HYPERBOLIC DYNAMICS OF ENDOMORPHISMS

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Abstract. We present the theory of hyperbolic dynamics of endomorphisms in. Topics covered are hyperbolic sets, stable manifolds, local product structure, shadowing, spectral decomposition and $\hat{\Omega}$-stability.

0. Introduction

In this paper we study a smooth mapping f of a manifold M as a dynamical system. We will discuss both semilocal and global dynamical properties of f, but always under some hyperbolicity assumption. The main examples we have in mind are holomorphic endomorphisms of complex projective space $\mathbf{P}^k$, $k \geq 1$ but we will state the results in greater generality.

There are many excellent and detailed expositions on differentiable dynamics, e.g. [S], but they usually consider only invertible systems, such as diffeomorphisms of a compact manifold. As for noninvertible maps, the attitude seems to be that most results for diffeomorphisms continue to hold when interpreted correctly, but it is difficult to find a detailed written account; the purpose of this paper is to improve upon that. We do not claim that our results are new. Our main references are [R] and [PS].

The building blocks in hyperbolic dynamics are *hyperbolic sets*. These are generalizations of hyperbolic fixed points, i.e. fixed points where the derivative has no eigenvalue of modulus one. For the precise definition of what it means for a compact, invariant set Λ to be hyperbolic, we refer to section 1, but the definition involves the set

$$\hat{\Lambda} = \{(x_i)_{i \leq 0}; x_i \in \Lambda, f(x_i) = x_{i+1}\}.$$

of *histories* in Λ.

A hyperbolic set Λ has *local stable and unstable manifolds* at each point; see Theorem 1.2 for details. Another basic feature of hyperbolic sets is *persistence* under perturbations. This means that if f is hyperbolic on $\Lambda = \Lambda_f$ and g is close to f, then g has a hyperbolic set Λ_g close to Λ_f such that $\hat{f}|_{\hat{\Lambda}_f}$ and $\hat{g}|_{\hat{\Lambda}_g}$ are conjugate. Here $\hat{f}$ is the shift $f((x_i)) = (f(x_i))$. For more details see Proposition 1.4. Note that the sets Λ_f and Λ_g themselves need not be homeomorphic.

Many results on the dynamics near a hyperbolic set Λ are best formulated in terms of $\hat{\Lambda}$. With this in mind we introduce the concept of *local product structure* for $\hat{\Lambda}$. The definition says that if $(\hat{p}^{(i)})_{i\in\mathbf{Z}}$ and $(\hat{q}^{(i)})_{i\in\mathbf{Z}}$ are orbits in $\hat{\Lambda}$ and $(\hat{x}^{(i)})_{i\in\mathbf{Z}}$

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is an orbit which follows $(\hat{p}^{(i)})$ in positive time and follows $(\hat{q}^{(i)})$ in negative time, then $\hat{x}^{(i)}$ is in fact an orbit in $\hat{\Lambda}$.

Under the assumptions of local product structure for $\hat{\Lambda}$ we prove *shadowing* results for $\hat{\Lambda}$ and Λ, saying that an approximate orbit in $\hat{\Lambda}$ (Λ) is always close to an honest orbit in $\hat{\Lambda}$ (Λ). It seems difficult to prove this result for Λ without first proving it for $\hat{\Lambda}$.

Hyperbolicity of a compact set Λ is a semilocal condition, only involving the dynamics in a neighborhood of Λ. Axiom A, however, is a global condition, i.e. a condition on the dynamics of f on all of M. For most results on Axiom A maps we will make two assumptions, namely that M is compact, and that f is an open mapping. These assumptions are needed in some of the proofs; they are always satisfied for nonconstant holomorphic endomorphisms of $\mathbf{P}^k$.

The *nonwandering set* Ω of f is, by definition, the set of points $x \in M$ having no neighborhood U such that $f^n(U) \cap U = \emptyset$ for all $n \geq 1$. If M is compact, then all orbits of f converge to Ω in forward and backward time. We say that f is Axiom A if periodic points are dense in Ω and f is hyperbolic on Ω.

The first consequence of Axiom A is that $\hat{\Omega}$ has local product structure; thus the shadowing results mentioned above apply. We use this to prove versions of Smale's *spectral decomposition theorem* for $\hat{\Omega}$ and Ω, saying that $\hat{\Omega}$ (Ω) is the finite disjoint union of compact invariant sets, called basic sets, on which $\hat{f}$ (f) is topologically transitive. Again it seems difficult to prove this for Ω without going via $\hat{\Omega}$.

Finally we address stability. An endomorphism f is called *$\hat{\Omega}$-stable* if $\hat{f}|_{\hat{\Omega}_f}$ is conjugate to $\hat{g}|_{\hat{\Omega}_g}$ for all g sufficiently close to f. Define a relation on the basic sets of an Axiom A endomorphism f by saying that $\Omega_j > \Omega_k$ if there is an orbit $(x_i)_{i \in \mathbf{Z}}$ such that $x_i \to \Omega_j$ as $i \to -\infty$ and $x_i \to \Omega_k$ as $i \to \infty$. Then f is said to have *no cycles* if there is no nontrivial sequence of basic sets $\Omega_{i_0} < \Omega_{i_1} < \cdots < \Omega_{i_k} = \Omega_{i_0}$ We prove that if f is Axiom A and has no cycles, then f is $\hat{\Omega}$-stable. Axiom A in itself does not imply $\hat{\Omega}$-stability.

The paper starts by recalling the definition of a hyperbolic set for an endomorphism and stating some basic properties, including the stable manifold theorem and persistence. This is done in section 1. The proofs here are only sketched, as the (long) details can be found elsewhere. In section 2 we consider local product structure for a hyperbolic set and prove shadowing results. Then, in section 3, we define Axiom A endomorphisms, show that their nonwandering sets have the suitable local product structure and prove the spectral decomposition theorem. Finally, in the last section we study $\hat{\Omega}$-stability and prove that an open Axiom A endomorphism f of a compact manifold M with no cycles is $\hat{\Omega}$-stable.

1. Hyperbolic sets and the stable manifold theorem

In this section we will give the definition of a hyperbolic set and state some basic facts about them. In particular we will be concerned with persistence under perturbations and existence of local stable and unstable manifolds.

Suppose f is a C^∞ endomorphism of a C^∞ finite-dimensional Riemannian manifold M. Let Λ be a compact subset of M with $f(\Lambda) = \Lambda$ and define $\hat{\Lambda}$ to be the set of histories in Λ, i.e.

$$\hat{\Lambda} = \{(x_i)_{i \leq 0}; x_i \in \Lambda, f(x_i) = x_{i+1}\}.$$

Then $\hat{\Lambda}$ is a closed subset of $\Lambda^{\mathbf{N}}$, hence compact. We will often use the notation $\hat{x}$ for a point $(x_i)_{i\leq 0}$ in $\hat{\Lambda}$. Every distance d on Λ defines a distance on $\hat{\Lambda}$, also denoted by d, by

$$d(\hat{x},\hat{y}) = \sum_{i\leq 0} 2^i d(x_i, y_i).$$

The restriction $f|_\Lambda$ lifts to a homeomorphism $\hat{f}$ of $\hat{\Lambda}$ given by $\hat{f}((x_i)) = (x_{i+1})$. There is a natural projection π from $\hat{\Lambda}$ to Λ sending $\hat{x}$ to x_0 and the pullback under π of the restriction to Λ of the tangent bundle of M is a bundle on $\hat{\Lambda}$ which we call the tangent bundle $T_{\hat{\Lambda}}$. Explicitly, a point in $T_{\hat{\Lambda}}$ is of the form $(\hat{x}, v)$ where $\hat{x} \in \hat{\Lambda}$ and v is a tangent vector in $T_{x_0}M$. The derivative Df lifts to a map $D\hat{f}$ of $T_{\hat{\Lambda}}$ in a natural way.

Now f is said to be *hyperbolic* on Λ, or that Λ is a *hyperbolic set*, if there exists a continuous splitting $T_{\hat{\Lambda}} = E^u \oplus E^s$ which is invariant under $D\hat{f}$ and such that $D\hat{f}$ is expanding on E^u and contracting on E^s. More precisely, $D\hat{f}(E^{u/s}) \subset E^{u/s}$ and there exist constants $c > 0$ and $\lambda > 1$ such that for all $n \geq 1$

$$|D\hat{f}^n(v)| \geq c\lambda^n |v| \quad v \in E^u$$
$$|D\hat{f}^n(v)| \leq c^{-1}\lambda^{-n} |v| \quad v \in E^s.$$

Remark 1.1. It is possible to make a smooth change of metric in a neighborhood of Λ and obtain $c = 1$ in the equation above.

