PLURICOMPLEX DYNAMICS

1. Motivation and examples.
2. Polynomial dynamics on $\mathbf{C}$.
3. Hénon mappings.
4. Regular polynomial mappings of $\mathbf{C}^2$.

DYNAMICAL SYSTEMS

(way too) general definition

X phase space.
G group acting on X.

In these lectures:
$X =$ complex manifold ($\mathbf{P}^k$ or $\mathbf{C}^k$)
$G = \{f^n\}_{n \geq 0}$ or $G = \{f^n\}_{n \in \mathbf{Z}}$.

Here $f : X \to X$ is a holomorphic mapping (or biholomorphism). Study behavior of iterates $f^n = f \circ \cdots \circ f$ as $n \to \pm\infty$.

Interesting things to look at:
- Orbits $\{f^n p\}_{n \geq 0}$ or $\{f^n p\}_{n \in \mathbf{Z}}$.
- Invariant "objects" (measures, currents).

Of special interest are "recurrent" points, e.g. *periodic points*, $f^n p = p$.

Different aspects of dynamics.

1. **Local dynamics**.
Example: linearization at sinks.
$f : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ germ.
$f(z) = \lambda z + O(z^2)$, $0 < |\lambda| < 1$. Then $\exists$ $\phi(z) = z + o(1)$ such that $\phi(f(z)) = \lambda\phi(z)$.

2. **Global dynamics**.
Example: finitely many sinks.
If $f : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is a rational map of degree d, then f has at most $2d-2$ attracting periodic cycles.

3. **Semilocal dynamics**.
Example: horseshoes.

SYMBOLIC DYNAMICS

$$\Sigma^+ := \{0,1\}^{\mathbf{Z}_+} = \{(\epsilon_n)_{n\geq 0}\}$$
$$\Sigma \ := \{0,1\}^{\mathbf{Z}} \ = \{(\epsilon_n)_{n\in \mathbf{Z}}\}$$

are compact metric spaces (Cantor sets):

$$d((\epsilon_n),(\epsilon'_n)) = \sum 2^{-|n|} d(\epsilon_n,\epsilon'_n)$$

We have natural maps

$$\sigma^+ : \Sigma^+ \to \Sigma^+$$
$$\sigma \ : \Sigma \ \to \Sigma$$

defined by left shift:

$$\sigma^+((\epsilon_n)) = (\epsilon_{n+1})$$
$$\sigma((\epsilon_n)) = (\epsilon_{n+1})$$

(Σ^+, σ^+): (full)*1-sided shift* on two symbols.
(Σ, σ) : (full)*2-sided shift* " .

Note: σ^+ and σ are continuous. σ is invertible (homeomorphism) but σ^+ is 2–1.

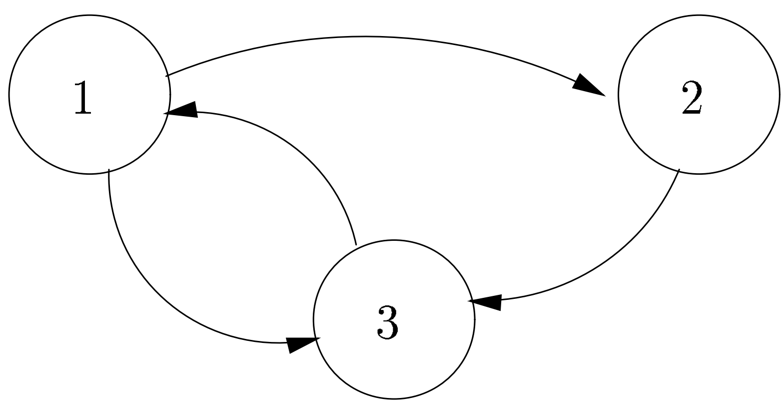
Σ and Σ^+ carry natural invariant measures (Bernoulli measures).

Idea: Shift maps are models for behavior of differentiable or complex dynamics.

