Some remarks on the Jacobian Conjecture

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Generalities

Def. A polynomial map $F : \mathbf{C}^n \to \mathbf{C}^n$ is a Keller map if it is a local biholomorphism, i.e. $JF :=$ det DF is a nonzero constant.

Jacobian Conjecture (JC$_n$) [Keller, 1939]. Every Keller map is invertible.

Prop. For $F : \mathbf{C}^n \to \mathbf{C}^n$ polynomial, TFAE:
(i) F is bijective (onto $\mathbf{C}^n$);
(ii) F is injective;
(iii) F is bijective and F^{-1} is a polynomial map.

Def. A polynomial map $F : \mathbf{C}^n \to \mathbf{C}^n$ satisfying (i)-(iii) is called a polynomial automorphism.

Rem. JC$_1$ is trivially true, JC$_n$ is open for $n > 1$ and JC$_n \implies$ JC$_m$ for $n > m$.

Rem. There are at least 5 published incorrect proofs of JC$_2$!

Rem. JC$_2$ fails in the transcendental category: $F(X,Y) = (e^X, Ye^{-X})$.

Reduction

Thm [Bass, Connell, Wright 1982; Yagzhev 1980; Drużkowski 1983]. In order to prove JC_n for all $n \geq 2$ it suffices to consider degree 3 Keller maps $F : \mathbf{C}^n \to \mathbf{C}^n$ of the form

$$F_i(X) = X_i + L_i(X)^3, \quad 1 \leq i \leq n,$$

where L_i are homogeneous linear forms. Must again do this *for all* $n \geq 2$.

Thm [Wang 1980]. Every Keller map F of degree ≤ 2 is a polynomial automorphism.

Proof. Assume F not injective, say

$$F(a) = F(0) = 0 \neq a.$$

Write $F = F_1 + F_2$, F_i homog. of degree i.

$$\begin{aligned} 0 &= F_1(a) + F_2(a) \\ &= \frac{d}{dt}(tF_1(a) + t^2F_2(a))\Big|_{t=1/2} \\ &= \frac{d}{dt}F(ta)\Big|_{t=1/2} \\ &= DF(a/2) \cdot a \neq 0. \end{aligned}$$

□

Rem. $2 \neq 3$.

Partial results

Most of the (few) known partial results are in dimension $n = 2$.

Thm [Magnus, Appelgate, Onishi, Nagata]. If $F = (P, Q)$ is a Keller map and $\gcd(\deg P, \deg Q)$ is at most 8, or a prime number, then F is a polynomial automorphism.

Thm [Moh 1983]. If $F = (P, Q)$ is a Keller map and $\deg P, \deg Q \leq 100$, then F is a polynomial automorphism.

Rem. Very few people have read this paper and there is no complete independent verification!

A sort of generalized Jacobian Conjecture in $\mathbf{R}^2$ was disproved by Pinchuk.

Thm [Pinchuk 1994]. There exists a non-invertible polynomial map $F : \mathbf{R}^2 \to \mathbf{R}^2$ with $JF(X) \neq 0$ for all $X \in \mathbf{R}^2$.

The example is explicit (but a bit complicated).

Jung's Theorem

Thm [Jung 1942]. Every polynomial automorphism of $\mathbf{C}^2$ is a finite composition of affine maps

$$(X,Y) \to (aX + bY + c, dX + eY + f)$$

and shears

$$(X,Y) \to (X, Y + P(X)), \quad \deg P \geq 2.$$

A beautiful result with many proofs.

Rem. Polynomial automorphisms of $\mathbf{C}^n$, $n > 2$ are not very well understood.

Rem. Birational polynomial mappings of $\mathbf{C}^2$, are not very well understood either.

Rem. To study JC_2, can use coordinate changes on $\mathbf{C}^2$ by polynomial automorphisms.

Rem. Most (failed) attempts to prove JC_2 are based on starting with a noninvertible Keller map F of minimal degree. Jung's Theorem gives some restrictions on F.