Note that whereas the fiber of the unstable bundle E^u at a point $\hat{x} \in \hat{\Lambda}$ depends on the whole history $\hat{x}$ of x_0, the fiber of E^s at $\hat{x}$ depends only on the point x_0. Hence the dimension of the fiber of E^u at a point $\hat{x}$ depends only on x_0, so the dimensions of the fibers of the bundles E^u and E^s are locally constant.

As a special case of the above we say that f is *expanding* on Λ if the bundle E^s is trivial. This means that there exist constants $c > 0$ and $\lambda > 1$ such that $|D\hat{f}^n(x)v| \geq c\lambda^n|v|$ for all $x \in \Lambda$, $v \in T_xM$ and all $n \geq 1$.

Perhaps the most fundamental result in hyperbolic dynamics is the stable manifold theorem. For each point p in Λ and each history $\hat{q}$ in $\hat{\Lambda}$, we define *local stable and unstable manifolds* by

$$W^s_\delta(p) = \{y \in M; d(f^i(y), f^i(p)) < \delta \ \forall i \geq 0\}$$
$$W^u_\delta(\hat{q}) = \{y \in M; \exists \hat{y}, \pi(\hat{y}) = y, d(y_i, q_i) < \delta \ \forall i \leq 0\}$$

for small $\delta > 0$. The following theorem asserts that the (un)stable manifolds are indeed nice objects.

Theorem 1.2. ***(Stable Manifold Theorem)*** *If δ is small enough, then*

(i) *For all $p \in \Lambda$ and all $\hat{q} \in \hat{\Lambda}$, $W^s_\delta(p)$ and $W^u_\delta(\hat{q})$ are embedded C^∞ disks in M tangent to $E^s(p)$ and $E^u(\hat{q})$ at p and q_0, respectively.*
(ii) *$W^s_\delta(p)$ and $W^u_\delta(\hat{q})$ depend continuously on p and $\hat{q}$, respectively.*
(iii) *If $x \in W^s_\delta(p)$, then $d(f^n(x), f^n(p)) \to 0$ exponentially fast as $n \to \infty$. Similarly, every point x in $W^u_\delta(\hat{q})$ has a unique history $\hat{x}$ such that $x_j \in W^u_\delta(\hat{f}^j(\hat{q}))$ for all $j \leq 0$ and $d(x_j, q_j) \to 0$ exponentially fast as $j \to -\infty$.*

Let us sketch a proof of Theorem 1.2. The idea is to consider the set $B(\hat{\Lambda}, M)$ of bounded maps of $\hat{\Lambda}$ into M. This is a Banach manifold modeled on the Banach space of bounded sections of $T_{\hat{\Lambda}}$. Define a map $\mathcal{F}$ of $B(\hat{\Lambda}, M)$ by $\mathcal{F}(h) = f \circ h \circ \hat{f}^{-1}$. Then the projection π is a fixed point of $\mathcal{F}$ and the assumption that f was hyperbolic on Λ means exactly that π is a hyperbolic fixed point. By a general stable manifold theorem for hyperbolic fixed points in Banach spaces it follows that $\mathcal{F}$ has a local (un)stable manifold. The (un)stable manifolds of f are then obtained as $\{h(x)\}$, where h runs over the (un)stable manifold of $\mathcal{F}$. To do all of this precisely, and to verify that (i)–(iii) holds, requires a nontrivial amount of work, which we will not go into here. A proof of a more general theorem can be found in [PS].

A special case of a hyperbolic set Λ is a *hyperbolic fixed point* p. This means that $f(p) = p$ and Df_p has no eigenvalue of modulus one. Theorem 1.2 is then easier to prove and the method of proof yields the following "Lambda Lemma" or "Inclination Lemma". For an outline of the proof see [R].

Proposition 1.3. *If p is a* hyperbolic fixed point *of f and Σ is an embedded C^1 submanifold of M intersecting $W^s_\delta(p)$ transversely near p, then for n large enough $f^n(\Sigma)$ contains an embedded manifold Σ_n, which is C^1-close to $W^u_\delta(\hat{p})$, where $\hat{p} = (\dots, p, p)$. Similarly, if Σ' is an embedded C^1 submanifold of M intersecting $W^u_\delta(\hat{p})$ transversely near p, then $f^{-n}(\Sigma')$ contains a submanifold Σ'_n, which is C^1-close to $W^s_\delta(p)$ for large n.*

We close this section by stating a persistence property for hyperbolic sets.

Proposition 1.4. *If f is hyperbolic on $\Lambda = \Lambda_f$ and g is C^1-close to f, then there exists a continuous map $h : \hat{\Lambda} \to M$ close to the projection $\pi(\hat{x}) = x_0$ such that $g \circ h = h \circ \hat{f}$ and that g is hyperbolic on $\Lambda_g := h(\Lambda_f)$. The map h lifts to a homeomorphism $\hat{h} : \widehat{\Lambda_f} \to \widehat{\Lambda_g}$ with $\hat{g} \circ \hat{h} = \hat{h} \circ \hat{f}$, and h depends continuously on g in the C^r topology, $1 \le r \le \infty$.*

Let us sketch a proof of this. Consider the Banach manifold $C(\hat{\Lambda}, M)$ of continuous maps of $\hat{\Lambda}$ into M and define a selfmap $\mathcal{F}_g$ of $C(\hat{\Lambda}, M)$ for each g by $\mathcal{F}_g(h) = g \circ h \circ \hat{f}^{-1}$. Again π is a hyperbolic fixed point of $\mathcal{F}_f$, so for g sufficiently close to f, $\mathcal{F}_g$ has a hyperbolic fixed point h_g, depending continuously on g. This is the map h above.

2. Local product structure and shadowing

We now use the local stable and unstable manifolds to analyze the dynamics near a hyperbolic set Λ. In particular we will define the notion of local product structure on $\hat{\Lambda}$ and show how this implies that pseudoorbits in $\hat{\Lambda}$ (Λ) can be shadowed by real orbits in $\hat{\Lambda}$ (Λ).

Let Λ be a hyperbolic set for an endomorphism f. If δ is small enough, then by continuity $W^s_\delta(p)$ and $W^u_\delta(\hat{q})$ are almost flat, i.e. C^1-close to the tangent at p and q_0, respectively for all $p \in \Lambda$ and all $\hat{q} \in \hat{\Lambda}$. Therefore, by the continuity of E^u and E^s, $W^s_\delta(p)$ and $W^u_\delta(\hat{q})$ intersect in at most one point. In particular, if $p = q_0$, then $W^s_\delta(p) \cap W^u_\delta(\hat{q}) = \{q_0\}$, which implies

Proposition 2.1. *If f is hyperbolic on Λ, then $f|_\Lambda$ is expansive, i.e. there is a $\delta > 0$ such that if $(x_i)_{i \in \mathbf{Z}}$ and $(y_i)_{i \in \mathbf{Z}}$ are two orbits in Λ with $d(x_i, y_i) < \delta$ for all i, then $x_i = y_i$ for all i. The same result holds if only (x_i) is assumed to be in Λ.*

More generally we say that Λ has *local product structure* if δ can be chosen so that $W^s_\delta(p) \cap W^u_\delta(\hat{q}) \subset \Lambda$.

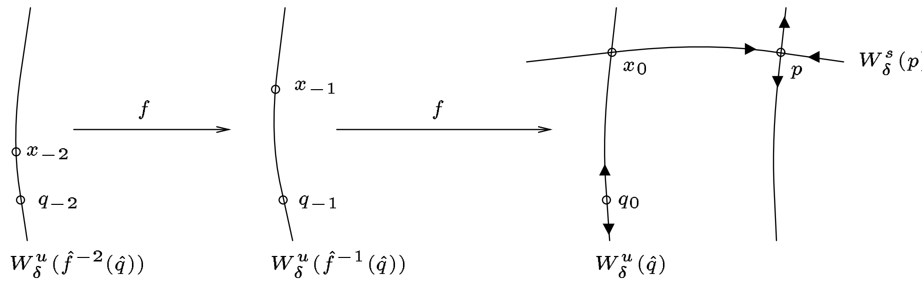

FIGURE 1. Local product structure for $\hat\Lambda$.

If Λ has local product structure, if $p \in \Lambda$, $\hat q \in \hat\Lambda$ and if p,q_0 are sufficiently close, then $W^s_\delta(p)$ and $W^u_\delta(\hat q)$ intersect in exactly one point $x \in \Lambda$ and x has a history $\hat x$ such that $x_j \in W^u_\delta(\hat f^j(\hat q))$ for all $j \le 0$. It is not a priori clear that $\hat x \in \hat\Lambda$, i.e. that $x_j \in \Lambda$ for all $j \le 0$. It will be useful in the sequel to assume this, so we state the following definition.