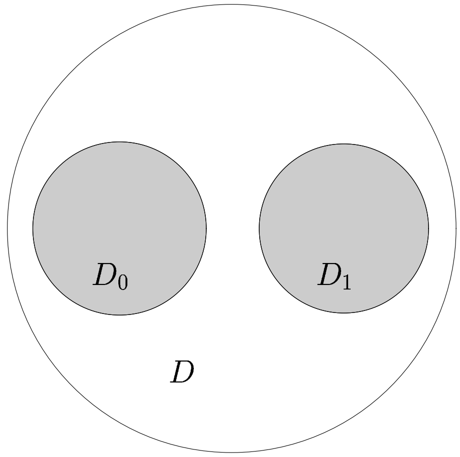
Generalizations:

- (Full) shifts on N symbols.
- Subshifts of finite type:
  $M = N \times N$ binary matrix.

$$\begin{aligned}\Sigma_M &= \{(\epsilon_n) : M_{\epsilon_n, \epsilon_{n+1}} = 1\} \\ &= \{\text{set of paths in finite directed graph}\}\end{aligned}$$



TOPOLOGICAL MODEL FOR 1-SIDED SHIFT



Define $f : \mathbf{C} \to \mathbf{C}$ continuous such that

1. f maps D_i affinely onto D, $i = 0, 1$.
2. $f(D - (D_1 \cup D_2)) \subset \mathbf{C} - D$.
3. $f(\mathbf{C} - D) \subset \mathbf{C} - D$ and $f^n \to \infty$ on $\mathbf{C} - D$.

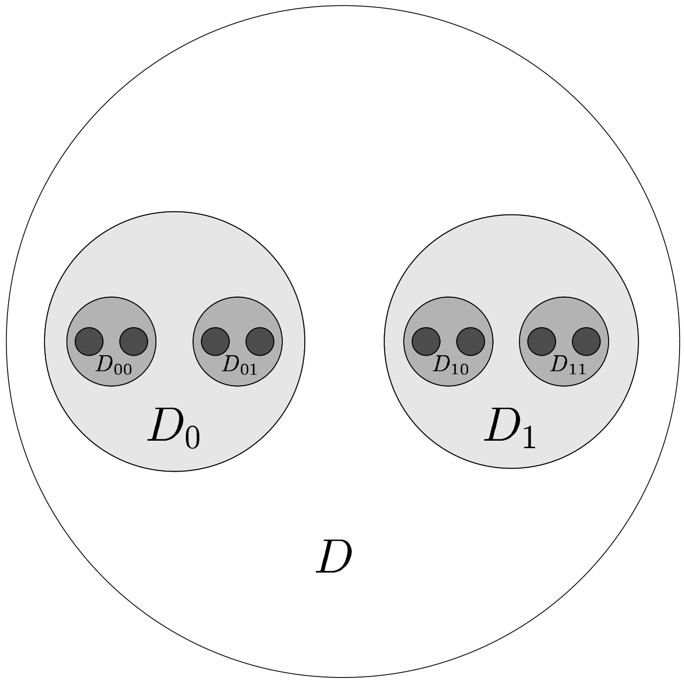
$K := \{z \in \mathbf{C} : f^n z \text{ bounded}\}$

Claim. $f|_K$ is conjugate to $\sigma^+|_{\Sigma^+}$.

Proof. Let $x \in K$. Define $\phi(x) = (\epsilon_n)_{n \geq 0}$ by

$$\epsilon_n = i \text{ if } f^n x \in D_i, i = 0, 1.$$

Then ϕ is a homeomorphism of K onto Σ^+ and $\sigma^+ \circ \phi = \phi \circ f$. □

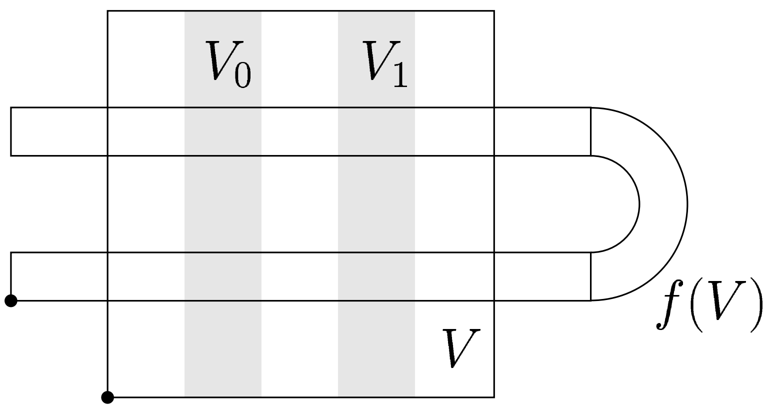


Consequences.