The Line Embedding Theorem

Closely related to Jung's Theorem is the Line Embedding Theorem, proved independently by Abhyankar-Moh (1973) and Suzuki (1974):

Thm. If $\phi : \mathbf{C} \to \mathbf{C}^2$ is a polynomial embedding, then there exists $F \in \mathrm{Aut}(\mathbf{C}^2)$ such that

$$F \circ \phi(T) = (0, T).$$

Thus the image $\phi(\mathbf{C})$ can be "straightened".

Also a beautiful result with many proofs, often paired with proofs of Jung's Theorem.

Rem. The image $\phi(\mathbf{C})$ can be characterized as a smooth rational curve with one place at infinity.

Rem. No known normal form for $\phi(\mathbf{C})$ for general polynomial maps $\phi : \mathbf{C} \to \mathbf{C}^2$.

Rem. The Line Embedding Theorem fails in the transcendental category [Buzzard, Fornæss 1996].

Injectivity

Recall that an injective Keller map must be a polynomial automorphism.

Thm [Gwoździewicz 1993]. If $F : \mathbf{C}^2 \to \mathbf{C}^2$ is a Keller map and F is injective on one line $L \subset \mathbf{C}^2$, then F is a polynomial automorphism.

Proof. By the Line Embedding Theorem, may assume that $L = F(L) = \{Y = 0\}$, and that in fact $F(X, 0) = (X, 0)$. Hence

$$F(X,Y) = (X, YQ(X,Y)).$$

Then $JF = (YQ)_Y \equiv \text{const} \implies Q \equiv \text{const}$. $\square$

Thus a potential counterexample to JC_2 must map every line onto a *singular* rational curve with one place at infinity. Unfortunately, very little is known about such curves.

Properness

Def. A continuous map $F : \mathbf{R}^m \to \mathbf{R}^m$ is proper if "$F(\infty) = \infty$", i.e.

$$F^{-1}(\text{compact}) = \text{compact}.$$

Thm [Hadamard 1906]. If $F : \mathbf{R}^m \to \mathbf{R}^m$ is C^1, $JF \neq 0$ on $\mathbf{R}^m$, and F is proper, then F is a diffeomorphism.

Proof. The Inverse Function Theorem implies that F is a local diffeomorphism. Properness implies that F is a covering map, hence a homeomorphism (and diffeomorphism) since $\mathbf{R}^m$ is simply connected. $\square$

Thus a potential counterexample to JC$_n$ must be non-proper. This is a source of many failed attempts to prove JC_2!!

Preimage of a line

Thm [Drużkowski 91, Chądzyński-Krasiński 92, Campbell 95]. If $F : \mathbf{C}^2 \to \mathbf{C}^2$ is a Keller map but not an automorphism, then for any line $L' \subset \mathbf{C}^2$ and any component $C \subset \mathbf{C}^2$ of $F^{-1}(L')$, we have:

(i) C is a smooth curve, not simply connected;

(ii) $F|_C$ maps C onto L' as a nonproper map.

Proof. Since F is a local biholomorphism, C is smooth and $F(C) = L'$. It suffices to show that $C \simeq \mathbf{C}$, for then $F|_C : C \to L' \simeq \mathbf{C}$ is a local biholomorphism, hence injective (and surjective) so that F is a polynomial automorphism.

(i) Compactify C to a smooth compact curve $\overline{C}$ by adding $q \geq 1$ points. If C is simply connected, $q = 1$ and $\overline{C} = \mathbf{P}^1$, hence $C \simeq \mathbf{C}$.

(ii) If $F|_C$ is proper, it gives a covering of $L' \simeq \mathbf{C}$ by C, hence $C \simeq \mathbf{C}$.

$\square$

Structure of a potential counterexample

Orevkov's Example

S. Orevkov produced a counterexample to the "Jacobian conjecture at infinity".

Thm [Orevkov 1990]. There exists a (noncompact) smooth complex surface U containing a smooth complex curve $L \simeq \mathbf{P}^1$ with $L^2 = +1$ and a meromorphic mapping $F : U \dashrightarrow \mathbf{C}^2$ such that $F|_{U \setminus L} : U \setminus L \to \mathbf{C}^2$ is a noninjective holomorphic immersion.

The proof is constructive.

The pair (U, L) "looks like" a neighborhood of the line at infinity in $L_\infty = \mathbf{P}^2 \setminus \mathbf{C}^2$. In fact, if (U, L) was embedded in $(\mathbf{P}^2, L_\infty)$, then by Hartogs F would extend to a noninjective Keller map.