Definition 2.2. We say that $\hat\Lambda$ has local product structure if δ can be chosen so that if the intersection $W^s_\delta(p) \cap W^s_\delta(\hat q)$ is nonempty, then it consists of a unique point $x \in \Lambda$ and the unique history $\hat x$ of x with $x_j \in W^u_\delta(\hat f^j(\hat q))$ for all $j \le 0$ is completely contained in $\hat\Lambda$. See Figure 1.

If $\hat\Lambda$ has local product structure, then there exist $\delta' > 0$ and $\kappa > 0$ such that if $p \in \Lambda$, $\hat q \in \hat\Lambda$ and $d(p, q_0) < \delta'$, then there is a unique history $\hat x \in \hat\Lambda$ such that $x_0 \in W^s_\delta(p) \cap W^u_\delta(\hat q)$ and $x_j \in W^u_\delta(\hat f^j(\hat q))$ for all $j \le 0$. Furthermore,

$$d(x_0, p) \le \kappa d(p, q_0), \tag{2.1}$$

$$d(\hat x, \hat q) \le \kappa d(p, q_0) \tag{2.2}$$

We define $[p, \hat q]$ to be this history $\hat x$.

Definition 2.3. Let $\eta > 0$. An η-pseudoorbit in M is a sequence $(x_i)_{[t_1,t_2]}$, where $-\infty \le t_1 < t_2 \le \infty$, such that $d(f(x_i), x_{i+1}) < \delta$ for $t_1 \le i < t_2$. An η-pseudoorbit $(x_i)_{[t_1,t_2]}$ is ϵ-shadowed by an orbit $(y_i)_{[t_1,t_2]}$ if $d(y_i, x_i) < \epsilon$ for all $i \in [t_1, t_2]$. In a similar way we define (shadowing of) pseudoorbits in $\hat M$ or $\hat\Lambda$.

Theorem 2.4. ***(Shadowing Lemma for*** $\hat\Lambda$***).*** *If Λ is a hyperbolic set for f and $\hat\Lambda$ has local product structure, then for each $\epsilon > 0$ there exists an $\eta > 0$ such that every η-pseudoorbit in $\hat\Lambda$ can be ϵ-shadowed by an orbit in $\hat\Lambda$.*

Proof. Since $\hat f$ is uniformly continuous on $\hat\Lambda$ it suffices to prove the result for an iterate of f (we may have to shrink η). Let $(\hat x^{(i)})_{[t_1,t_2]}$ be a η-pseudoorbit in $\hat\Lambda$, where $\hat x^{(i)} = (x^{(i)}_j)_{j\le 0}$. Using the compactness of $\hat\Lambda$ and a diagonal process we may assume that $-\infty < t_1 < t_2 < \infty$. After relabeling, then, we may assume that $t_2 = 0$ and $-\infty < t_1 < 0$.

We will construct points $\hat y^{(i)} \in \hat\Lambda$ for $t_1 \le i \le 0$ such that

$$(\hat y^{(i)}, \hat f(\hat y^{(i)}), \dots, \hat f^i(\hat y^{(i)}))$$

ϵ-shadows the η-pseudoorbit

$$(\hat x^{(i)}, \hat x^{(i+1)}, \dots, \hat x^{(0)}).$$

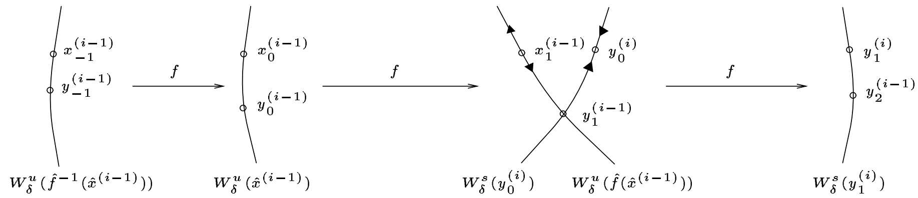

FIGURE 2. Definition of the shadowing orbit.

We define $\hat{y}^{(i)} = (y_j^{(i)})_{j\le 0}$ inductively by

$$\begin{aligned}\hat{y}^{(0)} &= \hat{x}^{(0)},\\ \hat{y}^{(i-1)} &= \hat{f}^{-1}([y_0^{(i)}, \hat{f}(\hat{x}^{(i-1)})]),\end{aligned}$$

see Figure 2. The idea behind this is that $\hat{f}^k(\hat{y}^{(i-1)})$ is close to $\hat{f}^{k-1}(\hat{y}^{(i)})$ for $k \ge 1$ and close to $\hat{f}^k(\hat{x}^{(i-1)})$ for $k \le 0$.

We have to check that the definition above makes sense. Let δ', δ and κ be the constants in (2.1) and (2.2). After replacing f by an iterate we may assume that there exists an $\alpha < 1/2$ such that f contracts stable directions by a factor α and expands unstable directions by a factor $\max(\kappa, 1)/\alpha$. Choose η,$\epsilon_0 > 0$ so small that

$$\begin{aligned}&\eta + \epsilon_0 < \delta'\\ &\alpha(\eta + \epsilon_0) < \epsilon_0.\end{aligned}$$

Assume inductively that $t_1 < i \le 0$, that $\hat{y}^{(i)}$ is well-defined, and that

$$d(\hat{x}^{(i)}, \hat{y}^{(i)}) = \sum_{j\le 0} 2^j d(x_j^{(i)}, y_j^{(i)}) < \epsilon_0.$$

Then

$$\begin{aligned}d(y_0^{(i)}, x_1^{(i-1)}) &< \eta + d(x_0^{(i)}, y_0^{(i)})\\ &< \eta + \epsilon_0\\ &< \delta',\end{aligned}$$

so $[y_0^{(i)}, \hat{f}(\hat{x}^{(i-1)})]$, and hence $\hat{y}^{(i-1)}$, is well-defined. Since $x_j^{(i-1)}$ and $y_j^{(i-1)}$ belong to the same local unstable manifold for all $j \le 0$ it follows that

$$\begin{aligned}d(\hat{x}^{(i-1)}, \hat{y}^{(i-1)}) &\le \frac{\alpha}{\kappa} d(\hat{f}(\hat{x}^{(i-1)}), \hat{f}(\hat{y}^{(i-1)}))\\ &\le \alpha d(x_1^{(i-1)}, \hat{y}_0^{(i)})\\ &\le \alpha(d(x_1^{(i-1)}, x_0^{(i)}) + d(x_0^{(i)}, y_0^{(i)}))\\ &\le \alpha(\eta + \epsilon_0),\end{aligned}$$

which by assumption is less than ϵ_0. Hence it follows inductively that $\hat{y}^{(i)}$ is well-defined for $t_1 \le i \le 0$.

We complete the proof by proving that $(\hat{y}^{(i)}, \dots, \hat{f}^i(\hat{y}^{(i)}))$ ϵ-shadows $(\hat{x}^{(i)}, \dots, \hat{x}^{(0)})$. First, if $t_1 < i \le 0$, then it follows from (2.1) that

$$\begin{aligned} d(y_0^{(i)}, y_1^{(i-1)}) &\le \kappa d(y_0^{(i)}, x_1^{(i-1)}) \\ &\le \kappa(d(y_0^{(i)}, x_0^{(i)}) + d(x_0^{(i)}, x_1^{(i-1)})) \\ &\le \kappa(\epsilon_0 + \eta). \end{aligned}$$

Now we let $i \le t \le 0$ and estimate

$$d(\hat{x}^{(t)}, \hat{f}^{t-i}(\hat{y}^{(i)})) \le d(\hat{x}^{(t)}, \hat{y}^{(t)}) + \sum_{j=0}^{t-i-1} d(\hat{f}^j(\hat{y}^{(t-j)}), \hat{f}^{j+1}(\hat{y}^{(t-j-1)})).$$

The first term is bounded by ϵ_0. The terms in the last sum can be written as

$$\begin{aligned} d(\hat{f}^j(\hat{y}^{(t-j)}), \hat{f}^{j+1}(\hat{y}^{(t-j-1)})) &= \sum_{s \le 0} 2^s d(y_{s+j}^{(t-j)}, y_{s+j+1}^{(t-j-1)}) \\ &= \sum_{-j \le s \le 0} + \sum_{s < -j}. \end{aligned}$$

