of the linear system $f^*|\mathfrak{L}|$ on $\mathbf{P}^2$. By this we mean an admissible compactification X of $\mathbf{A}^2$ such that the strict transform of $f^*|\mathfrak{L}|$ to X has no base points on $X \setminus \mathbf{A}^2$. Such an admissible compactification exists by resolution of singularities. At least when f is proper, we get a well defined selfmap

$r_X f_\bullet : |\Delta(X)| \to |\Delta(X)|$. However, a fixed point v of this map does not have an immediate bearing on Theorem C. Indeed, we have seen in §10.4 that even when v is actually fixed by $f_\bullet$, so that $f(v) = dv$ for some $d > 0$, it may happen that $d < d_\infty$.

One way around this problem would be to ensure that the compactification X is tight, in the sense of §9.7.2. Unfortunately, it is not always possible, even for f proper, to find a tight X that defines a log resolution of infinity of $f^*|\mathfrak{L}|$.

Instead we use a recursive procedure. The proof below in fact works also when f is merely dominant, and not necessarily proper. Before starting the procedure, let us write down a few cases where we actually obtain a proof of Theorem C.

Lemma 10.8. *Let X be a tight compactification of $\mathbf{A}^2$ with associated retraction $r_X : \mathcal{V}_\infty \to |\Delta(X)|$. Consider a fixed point $v \in |\Delta(X)|$ of the induced selfmap $r_X f_\bullet : |\Delta(X)| \to |\Delta(X)|$. Assume that we are in one of the following three situations:*

- (a) *$f_\bullet v = v$ and $\alpha(v) > 0$;*
- (b) *$f_\bullet v \neq v$, $\alpha(v) > 0$, v is divisorial and $b(v) > \deg(f)$;*
- (c) *$\alpha(v) = 0$ and $(r_X f_\bullet)^n w \to v$ as $n \to \infty$ for $w \in |\Delta(X)|$ close to v.*

Then Theorem C holds.

Proof. Case (a) is treated as in the local situation. Since $\alpha := \alpha(v) > 0$ we have $\alpha^{-1} v \le \operatorname{ord}_\infty \le v$ on R. Write $f(v) = dv$, where $d = d(f, v) > 0$. Then

$$\deg f^n = -\operatorname{ord}_\infty(f^{n*}|\mathfrak{L}|) \le -\alpha^{-1} v(f^{n*}|\mathfrak{L}|) = -\alpha^{-1} d^n v(|\mathfrak{L}|) = \alpha^{-1} d^n.$$

Similarly, $\deg f^n \ge d^n$. This proves statement (a) of Theorem C (and that $d_\infty = d$). The fact that $d = d_\infty$ is a quadratic integers is proved exactly as in the local case, using value groups. Indeed, one obtains $d\Gamma_v \subseteq \Gamma_v$. Since $\Gamma_v \simeq \mathbf{Z}$ or $\Gamma_v \simeq \mathbf{Z} \oplus \mathbf{Z}$, d must be a quadratic integer.

Next we turn to case (b). By the analogue of Lemma 7.12 we may assume that the center of $f_\bullet v$ on X is a free point ξ of E. By Corollary 10.7 we have $d(f, \cdot) \equiv d := d(f, v)$ on $U(\xi)$. As in the local case, this implies that $f_\bullet \overline{U(\xi)} \subseteq U(\xi)$, $d(f^n, v) = d^n$, $d^n \le \deg(f^n) \le \alpha^{-1} d^n$, so that we are in case (a) of Theorem C, with $d_\infty = d$. The fact that $d = d_\infty$ is a quadratic integer follows from $d\Gamma_v \subseteq \Gamma_v \simeq \mathbf{Z}$. In fact, $d \in \mathbf{N}$.

Finally we consider case (c). Recall that the statements of Theorem C are invariant under conjugation by polynomial automorphisms. Since X is tight and $\alpha(v) = 0$, we may by Theorem 9.18 choose coordinates (z_1, z_2) on $\mathbf{A}^2$ in which v is monomial with $v(z_1) = 0$, $v(z_2) = -1$. Since v is an end in the $f_\bullet$-invariant tree $\mathcal{V}'_\infty$ and $r_X f_\bullet v = v$, we must have $f_\bullet v = v$. In particular, $f_\bullet v(z_1) = 0$, which implies that f is a skew product of the form

$$f(z_1, z_2) = (\phi(z_1), \psi(z_1) z_2^d + O_{z_1}(z_2^{d-1})),$$

where $d \ge 1$ and ϕ, ψ are nonzero polynomials. The valuations in $|\Delta(X)|$ close to v must also be monomial valuations, of the form w_t, with $w_t(z_1) = -t$ and $w_t(z_2) = -1$, where $0 \le t \ll 1$. We see that $f(w_t)(z_1) = -t \deg \phi$ and $f(w_t)(z_2) = -(d + t \deg q)$. When t is irrational, $f_\bullet w_t$ must be monomial, of the form $w_{t'}$, where $t' = t \frac{\deg p}{d + t \deg q}$. By continuity, this relationship must hold for all real t, $0 \le t \ll 1$. By our assumptions, $t' < t$ for $0 < t \ll 1$. This implies that either $\deg p < d$ or that $\deg p = d$, $\deg q > 0$. It is then clear that $d_\infty = d$ is an integer, proving the first statement in Theorem C. Finally, from a direct computation, that we leave as an exercise to the reader, it follows that $\deg f^n \sim nd^n$. □

The main case not handled by Lemma 10.8 is the case (b) but without the assumption that $b_E > \deg f$. In this case we need to blow up further.