- Periodic points are dense in K.
- f has periodic points of all orders.
- "Most" points in K have dense orbits in K.

TOPOLOGICAL MODEL FOR 2-SIDED SHIFT
(Smale's horseshoe)

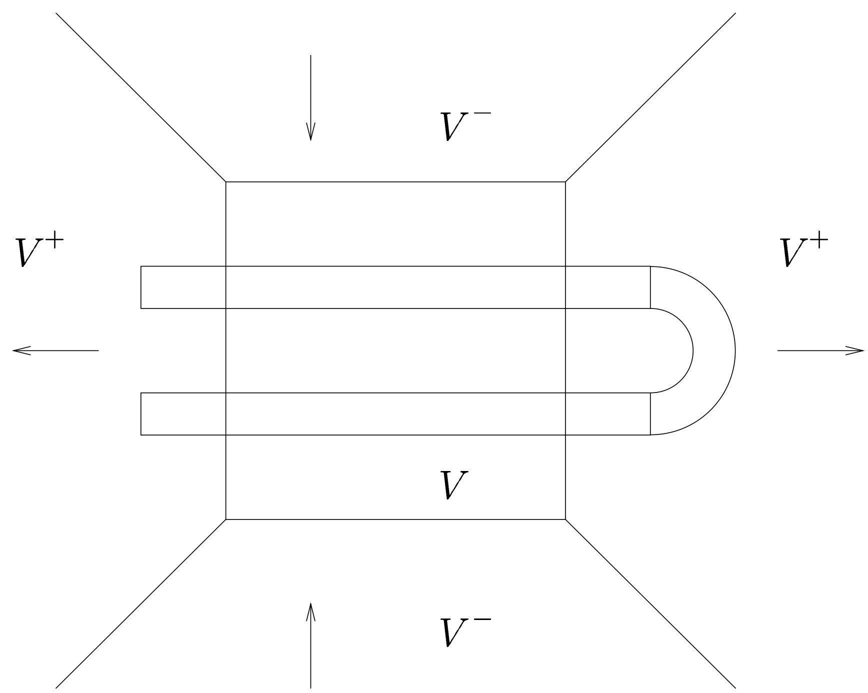
Define $f : V \to \mathbf{R}^2$.



$f^{-1}(V) \cap V = V_0 \cup V_1$.
f is affine on V_i, $i = 0, 1$.

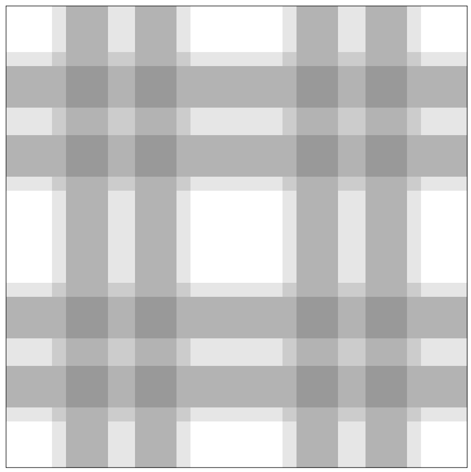
Extend f to a diffeomorphism of $\mathbf{R}^2$ such that:

1. $f(V^+) \subset V^+$ and $f^n \to \infty$ on V^+.
2. $f^{-1}(V^-) \subset V^-$ and $f^{-n} \to \infty$ on V^-.



$K^\pm := \{p \in \mathbf{R}^2 : f^n p$ bounded as $n \to \pm\infty\}$.
$K := K^+ \cap K^-$.

Then
$K^+ \cap V =$ Cantor set $\times$ interval.
$K^- \cap V =$ interval $\times$ Cantor set.
$K =$ Cantor set $\times$ Cantor set.