Note that $y_{s+j}^{(t-j)}$ and $y_{s+j+1}^{(t-j-1)}$ are on the same local stable manifold if $s + j \ge 0$, so the first sum is bounded by

$$\sum_{-j \le s \le 0} 2^s \alpha^{s+j} d(y_0^{(t-j)}, y_1^{(t-j-1)}) \le \frac{2^{-j}}{1 - 2\alpha} \kappa(\epsilon_0 + \eta).$$

The second sum is bounded by

$$\begin{aligned} &\sum_{s < -j} 2^s (d(y_{s+j}^{(t-j)}, x_{s+j}^{(t-j)}) + d(x_{s+j}^{(t-j)}, x_{s+j+1}^{(t-j-1)}) + d(x_{s+j+1}^{(t-j-1)}, y_{s+j+1}^{(t-j-1)})) \\ &\le 2^{-j}(d(\hat{y}^{(t-j)}, \hat{x}^{(t-j)}) + d(\hat{x}^{(t-j)}, \hat{f}(\hat{x}^{(t-j-1)})) + d(\hat{x}^{(t-j-1)}, \hat{y}^{(t-j-1)})) \\ &\le 2^{-j}(\epsilon_0 + \eta + \epsilon_0). \end{aligned}$$

Thus $d(\hat{x}^{(t)}, \hat{f}^{t-i}(\hat{y}^{(i)})) < \epsilon$, where $\epsilon = 5\epsilon_0 + 2\eta + 2\kappa(\epsilon_0 + \eta)/(1 - 2\alpha)$ can be made arbitrarily small by choosing η and ϵ_0 appropriately. □

Once we can shadow orbits in $\hat{\Lambda}$ it is fairly easy to do shadowing in Λ.

Corollary 2.5. ***(Shadowing Lemma for*** Λ***).*** *Suppose that $\hat{\Lambda}$ has local product structure. Then for each $\epsilon > 0$ there exists an $\eta > 0$ such that every η-pseudoorbit in Λ can be ϵ-shadowed by an orbit in Λ.*

Proof. By Theorem 2.4 there exists an $\eta' > 0$ such that every η'-pseudoorbit in $\hat{\Lambda}$ can be $(\epsilon/2)$-shadowed by an orbit in $\hat{\Lambda}$. Fix $m > 0$ so that $2^{1-m}\mathrm{diam}(\hat{\Lambda}) < \eta'/2$. Let $A \ge 2$ be larger than the Lipschitz constant for f on Λ and let $\eta < A^{-m-1}\min(\eta', \epsilon)/2$.

Now suppose $(x_i)_{[t_1,t_2]}$ is an η-pseudoorbit in Λ. If $t_2 < \infty$, then we define $x_i = f^{i-t_2}(x_{t_2})$ for $i \ge t_2$ and if $t_1 > -\infty$, then we pick any history $\hat{q}$ of x_{t_1} in $\hat{\Lambda}$ and declare $x_i = q_{i-t_1}$ for $i \le t_1$. In this way we obtain an η-pseudoorbit $(x_i)_{i \in \mathbf{Z}}$ in Λ.

Define a sequence $(\hat{x}^{(i)})_{i \in \mathbf{Z}}$ of points in $\hat{\Lambda}$ by

$$\hat{x}^{(i)} = (\hat{z}^{(i)}, f(x_{i-m}), \dots, f^{m-1}(x_{i-m}), f^m(x_{i-m})),$$

where $\hat{z}^{(i)}$ is any history of x_{i-m} in $\hat{\Lambda}$. We claim that $(\hat{x}^{(i)})$ is an η'-pseudoorbit in $\hat{\Lambda}$. Indeed, for any $i \in \mathbf{Z}$ we have

$$\begin{aligned} d(\hat{f}(\hat{x}^{(i-1)}), \hat{x}^{(i)}) &\leq 2^{1-m} d(\hat{f}(\hat{z}^{(i-1)}), \hat{z}^{(i)}) + \sum_{1-m \leq j \leq 0} 2^j d(x_{j+1}^{(i-1)}, x_j^{(i)}) \\ &\leq 2^{1-m} \operatorname{diam}(\hat{\Lambda}) + \sum_{1 \leq k \leq m} 2^{k-m} d(f^{k+1}(x_{i-m-1}), f^k(x_{i-m})) \\ &< \eta'/2 + \sum_{1-m \leq j \leq 0} 2^j A^{m+j} d(f(x_{i-m-1}), x_{i-m}) \\ &\leq \eta'/2 + A^{m+1}\eta \\ &\leq \eta'. \end{aligned}$$

By Theorem 2.4 we can find an orbit $(\hat{y}^{(i)})_{i \in \mathbf{Z}}$ in $\hat{\Lambda}$ which $\epsilon/2$-shadows $(\hat{x}^{(i)})$. If we let $y_i = y_0^{(i)}$, then y_i is an orbit in Λ and we have

$$\begin{aligned} d(y_i, x_i) &\leq d(y_0^{(i)}, x_0^{(i)}) + d(x_0^{(i)}, x_i) \\ &\leq d(\hat{y}^{(i)}, \hat{x}^{(i)}) + d(f^m(x_{i-m}), x_i) \\ &\leq \epsilon/2 + \sum_{j=1}^{m-1} d(f^j(x_{i-j}), f^{j-1}(x_{i-j+1})) \\ &\leq \epsilon/2 + \sum_{j=1}^{m-1} A^{j-1}\eta \\ &< \epsilon. \end{aligned}$$

Hence (y_i) ϵ-shadows (x_i) and we are done. □

Using shadowing we can control the orbits of f staying near Λ in positive or negative time. A neighborhood U of Λ with the properties in the following corollary will be called a *fundamental neighborhood*.

Corollary 2.6. ***(Fundamental neighborhood)***. *Let Λ be a hyperbolic set for a map f such that $\hat{\Lambda}$ has local product structure. Then, for any sufficiently small neighborhood U of Λ in M we have*

(i) *If $x \in U$ and $f^j(x) \in U$ for all $j \geq 0$, then $x \in W^s_\delta(p)$ for some $p \in \Lambda$.*
(ii) *If $x \in U$ and x has a history $\hat{x}$ with $x_i \in U$ for all $i \leq 0$, then $x \in W^u_\delta(\hat{q})$ for some $\hat{q} \in \hat{\Lambda}$.*
(iii) *If $(x_i)_{i \in \mathbf{Z}}$ is a complete orbit in U then $x_i \in \Lambda$ for all i.*
(iv) *If g is C^1-close to f, then the set Λ_g in Proposition 1.4 is given by*

$$\Lambda_g = \{x_0; (x_i)_{i \in \mathbf{Z}} \text{ is a } g\text{-orbit completely contained in } U\}.$$

In particular, $\hat{\Lambda}_g$ has local product structure.

Proof. We will apply Corollary 2.5 with $\epsilon = \delta/2$. Assume that $\eta \leq \delta$ and define $U := \{x \in M; d(x, \Lambda) < \eta/2\}$, with η from Corollary 2.5.

(i) Pick points z_i in Λ for $i \geq 0$ with $d(x_i, z_i) < \eta/2$. Then $(z_i)_{i \geq 0}$ is an η-pseudoorbit in Λ so by Corollary 2.5 there is an orbit $(p_i)_{i \geq 0}$ in Λ which $\delta/2$-shadows (z_i). It follows that $d(p_i, x_i) < \eta/2 + \delta/2 \leq \delta$ for all $i \geq 0$ so $x \in W^s_\delta(p)$, where $p = p_0$.

(ii) As in (i) we construct an η-pseudoorbit $(z_i)_{i\leq 0}$ in Λ such that $d(x_i, z_i) < \eta/2$ for all $i \leq 0$. Corollary 2.5 provides us with a point $\hat{q} \in \hat{\Lambda}$ such that $d(q_i, x_i) < \eta/2 + \delta/2 \leq \delta$ for all i, so $x \in W^u_\delta(\hat{q})$.

(iii) From (i) and (ii) we find $p \in \Lambda$ and $\hat{q} \in \hat{\Lambda}$ such that $x \in W^s_\delta(p) \cap W^u_\delta(\hat{q})$. Since $\hat{\Lambda}$ has local product structure this implies that $x \in \Lambda$.

(iv) We have $\Lambda_g \subset U$ by Proposition 1.4 so we only have to prove the reverse inclusion. Let $(x_i)_{i\in\mathbf{Z}}$ be a g-orbit completely contained in U. By shrinking U we may assume that if g is close to f, then (x_i) may be $\delta/2$-shadowed by an f-orbit (y_i) in Λ and hence δ-shadowed by the g-orbit (z_i) in Λ_g coming from the conjugacy in Proposition 1.4. Thus

$$x_0 \in W^s_\delta(z_0) \cap W^u_\delta((z_i)_{i\leq 0}) = \{z_0\} \in \Lambda_g.$$

□

3. Axiom A endomorphisms

The results up to now have been of a semilocal nature, i.e. they concern the dynamics near a compact set. In order to study global dynamical properties we now restrict our attention to Axiom A endomorphisms. Our goal here is to prove the spectral decomposition theorem, which allows us to understand the dynamics of f near its nonwandering set. For the proof we will assume that f is an open mapping.