Lemma 10.9. *Let X be a tight compactification of $\mathbf{A}^2$ with associated retraction $r_X : \mathcal{V}_\infty \to |\Delta(X)|$. Assume that $v = v_E = b_E^{-1}\operatorname{ord}_E \in |\Delta(X)|$ is a divisorial valuation such that $r_X f_\bullet v_E = v_E$ but $f_\bullet v_E \neq v_E$. Then there exists a tight compactification X' of $\mathbf{A}^2$ dominating X and a valuation $v' \in |\Delta(X')| \setminus |\Delta(X)|$ such that $r_{X'} f_\bullet v' = v'$ and such that we are in one of the following cases:*

(a) *$f_\bullet v' = v'$ and $\alpha(v') > 0$;*
(b) *$f_\bullet v' \neq v'$, v' is divisorial, $\alpha(v') > 0$ and $b(v') > b(v)$;*
(c) *$\alpha(v') = 0$ and $(r_{X'} f_\bullet)^n w \to v'$ as $n \to \infty$ for $w \in |\Delta(X)|$ close to v'.*

It is clear that repeated application of Lemma 10.8 and Lemma 10.9 leads to a proof of Theorem C. The only thing remaining is to prove Lemma 10.9.

Proof. Write $v_0 = v$. By (the analogue at infinity of) Lemma 7.12 we may find an admissible compactification X_0 dominating X, such that $|\Delta_0| := |\Delta(X_0)| = |\Delta(X)|$, $r_0 := r_{X_0} = r_X$ and such that the center of $v_0 = v$ on X_0 is a prime E_0 of X_0 at infinity. Since $f_\bullet v_0 \neq v_0$, the center of $f_\bullet v_0$ must be a free point $\xi_0 \in E_0$. Let X_1 be the blowup of X_0 at ξ_0, E_1 the exceptional divisor and $v_1 = b_1^{-1}\operatorname{ord}_{E_1}$ the associated divisorial valuation. Note that $b_1 = b_0$ and $\alpha(v_1) = \alpha(v_0) - b_0^{-1}$ by (9.4). In particular, X_1 is still tight. Write $|\Delta_1| = |\Delta(X_1)|$ and $r_1 := r_{X_1}$. We have $r_1 f_\bullet v_0 \in |\Delta_1| \setminus |\Delta_0| =]v_0, v_1]$. Thus there are two cases:

(1) there exists a fixed point $v' \in]v_0, v_1[$ for $r_1 f_\bullet$;
(2) $(r_1 f_\bullet)^n \to v_1 = r_1 f_\bullet v_1$ as $n \to \infty$;

Let us first look at case (1). Note that $\alpha(v') > \alpha(v_1) \geq 0$. If $f_\bullet v' = v'$, then we are in situation (a) and the proof is complete. Hence we may assume that $f_\bullet v' \neq v'$. Then v' is necessarily divisorial. By Lemma 7.12 we have $b(v') > b_0 = b(v)$. We are therefore in situation (b), so the proof is complete in this case.

It remains to consider case (2). If $\alpha(v_1) = 0$, then we set $X' = X_1$, $v' = v_1$ and we are in situation (c). We can therefore assume that $\alpha(v_1) > 0$. If $f_\bullet v_1 = v_1$, then we set $X' = X_1$, $v' = v_1$ and we are in situation (a). If $f_\bullet v_1 \neq v_1$, so that the center of $f_\bullet v_1$ is a free point $\xi_1 \in E_1$, then we can repeat the procedure above. Let X_2 be the blowup of X_1 at ξ_1, let E_2 be the exceptional divisor and $v_2 = b_2^{-1}\operatorname{ord}_{E_2}$ the associated divisorial valuation. We have $b_2 = b_1 = b$ and $\alpha(v_2) = \alpha(v_1) - b^{-1} = \alpha(v) - 2b^{-1}$ by (9.4).

Continuing the procedure above must eventually lead us to the situation in (a) or (c). Indeed, all of our compactifications are tight, so in particular all valuations v_n satisfy $\alpha(v_n) \geq 0$. But $\alpha(v_n) = \alpha(v) - nb^{-2}$. This completes the proof. □

10.7. **Further properties.** The presentation above was essentially optimized to give a reasonably short proof of Theorem C. While it is beyond the scope of these notes to present the details, let us briefly summarize some further results from [FJ07, FJ11]. Let $f : \mathbf{A}^2 \to \mathbf{A}^2$ be a polynomial mapping and write f also for its extension $f : \mathbf{A}^2_{\mathrm{Berk}} \to \mathbf{A}^2_{\mathrm{Berk}}$.