However, the examples given by Orevkov are not counterexamples to JC_2!

Analysis at infinity I

View a general polynomial map $F : \mathbf{C}^2 \to \mathbf{C}^2$ as a rational map

$$F : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2.$$

Resolve singularities of F by finitely many blowups at infinity on the "source" $\mathbf{P}^2$,

$$\pi : S \to \mathbf{P}^2 :$$

get holomorphic map

$$F : S \to \mathbf{P}^2.$$

If $F : \mathbf{C}^2 \to \mathbf{C}^2$ is not proper, then some irreducible "dicritical" components of $S \setminus \mathbf{C}^2$ will be mapped onto curves in $\mathbf{C}^2$.

Analysis at infinity II

Get more information at infinity by blowing up also at the "target" $\mathbf{P}^2$.

Start with $\pi' : S' \to \mathbf{P}^2$, a finite composition of point blowups above infinity.

Then $F : \mathbf{C}^2 \to \mathbf{C}^2$ lifts to a rational map

$$F : \mathbf{P}^2 \dashrightarrow S'.$$

Resolve singularities of F, i.e. find finite composition of blowups $\pi : S \to \mathbf{P}^2$ above infinity such that F lifts to a holomorphic map

$$F : S \to S'.$$

In general, no obvious choice for π' and π but may want to blow up "enough" times.

For instance, can make sure that F maps the line at infinity in S onto a curve in S'.

Can do this also for Orevkov maps $F : U \dashrightarrow \mathbf{P}^2$.

A valuative approach I

Another idea: blow up all points above infinity. Can be formalized in terms of valuations.

Def. Let $\mathcal{V}_0$ be the set of all valuations

$$\nu : \mathbf{C}[X,Y] \to (-\infty, +\infty]$$

with $\min\{\nu(X), \nu(Y)\} = -1$.

Def. A valuation ν is divisorial if $\nu(P) \sim \mathrm{ord}_E(P)$ for some irreducible component $E \subset S \setminus \mathbf{C}^2$ and some compactification $S \supset \mathbf{C}^2$.

Thm [Favre-J]. $\mathcal{V}_0$ is an $\mathbf{R}$-tree. The divisorial valuations are the "rational" points on $\mathcal{V}_0$.

Thm [Favre-J]. If $F : \mathbf{C}^2 \to \mathbf{C}^2$ is proper, then F induces a surjective tree map $F_\bullet : \mathcal{V}_0 \to \mathcal{V}_0$ preserving the divisorial valuations.
If $F : \mathbf{C}^2 \to \mathbf{C}^2$ is nonproper, $F_\bullet$ is only defined on a subtree $\mathcal{D}_F \subsetneq \mathcal{V}_0$ and maps $\mathcal{D}_F$ onto $\mathcal{V}_0$.

A valuative approach II

Assume $F : \mathbf{C}^2 \to \mathbf{C}^2$ is a noninjective Keller map.

The Jacobian condition implies strong restrictions on the tree map $F_\bullet : \mathcal{D}_F \to \mathcal{V}_0$.

For instance, there are 6 possible "types" of tree maps associated to $F = (P, Q)$ with $\deg P, \deg Q \leq 100$. (Need computer search to find them.)

For these types, $(\deg P, \deg Q)$ are among

$$(64, 48), (75, 50), (84, 56), (99, 66)$$

Keep Orevkov's examples in mind: a "counterexample at infinity" also gives rise to a tree map with the same restrictions. Must use that F is a polynomial mapping!

Moh's approach is essentially equivalent to the valuative one. So is the one by Heitmann (1990) who obtained some of the restrictions on the tree map.

Moh excluded the six types mentioned above by clever/complicated/... computations.

Possible approach to JC_2.

Pictures

Shears.

The Line Embedding Theorem

Campbell's proof.

$F(L)$ and $F^{-1}(L')$.

Orevkov's construction

Liftings: $F : S \to \mathbf{P}^2$ or $F : S \to S'$.

$F_\bullet : \mathcal{D}_F \to \mathcal{V}_0$

Orevkov's or Moh's example.