Let f be an C^∞ endomorphism of a C^∞ manifold M. A point $x \in M$ is *nonwandering* if it has no neighborhood V such that $f^n(V) \cap V = \emptyset$ for all $n \geq 1$. The *nonwandering set* Ω of f is the set of all nonwandering points; it is a closed set.

Definition 3.1. f is said to satisfy Axiom A if its nonwandering set satisfies

(i) Ω is compact.
(ii) Periodic points are dense in Ω.
(iii) f is hyperbolic on Ω.

Remark 3.2. If Ω satisfies (i) and (ii), then $f(\Omega) = \Omega$, so (iii) makes sense. Also, if f is Axiom A, then periodic points (under $\hat{f}$) are dense in $\hat{\Omega}$.

The following proposition shows that the results in section 2 apply to open Axiom A endomorphisms.

Proposition 3.3. *If f is an open Axiom A map, then $\hat{\Omega}$ has local product structure.*

Proof. Choose $\delta^* > \delta > 0$ so small that if $\hat{p}, \hat{q} \in \hat{\Omega}$ and $W^s_\delta(p_0)$ and $W^u_\delta(\hat{q})$ intersect in a unique point, then $W^s_{\delta^*}(q_0)$ and $W^u_{\delta^*}(\hat{p})$ intersect in a unique point. Now let $\hat{p}$ and $\hat{q}$ be any two points in $\hat{\Omega}$ such that $W^s_\delta(p_0)$ and $W^u_\delta(\hat{q})$ intersect in a unique point x. Then x has a history $\hat{x}$ such that $x_j \in W^u_\delta(\hat{f}^j(\hat{q}))$ for all $j \leq 0$. We have to prove that $\hat{x} \in \hat{\Omega}$.

We first consider the case when $\hat{p}$ and $\hat{q}$ are periodic, say of periods l and m, respectively. Let $g = f^{lm}$ and let U be any neighborhood of x. By Proposition 1.3, $g^j(U)$ contains a manifold C^1-close to $W^u_\delta(\hat{q})$ and $g^{-j}(U)$ contains a manifold C^1-close to $W^s_\delta(p_0)$ for all large j. Therefore $g^j(U)$ and $g^{-j}(U)$ intersect in a point near $x^* := W^s_{\delta^*}(q_0) \cap W^u_{\delta^*}(\hat{p})$ for all large j, so x is nonwandering, i.e. $x \in \Omega$.

For general $\hat{p}$, $\hat{q}$ let $\hat{x}$ be the history defined above, let $i \leq 0$ and let U be any neighborhood of x_i. Then $f^{-i}(U)$ is a neighborhood of x, because f is open. Since

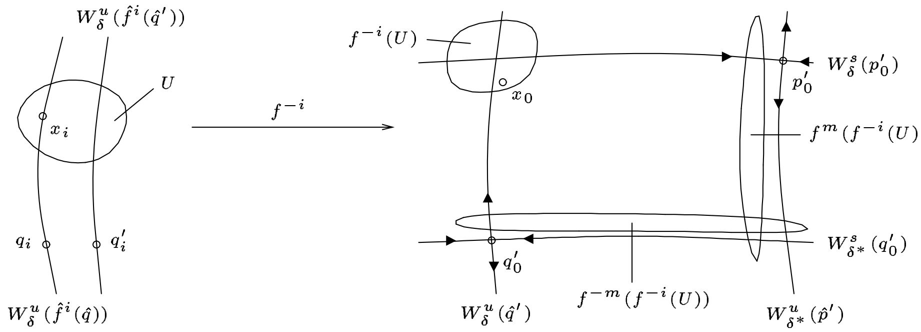

FIGURE 3. Local product structure for $\hat{\Omega}$.

periodic points are dense in $\hat{\Omega}$ we may find periodic points $\hat{p}'$, $\hat{q}'$ in $\hat{\Omega}$ close to $\hat{p}$, $\hat{q}$ such that $W^s_\delta(p_0')$ intersects $W^u_\delta(\hat{q}')$ in $f^{-i}(U)$ and $W^u_\delta(\hat{f}^i(\hat{q}'))$ intersects U. Then the above argument shows that $f^k(U)$ intersects $f^{-k}(U)$ for infinitely many $k \geq 0$. Hence x_i is nonwandering for all $i \leq 0$, so $\hat{x} \in \hat{\Omega}$. See Figure 3 for an illustration of the proof. □

Theorem 3.4. ***(Spectral decomposition of*** $\hat{\Omega}$***).*** *If f is an open Axiom A endomorphism, then $\hat{\Omega}$ can be written in a unique way as a disjoint union $\hat{\Omega} = \cup_{i=1}^{l} \hat{\Omega}_i$, where each $\hat{\Omega}_i$ is compact, satisfies $\hat{f}(\hat{\Omega}_i) = \hat{\Omega}_i$ and $\hat{f}$ is transitive on $\hat{\Omega}_i$. The sets $\hat{\Omega}_i$ are called the basic sets of $\hat{f}$. Morover, each $\hat{\Omega}_i$ can be further decomposed into a finite disjoint union $\hat{\Omega}_i = \cup_{1 \leq j \leq n_i} \hat{\Omega}_{i,j}$, where $\Omega_{i,j}$ is compact, $\hat{f}(\hat{\Omega}_{i,j}) = \hat{\Omega}_{i,j+1}$ ($\hat{\Omega}_{i,n_i+1} = \hat{\Omega}_{i,1}$) and $\hat{f}^{n_i}$ is mixing on each $\hat{\Omega}_{i,j}$.*

Proof. From Proposition 3.3 we know that $\hat{\Omega}$ has local product structure. Choose $\delta, \delta' > 0$ as in the discussion preceding (2.1) and (2.2). If $\hat{p} \in \hat{\Omega}$ is a periodic history, say of period l, then we let $\hat{W}^u_\delta(\hat{p})$ be the set of histories $\hat{x} \in \hat{\Omega}$ such that $d(x_i, p_i) < \delta$ for all $i \leq 0$. Similarly, we let $\hat{W}^u(\hat{p})$ be the set of histories $\hat{x} \in \hat{\Omega}$ such that $d(x_i, p_i) \to 0$ as $i \to -\infty$. Then $\hat{W}^u(\hat{p}) = \cup_{j \geq 0} \hat{f}^{jl}(\hat{W}^u_\delta(\hat{p}))$. Let $X_{\hat{p}}$ be the closure of $\hat{W}^u(\hat{p})$ in $\hat{\Omega}$.

Suppose that $\hat{p} \in \hat{\Omega}$ is periodic of period l. We first prove that if $\hat{y} \in \hat{\Omega}$ and $d(\hat{y}, X_{\hat{p}}) < \delta'$, then $\hat{y} \in X_{\hat{p}}$. We may assume that $\hat{y}$ is periodic, say of period m. Take any point $\hat{x} \in \hat{W}^u(\hat{p})$ with $d(\hat{y}, \hat{x}) < \delta'$ and let $\hat{z} = [y_0, \hat{x}]$. Then $\hat{z} \in \hat{W}^u(\hat{p})$, which implies that $\hat{f}^j(\hat{z}) \in \hat{W}^u(\hat{p})$ if $j \geq 0$ and l divides j. But $\hat{f}^j(\hat{z})$ is close to $\hat{y}$ if j is large and m divides j, so $\hat{y} \in X_{\hat{p}}$.

The next step is to prove that if $\hat{p}$ and $\hat{q}$ are two periodic points in $\hat{\Omega}$ of periods l and m, respectively, then either $X_{\hat{p}} = X_{\hat{q}}$ or $X_{\hat{p}} \cap X_{\hat{q}} = \emptyset$. First suppose $\hat{q} \in X_{\hat{p}}$. By the preceding paragraph $X_{\hat{p}}$ is open, so we may find $\gamma \in (0, \delta)$ such that $\hat{W}^u_\gamma(\hat{q}) \subset X_{\hat{p}}$. Then $f^{jlm}\hat{W}^u_\gamma(\hat{q}) \subset X_{\hat{p}}$ for all $j \geq 0$, so $X_{\hat{q}} \subset X_{\hat{p}}$. On the other hand, $X_{\hat{q}}$ is open and intersects $X_{\hat{p}}$, so we may find $\hat{x} \in X_{\hat{q}} \cap \hat{W}^u(\hat{p})$. But it is easy to see that $\hat{f}^m(X_{\hat{q}}) = X_{\hat{q}}$ so $\hat{f}^{-jlm}(\hat{x}) \in X_{\hat{q}}$ for all $j \geq 0$, which implies that $\hat{p} \in X_{\hat{q}}$. Therefore $\hat{q} \in X_{\hat{p}}$ implies $X_{\hat{p}} = X_{\hat{q}}$. Now suppose $\hat{p}$ and $\hat{q}$ are periodic and that $X_{\hat{p}}$ and $X_{\hat{q}}$ are not disjoint. Then they intersect in an open set, which contains a periodic history $\hat{r}$, so the previous argument shows that $X_{\hat{p}} = X_{\hat{r}} = X_{\hat{q}}$.