Define $\phi(x) = (\epsilon_n)_{n \in \mathbf{Z}}$, where

$$\epsilon_n = i \text{ if } f^n x \in V_i, i = 0, 1.$$

Claim. ϕ conjugates $f|_K$ to $\sigma|_\Sigma$.
Proof. Same as before. $\square$

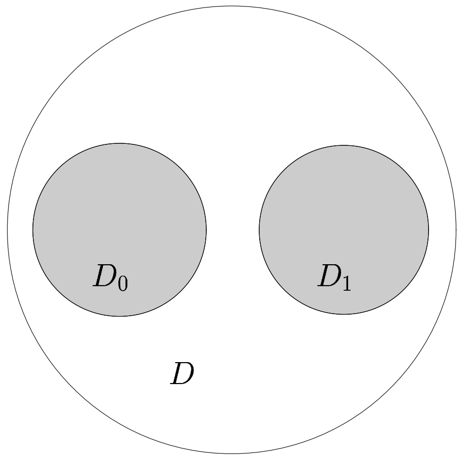
Consequences.

- Periodic points are dense in K.
- f has periodic points of all orders.

A QUADRATIC POLYNOMIAL ON $\mathbf{C}$

$$f(z) = z^2 + 10.$$

If $|z| \geq 5$, then $|f(z)| \geq 5$ and $|f^n(z)| \to \infty$.
If $D := \{|z| \leq 5\}$, then $f^{-1}(D) = D_0 \cup D_1$.



Here $f : D_i \to D$ is univalent ($\approx$ affine).
$K := \{z \in \mathbf{C} : f^n z \text{ bounded}\}$
$J := \partial K (= K)$ — Julia set of f.
Topological model $\Rightarrow$ $f|_J \simeq \sigma^+|_{\Sigma^+}$.

Conclusions.

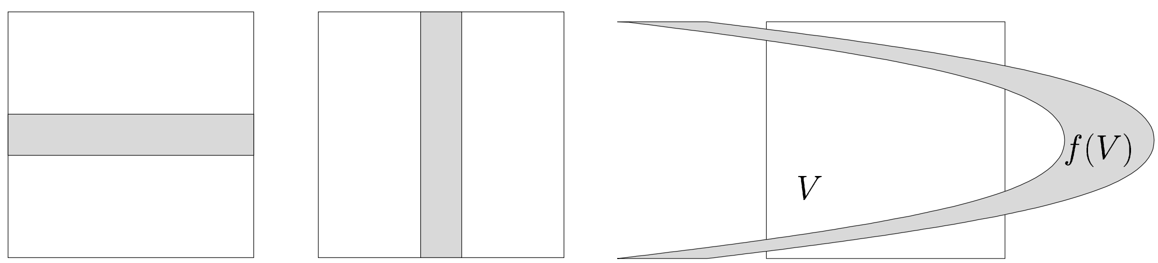
- Periodic points are dense in J.
- f has periodic points of all orders.

A QUADRATIC HÉNON MAPPING

Define $f : \mathbf{R}^2 \to \mathbf{R}^2$ (or $\mathbf{C}^2 \to \mathbf{C}^2$)

$f(x, y) = (-x^2 + a - by, x) \quad 0 < b \ll 1$ and $a \gg 1$

Decompose $f = f_3 \circ f_2 \circ f_1$.
$f_1(x, y) = (x, by)$
$f_2(x, y) = (-y, x)$
$f_3(x, y) = (x + (-y^2 + a), y)$.



Thus f "is" a horseshoe.

$K^{\pm} := \{p : f^n p$ bounded as $n \to \pm\infty\}$.
$K := K^+ \cap K^-$.
$J := \partial K (= K)$

Topological model $\Rightarrow f|_J \simeq \sigma|_\Sigma$.

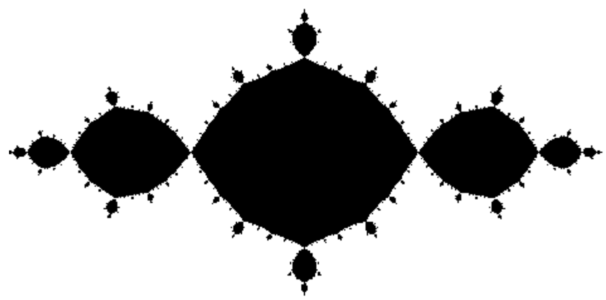
Conclusions.