To begin, f interacts well with the classification of points: if $v \in \hat{\mathcal{V}}_\infty$ and $f(v) \in \hat{\mathcal{V}}_\infty$ then $f(v)$ is of the same type as v (curve, divisorial, irrational or infinitely singular). This is proved using numerical invariants in the same way as in §4.4.

At least when f is proper the induced map $f_\bullet : \mathcal{V}_\infty \to \mathcal{V}_\infty$ is continuous, finite and open. This follows from general results on Berkovich spaces, just as in Proposition 4.3. As a consequence, the general results on tree maps in §2.6 apply.

In [FJ07, FJ11], the existence of an *eigenvaluation* was emphasized. This is a valuation $v \in \mathcal{V}_\infty$ such that $f(v) = d_\infty v$. One can show from general tree arguments that there must exist such a valuation in the tight tree $\mathcal{V}'_\infty$. The proof of Theorem C gives an alternative construction of an eigenvaluation in $\mathcal{V}'_\infty$.

Using a lot more work, the global dynamics on $\mathcal{V}'_\infty$ is described in [FJ11]. Namely, the set $\mathcal{T}_f$ of eigenvaluations in $\mathcal{V}'_\infty$ is either a singleton or a closed interval. (The "typical" case is that of a singleton.) In both cases we have $f^n_\bullet v \to \mathcal{T}_f$ as $n \to \infty$, for all but at most one $v \in \mathcal{V}'_\infty$. This means that the dynamics on the tight tree $\mathcal{V}_\infty$ is globally contracting, as opposed to a rational map on the Berkovich projective line, which is globally expanding.

Using the dynamics on $\mathcal{V}'_\infty$, the cocycle $d(f^n, v)$ can be very well described: for any $v \in \mathcal{V}'_\infty$ the sequence $(d(f^n, v))_{n \ge 0}$ satisfies an integral recursion relation. Applying this to $v = \operatorname{ord}_\infty$ we see that the degree growth sequence $(\deg(f^n))$ satisfies such a recursion relation.

As explained in the introduction, one motivation for the results in this section comes from polynomial mappings of the complex plane $\mathbf{C}^2$, and more precisely understanding the rate at which orbits are attracted to infinity. Let us give one instance of what can be proved. Suppose $f : \mathbf{C}^2 \to \mathbf{C}^2$ is a dominant polynomial mapping and assume that f has "low topological degree" in the sense that the asymptotic degree $d_\infty(f)$ is strictly larger than the topological degree of f, i.e. the number of preimages of a typical point. In this case, we showed in [FJ11] that the functions

$$\frac{1}{d_\infty^n} \log^+ \|f^n\|$$

converge uniformly on compact subsets of $\mathbf{C}^2$ to a plurisubharmonic function G^+ called the *Green function* of f. Here $\|\cdot\|$ is any norm on $\mathbf{C}^2$ and we write $\log^+\|\cdot\| := \max\{\log|\cdot|, 0\}$. This Green function is important for understanding the ergodic properties of f, as explored by Diller, Dujardin and Guedj [DDG1, DDG2, DDG3].

10.8. **Other ground fields.** Throughout the section we assumed that the ground field was algebraically closed and of characteristic zero. Let us briefly discuss what happens when one or more of these assumptions are not satisfied.

First, the assumption on the characteristic was only used for the monomialization result Theorem 9.18. Granted this theorem, everything in §10 holds over any algebraically closed field.

Second, the assumption above that K be algebraically closed is unimportant for Theorem C, at least for statements (a) and (b). Indeed, if K^a is the algebraic closure of K, then any polynomial mapping $f : \mathbf{A}^2(K) \to \mathbf{A}^2(K)$ induces a polynomial mapping $f : \mathbf{A}^2(K^a) \to \mathbf{A}^2(K^a)$. Further, an embedding $\mathbf{A}^2(K) \hookrightarrow \mathbf{P}^2(K)$ induces an embedding $\mathbf{A}^2(K^a) \hookrightarrow \mathbf{P}^2(K^a)$ and the degree of f^n is then independent of whether we work over K or K^a. Thus statements (a) and (b) of Theorem C trivially follow from their counterparts over an algebraically closed field.

10.9. **Notes and further references.** The material in this section is adapted from the papers [FJ07, FJ11] joint with Charles Favre, but with a few changes in the presentation. In order to keep these lecture notes reasonably coherent, I have also changed some of the notation from the original papers. I have also emphasized a geometric approach that has some hope of being applicable in higher dimensions and the presentation is streamlined to give a reasonably quick proof of Theorem C.

Instead of working on valuation space, it is possible to consider the induced dynamics on divisors on the Riemann-Zariski space. By this we mean the data of one divisor at infinity for each admissible compactification of $\mathbf{A}^2$ (with suitable compatibility conditions when one compactification dominates another. See [FJ11] for more details and [BFJ08a] for applications of this point of view in a slightly different context.

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