The different sets $X_{\hat{p}}$ form a disjoint open covering of the compact set $\hat{\Omega}$ so they are finite in number. It is clear that $\hat{f}(X_{\hat{p}}) = X_{\hat{f}(\hat{p})}$ so $\hat{f}$ induces a permutation of the different sets $X_{\hat{p}}$. Let $\hat{\Omega}_{i,j}$, $i = 1, \dots, l$, $j = 1, \dots, n_i$ be the distinct sets $X_{\hat{p}}$, labeled so that $\hat{f}(\hat{\Omega}_{i,j}) = \hat{\Omega}_{i,j+1}$, for $j = 1, \dots, n_i$ where $\hat{\Omega}_{i,n_i+1} = \hat{\Omega}_{i,1}$. Let $\hat{\Omega}_i = \cup_{i=1}^{n_i} \hat{\Omega}_{i,j}$ for $i = 1, \dots, l$. Then $\hat{f}(\hat{\Omega}_i) = \hat{\Omega}_i$ and $\hat{f}^{n_i}(\hat{\Omega}_{i,j}) = \hat{\Omega}_{i,j}$ for all i, j.

We prove that $\hat{f}^{n_i}$ is mixing on $\hat{\Omega}_{i,j}$ for all i, j. Let U and V be two open sets in $\hat{\Omega}_{i,j}$. We have to show that $\hat{f}^{tn_i}(U) \cap V \neq \emptyset$ for all sufficiently large t. Let $\hat{p}$ be a periodic point in U, say of period l. Then $X_{\hat{f}^{sn_i}(\hat{p})} = \hat{\Omega}_{i,j}$ so we may find points $\hat{x}^{(s)}$ in $\hat{W}^u(\hat{f}^{sn_i}(\hat{p})) \cap V$ for $s = 0, \dots, l-1$. For every sufficiently large t we may then find $0 \leq s \leq l-1$ such that $\hat{f}^{-tn_i}(\hat{x}^{(s)}) \in U$ so $\hat{f}^{tn_i}(U) \cap V \neq \emptyset$. Hence $\hat{f}^{n_i}$ is mixing on $\hat{\Omega}_{i,j}$ for all i, j and this implies that $\hat{f}$ is transitive on Ω_i for all i. □

As we see next, the spectral decomposition of $\hat{\Omega}$ induces one of Ω.

Corollary 3.5. ***(Spectral decomposition of*** Ω***).*** *If f is an open Axiom A endomorphism, then Ω can be written in a unique way as a disjoint union $\Omega = \cup_{i=1}^{l} \Omega_i$, where each Ω_i is compact, satisfies $f(\Omega_i) = \Omega_i$ and f is transitive on Ω_i. The sets Ω_i are called the basic sets of f. Morover, each Ω_i can be further decomposed into a finite disjoint union $\Omega_i = \cup_{1 \leq j \leq n_i} \Omega_{i,j}$, where $\Omega_{i,j}$ is compact, $f(\Omega_{i,j}) = \Omega_{i,j+1}$ ($\Omega_{i,n_i+1} = \Omega_{i,1}$) and f^{n_i} is mixing on each $\Omega_{i,j}$.*

Proof. We define $\Omega_{i,j} = \pi(\hat{\Omega}_{i,j})$, where $\pi : \hat{\Omega} \to \Omega$ is the projection. We claim that the $\Omega_{i,j}$'s are pairwise disjoint. If not, then there exist periodic points p and q of periods l and m, respectively, such that $X_{\hat{p}} \cap X_{\hat{q}} = \emptyset$ but $\pi(X_{\hat{p}}) \cap \pi(X_{\hat{q}}) \neq \emptyset$. Let x be a point in Ω with two histories $\hat{x}^{(1)} \in X_{\hat{p}}$, $\hat{x}^{(2)} \in X_{\hat{q}}$. If $j \geq 0$, then $\hat{f}^{jlm}(\hat{x}^{(1)}) \in X_{\hat{p}}$, $\hat{f}^{jlm}(\hat{x}^{(2)}) \in X_{\hat{q}}$ and $d(\hat{f}^{jlm}(\hat{x}^{(1)}), \hat{f}^{jlm}(\hat{x}^{(1)})) \to 0$ as $j \to \infty$. This is a contradiction, because $d(\hat{X}_{\hat{p}}, \hat{X}_{\hat{q}}) \geq \delta'$.

Thus the sets $\Omega_{i,j}$ are pairwise disjoint. They are compact because $\Omega_{i,j}$ is compact for all i, j and π is continuous. It remains to be seen that f^{n_i} is mixing on $\Omega_{i,j}$. This is easy, because if U and V are two open subsets of $\Omega_{i,j}$, then $\hat{U} := \pi^{-1}(U)$ and $\hat{V} := \pi^{-1}(V)$ are open subsets of $\hat{\Omega}_{i,j}$ and $\hat{f}^{tn_i}(\hat{U}) \cap \hat{V} \neq \emptyset$ for sufficiently large t. It follows that $f^{tn_i}(U) \cap V \neq \emptyset$ for sufficiently large t, which completes the proof. □

It follows easily from the definition of the nonwandering set that if M is compact and $(x_i)_{i \in \mathbf{Z}}$ is a complete orbit in M, then $x_i \to \Omega$ as $i \to \pm\infty$. In the Axiom A case we can say more. Using the fact that the basic sets are compact, disjoint and f-invariant, we easily prove the following result.

Lemma 3.6. *Assume that M is compact and that f is an open Axiom A endomorphism. If $x \in M$, then there exists a basic set Ω_j such that $f^i(x) \to \Omega_j$ as $i \to \infty$. Similarly, if $\hat{x}$ is a history in $\hat{M}$, then there exists a (possibly different) basic set Ω_j such that $x_i \to \Omega_j$ as $i \to -\infty$.*

Combining Lemma 3.6 and Corollary 2.6 we obtain.

Proposition 3.7. *Assume that f is an open Axiom A endomorphism and that M is compact.*

1. *(i) If $x \in M$, then there exists a unique basic set Ω_j such that $f^j(x) \to \Omega_j$ as $j \to \infty$. Moreover, there exists a (not necessarily unique) $p \in \Omega_i$ such that $d(f^j(x), f^j(p)) \to 0$ as $j \to \infty$.*

2. *(ii) If $\hat{x} \in \hat{M}$, then there exists a unique basic set Ω_i such that $x_j \to \Omega_i$ as $j \to -\infty$. Moreover, there exists a (not necessarily unique) $\hat{q} \in \widehat{\Omega_i}$ such that $d(x_j, q_j) \to 0$ as $j \to -\infty$.*

4. $\hat{\Omega}$-STABILITY AND THE NO-CYCLE CONDITION

Given a dynamical system we may ask whether it is stable under perturbations. The answer to this fairly vague question depends on what we mean by stability. In this section we define the notion of $\hat{\Omega}$-stability and give sufficient conditions for it in terms of hyperbolicity.

Let $f : M \to M$ be an Axiom A endomorphism. For this section we will assume that f is open and M is compact. Let $\Omega = \bigcup_{1 \le i \le l} \Omega_i$ be the spectral decomposition for f. Define a relation $<$ among the basic sets Ω_i by declaring that $\Omega_i < \Omega_j$ if $W^s(\Omega_i) \cap W^u(\Omega_j) \neq \emptyset$. Here

$$W^s(\Omega_j) = \{x \in M; f^i(x) \to \Omega_j \text{ as } i \to \infty\}$$
$$W^u(\Omega_j) = \{x \in M; \exists \hat{x}, \pi(\hat{x}) = x, x_i \to \Omega_j \text{ as } i \to -\infty\}$$

Let us first show that there are no trivial cycles for the relation $<$.

Lemma 4.1. *For any i we have $W^s(\Omega_i) \cap W^u(\Omega_i) = \Omega_i$.*

Proof. The proof is similar to that of Proposition 3.3. Let $(x_k)_{k \in \mathbf{Z}}$ be a complete orbit with $x_k \to \Omega_i$ as $|k| \to \infty$. We have to show that $x_0 \in \Omega_i$ and it suffices to show that x_0 is nonwandering. Choose δ' as in the discussion preceding (2.1). By Proposition 3.7 there exist $k > 0$, $y \in \Omega_i$ and $\hat{z} \in \widehat{\Omega_i}$ such that $x_k \in W^s_\delta(y)$ and $x_{-k} \in W^u_\delta(\hat{z})$. Let U be an open neighborhood of x_0. Then $f^k(U)$ is open and intersects $W^s_\delta(y)$. Now f is transitive on Ω_i so we may find $j \ge 0$ and $y' \in \Omega_1$ such that $W^s_\delta(y') \cap f^k(U) \neq \emptyset$ and $d(f^j(y'), z_0) < \delta'$. We may replace $f^j(y')$ by a periodic point u of period m. Hence $W^s_\delta(u) \cap f^{k+j}(U) \neq \emptyset$. Similarly, we may find a periodic history $\hat{v} \in \widehat{\Omega_i}$ of period n such that $W^u_\delta(\hat{v}) \cap f^{-k}(U) \neq \emptyset$ and $d(v_0, u) < \delta'$. By Proposition 1.3 $f^{k+j+ml}(U)$ contains a manifold C^1-close to $W^u_\delta(\hat{v})$ and $f^{-k-nl}(U)$ contains a manifold C^1-close to $W^s_\delta(u)$ for large l. Hence $f^{2k+j+(m+n)l}(U) \cap U \neq \emptyset$ for large l, so x_0 is nonwandering. □

We say that f satisfies the *no-cycle condition* or, simply, that f has no cycles if there is no nontrivial chain

$$\Omega_{i_1} < \Omega_{i_2} < \cdots < \Omega_{i_n} = \Omega_{i_1}.$$

Definition 4.2. An endomorphism $f : M \to M$ is $\hat{\Omega}$-stable if there exists a neighborhood U of f and for every $g \in U$ a homeomorphism $\phi : \hat{\Omega}_f \to \hat{\Omega}_g$ with $\hat{g} \circ \phi = \phi \circ \hat{f}$. Here Ω_f and Ω_g are the nonwandering sets of f and g respectively.

We now come to the main result in this section. For simplicity we restrict our attention to compact manifolds M.

Theorem 4.3. *If M is compact and $f : M \to M$ is an open Axiom A endomorphism with no cycles, then f is $\hat{\Omega}$-stable.*

Remark 4.4. The proof will show that the conjugacy ϕ can be chosen close to the identity. Note that the conjugacy takes place on the level of histories — the sets Ω_f and Ω_g need not be homeomorphic.

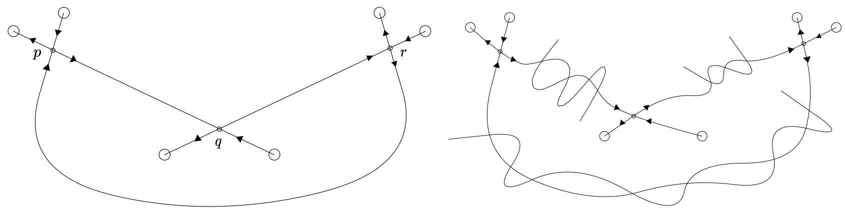

FIGURE 4. An Ω-explosion.

Let us make some observations before starting with the proof. By spectral decomposition, Ω is the disjoint union of the basic sets Ω_i, $1 \leq i \leq l$ and there are fundamental neighborhoods U_i of Ω_i in the sense of Corollary 2.6.

In particular, if g is C^1-close to f, then g has hyperbolic sets $\Omega_{i,g}$, $1 \leq i \leq l$ contained in U_i and there are homeomorphisms $\phi_i : \hat{\Omega}_{i,f} \to \hat{\Omega}_{i,g}$ conjugating $\hat{f}$ to $\hat{g}$. Thus $\Omega_{g,i}$ has local product structure, periodic points for g are dense in $\Omega_{i,g}$ and the restriction of g to $\Omega_{i,g}$ is transitive. In particular $\Omega_{i,g}$ is contained in the nonwandering set Ω_g of g. To prove that f is $\hat{\Omega}$-stable, it therefore suffices to prove that Ω_g is exactly the union of the sets $\Omega_{i,g}$. In general, there is no reason for this to be true. Picture 4 illustrates an Axiom A diffeomorphism f of, say, the two-dimensional sphere admitting an Ω-*explosion*, meaning that the nonwandering set for the original map f (a finite set) is much smaller than the nonwandering set for the perturbed map g (an infinite set). The nonwandering set of f consists of six sources and sinks, marked with big circles, and three saddle points p, q and r. These are the basic sets of f. The nonwandering set of g contain perturbations of these nine points, but also all the transverse intersection between unstable and stable manifolds in the second picture.

The main tool in proving Theorem 4.3 is the existence of a *filtration*, which we now describe. If f is Axiom A and has no cycles, then we may label the basic sets of f in such a way that $\Omega_i > \Omega_j$ implies $i > j$.

Proposition 4.5. *Let $f : M \to M$ be an open Axiom A map with no cycles, where M is compact. Then there is an integer $m \geq 1$, fundamental neighborhoods U_j of Ω_j and compact sets $\emptyset = M_0 \subset M_1 \subset \cdots \subset M_l = M$, such that $U_1 = int(M_1)$, $f^m(M_j) \subset int(M_j)$ for $1 \leq j \leq l$, and $f^m(M_j - U_j) \subset int(M_{j-1})$ for $2 \leq j \leq l$.*

We postpone the proof of Proposition 4.5 and show instead how to deduce $\hat{\Omega}$-stability.

Proof of Theorem 4.3. Let g be C^1-close to f. As mentioned above it suffices to show that the nonwandering set Ω_g of g is the union of the sets $\Omega_{j,g}$, $1 \leq j \leq l$, so let $(x_i)_{i \in \mathbf{Z}}$ be a g-orbit completely contained in Ω_g. If g is close enough to f, then Proposition 4.5 holds with f replaced by g. Hence there is a j, $1 \leq j \leq l$, such that $x_i \in U_j$ for all i. But then $x_i \in \Omega_{j,g}$ for all i by Corollary 2.6. $\square$

Thus it remains to construct the filtration in Proposition 4.5. Figure 5 illustrates the first two steps in the construction of the filtration. Here Ω_1 is an attracting set, by the labeling of Ω_i, and $M_1 = \overline{U}_1$ is a neighborhood of Ω_1. Next, $W^u(\Omega_2)$ is in the stable set of Ω_1 and M_2 is the union of M_1 and a neighborhood of $W^u(\Omega_2) - M_1$. It will take some care to define this neighborhood so that the properties in Proposition 4.5 hold.

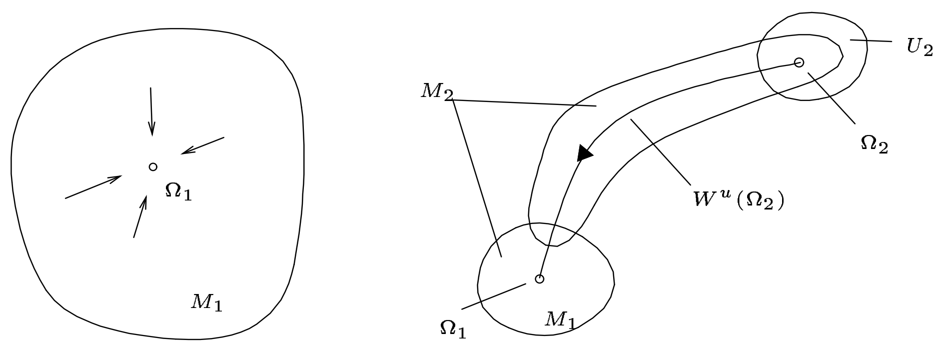

FIGURE 5. Construction of the filtration.

We start the proof of Proposition 4.5 with a preliminary result.

Lemma 4.6. *The set* $\Lambda_k := \bigcup_{i\leq k} W^u(\Omega_i)$ *is compact and* $\bigcup_{i\leq k} W^s(\Omega_i)$ *is an open neighborhood of* Λ_k *for* $1 \leq k \leq l$.

Proof. We first show that Λ_k is closed, hence compact. Let $x \in \overline{W^u(\Omega_{i_0})}$ for some $i_0 \leq k$. We must show that $x \in W^u(\Omega_i)$ for some $i \leq k$. Pick histories $\hat{y}^{(\mu)}$, $\mu \geq 1$, such that $y_0^{(\mu)} \to x$ as $\mu \to \infty$ and $y_s^{(\mu)} \to \Omega_{i_0}$ as $s \to -\infty$ for all μ. By passing to a subsequence we may assume that $\hat{y}^{(\mu)}$ converges to a history $\hat{z}$.

Let I be the set of i such that $\hat{y}^{(\mu)}$ accumulates on Ω_i as $\mu \to \infty$. More precisely, $i \in I$ if there exist $\mu_k \to \infty$ and $s_k \leq 0$ such that $y_{s_k}^{(\mu_k)} \to \Omega_i$ as $k \to \infty$. The proof now goes through a number of steps.

Lemma 4.7. *There is an* $i \in I$ *such that* $x \in W^u(\Omega_i)$.

Proof of Lemma 4.7. Recall that $\hat{y}^{(\mu)} \to \hat{z}$ as $\mu \to \infty$. We have $z_0 = x$ and there is an i such that $z_s \to \Omega_i$ as $s \to -\infty$. We claim that $i \in I$. To see this, pick s_k with $d(z_{s_k}, \Omega_i) < \frac{1}{k}$ for $k > 0$. If μ_k is large enough, then $d(y_{s_k}^{(\mu_k)}, \Omega_i) < \frac{1}{k}$, which proves that $i \in I$. □

Lemma 4.8. *If* $i \in I$, $i \neq i_0$, *then there is a* $j \in I$, $j \neq i$ *such that* $\Omega_j > \Omega_i$.

Proof of Lemma 4.8. Pick $\delta_0 > 0$ such that

$$\delta_0 < \frac{1}{2} \min_{1\leq i_1 < i_2 \leq l} d(\Omega_{i_1}, \Omega_{i_2})$$

By assumption there exist $\mu_k \to \infty$ and $s_k \leq 0$ such that $y_{s_k}^{(\mu_k)} \to \Omega_i$. Choose $t_k < s_k$ minimal such that $d(y_{t_k}^{(\mu_k)}, \Omega_i) < \delta_0$. This is possible because $i \neq i_0$. Define $\hat{w}^{(k)}$ by $w_s^{(k)} = y_{s+t_k}^{(\mu_k)}$. By passing to a subsequence we may assume that $\hat{w}^{(k)} \to \hat{w}$ as $k \to \infty$. We claim that $w_s \to \Omega_i$ as $s \to \infty$. To see this we first consider the case when $s_k - t_k \to \infty$. Then $d(w_s^{(k)}, \Omega_i) < \delta_0$ for $0 \leq s \leq s_k - t_k$, so we must have $w_s \to \Omega_i$ as $s \to \infty$. The second case is when $s_k - t_k$ is bounded as $k \to \infty$. By passing to a subsequence we may assume that $s_k - t_k = r \geq 0$ for all k. But then $d(w_r^{(k)}, \Omega_i) \to 0$ as $k \to \infty$ so $w_r \in \Omega_i$. Thus $w_s \to \Omega_i$ is this case too.

Similarly, we have $w_s \to \Omega_j$ as $s \to -\infty$ for some j. Hence $\Omega_j > \Omega_i$ and we have $j \neq i$ by Lemma 4.1. It remains to be seen that $j \in I$. But for each $m \geq 1$ we may choose $u_m < 0$ such that $d(z_{u_m}, \Omega_j) < \frac{1}{m}$. Then we find $\mu_m \to \infty$ such that $d(y_{u_m+t_m}^{(\mu_m)}, \Omega_j) < \frac{1}{m}$. This shows that $j \in I$. □

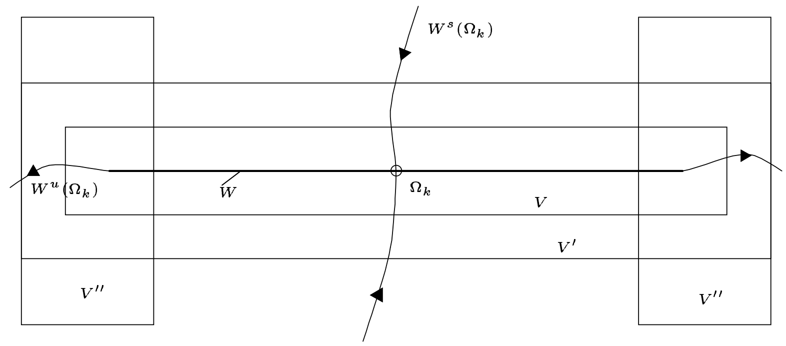

FIGURE 6. Dynamics near Ω_k

We now continue the proof of Lemma 4.6. By Lemma 4.7, Lemma 4.8 and the no-cycle property there exists a chain

$$\Omega_{i_0} > \Omega_{i_1} > \cdots > \Omega_{i_l},$$

such that $x \in W^u(\Omega_{i_l})$. Again by the no-cycle property we must have $i_l \leq i_0 \leq k$ so $x \in \Lambda_k$. This proves that Λ_k is compact. Similarly we may prove that $M - \bigcup_{i \leq k} W^s(\Omega_i) = \bigcup_{i > k} W^u(\Omega_i)$ is compact so $\bigcup_{i \leq k} W^s(\Omega_i)$ is open and it contains Ω_k by the labeling of the basic sets. □

Proof of Proposition 4.5. We construct the sets M_k and choose the fundamental neighborhoods U_k inductively. Compare with Figure 5. First note that Ω_1 is an attracting set, because

$$W^u(\Omega_1) = \bigcup_{1 \leq j \leq l} W^u(\Omega_1) \cap W^s(\Omega_j) = \Omega_1$$

by Lemma 4.1 and the labeling of the Ω_i. Hence, if U_1 is small enough and $M_1 = \overline{U_1}$, then we can find $m \geq 1$ such that $f^m(M_1) \subset U_1$. Note that $\Lambda_1 = \Omega_1 \subset \operatorname{int}(M_1)$. Now suppose that $2 \leq k \leq l$ and that we have an integer $m' \geq 1$ and compact sets $\emptyset = M_0 \subset M_1 \subset \cdots \subset M_{k-1}$ such that $\Lambda_j \subset \operatorname{int}(M_j)$, $f^{m'}(M_j) \subset \operatorname{int}(M_j)$ for $1 \leq j \leq k-1$, and $f^{m'}(M_j - U_j) \subset \operatorname{int}(M_{j-1})$ for $2 \leq j \leq k-1$.

If $x \in W^u(\Omega_k)$, then $x \in W^s(\Omega_i)$ for some $i < k$ by Lemma 4.1 and the labeling of the Ω_i. Given $\epsilon, \delta, \delta', \delta'' > 0$, define the sets W,V,V',V'' as follows. W is the open ϵ-neighborhood of Ω_k in $W^u(\Omega_k)$, V (V') is the closed δ-neighborhood (δ'-neighborhood) of W in M, and V'' is the closed δ''-neighborhood of $W^u(\Omega_k) - (M_{k-1} \cup W)$ in M. See Figure 6.

By the hyperbolicity of f on Ω_k we may choose $m_1 \geq 1$ and $\epsilon > 0$ such that for every $m \geq m_1$ and every $\delta'' > 0$ there exist $\delta' > \delta > 0$ such that $f^m(V') \subset V \cup \operatorname{int}(V'')$.

Now $W^u(\Omega_k) - (W \cup \operatorname{int}(M_{k-1}))$ is compact, so by the induction hypothesis there is an $m_2 \geq m_1$ such that $f^{m_2}(W^u(\Omega_k) - W) \subset \operatorname{int}(M_{k-1})$. Choose δ'' so small that $f^{m_2}(V'') \subset \operatorname{int}(M_{k-1})$ and find $\delta' > \delta > 0$ such that $f^{m_2}(V') \subset V \cup \operatorname{int}(V'')$.

Hence, if we let $m = m'm_2$, $U_k = \operatorname{int}(V')$ and $M_k = M_{k-1} \cup V'' \cup V'$, then M_k is an open neighborhood of Λ_k, $f^m(M_k) \subset \operatorname{int}(M_k)$, and $f^m(M_k - U_k) \subset \operatorname{int}(M_{k-1})$. This completes the induction. □

REFERENCES

[R] Ruelle, D. *Elements of differentiable dynamics and bifurcation theory.* Academic Press, 1989.
[PS] Pugh, C, Shub, M. Ergodic attractors. *Trans. Amer. Math. Soc.* **312**(1989), 1–54.
[S] Shub, M. *Global stability of dynamical systems.* Springer-Verlag, 1987.

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