- Periodic points are dense in J.
- f has periodic points of all orders.

POLYNOMIAL MAPPINGS OF $\mathbf{C}$

$$p(z) = z^d + a_{d-1}z^{d-1} + \dots$$

Want to understand dynamics of p.
Focus on points with recurrent behaviour.
$K := \{z \in \mathbf{C} : p^n z$ bounded as $n \to \infty\}$.
$J := \partial K$ (Julia set).



The picture shows K for $p(z) = z^2 - 1$.

In fact $J = \{z : \{p^n\}$ not normal at $z\}$.
What causes $\{p^n\}$ to be non-normal at J?

If z *repelling periodic point*, $p^n z = z$ and $|Dp^n z| > 1$, then $z \in J$.

Thm. $J = \overline{\{\text{repelling periodic points}\}}$

Two tools for analyzing global dynamics: Montel's Theorem and Potential Theory.

Montel's Theorem (Fatou, Julia, ...)
If $U \subset \widehat{\mathbb{C}}$ and $\mathcal{G}$ is a family of meromorphic functions on U with $\mathcal{G}(U) \subset \widehat{\mathbb{C}} - \{0, 1, \infty\}$, then $\mathcal{G}$ is normal.

Potential Theory (Brolin,Sibony, ...)

$$G(z) := \lim_{n\to\infty} \frac{1}{d^n} \log^+ |p^n(z)|.$$

$G \geq 0$ is continuous, subharmonic, harmonic off of J and $G(z) = \log |z| + o(1)$ as $|z| \to \infty$.

Thus G is the Green function of K and

$$\mu := \frac{1}{2\pi} \Delta G$$

is harmonic measure on K. Since $G \circ p = G$, μ is invariant, $\mu(p^{-1}A) = \mu(A)$.

Prop. J has no isolated points.
Proof. G is continuous and $\text{supp}(\mu) = J$. $\square$

Thm. Periodic points describes μ.

$$\lim_{n\to\infty} \frac{1}{d^n} \sum_{p^n z = z} \delta_z = \mu.$$

Remark: Also for repelling periodic points.

Proof. Let

$$H_n(z) = \frac{1}{d^n} \log |p^n z - z|.$$

Then H_n is a potential for LHS. It suffices to show that $H_n \to G$ in L^1_{loc}.

First, $H_n \to G$ on $\mathbf{C} - K$. Suppose $H_{n_j} \to H$ in L^1_{loc}. Then $H \leq G$ and $H = G$ on $\mathbf{C} - K$.

If $H \neq G$, then by Hartogs we have $\delta > 0$ and $\Omega \subset \mathrm{int}(K)$ such that $H_{n_j} \leq -\delta$ on Ω, i.e.

$$|p^{n_j} z - z| < \exp(-\delta d^{n_j})$$

on Ω. One can show this is impossible. $\square$

Cor. Periodic points are dense in J.

In a similar way one can show:

Thm. For (almost) all $w \in \mathbf{C}$

$$\frac{1}{d^n} \sum_{p^n z = w} \delta_z \to \mu$$

as $n \to \infty$. (Exception: $p(z) = z^d$, $w = 0$).

Cor. μ is ergodic, i.e.

$$p^{-1} A = A \Rightarrow \mu A = 0 \text{ or } \mu A = 1.$$

Thus, by the Ergodic Theorem.

Cor. For μ a.e. w

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{p^j w} = \mu.$$

In particular, the orbit of w is dense in J.

Compare with σ^+,

The results presented are very basic. Some further issues.

1. Classification of Fatou components.
2. Geometry of Julia sets.
3. Parameter space (Mandelbrot set).

HENON MAPPINGS

Goal. Understand dynamics of polynomial automorphisms of $\mathbf{C}^2$.

Friedland-Milnor showed that the interesting polynomial automorphisms of $\mathbf{C}^2$ are (conjugate to) compositions of *Hénon mappings*.

$$f(z,w) = (p(z) + bw, z),$$

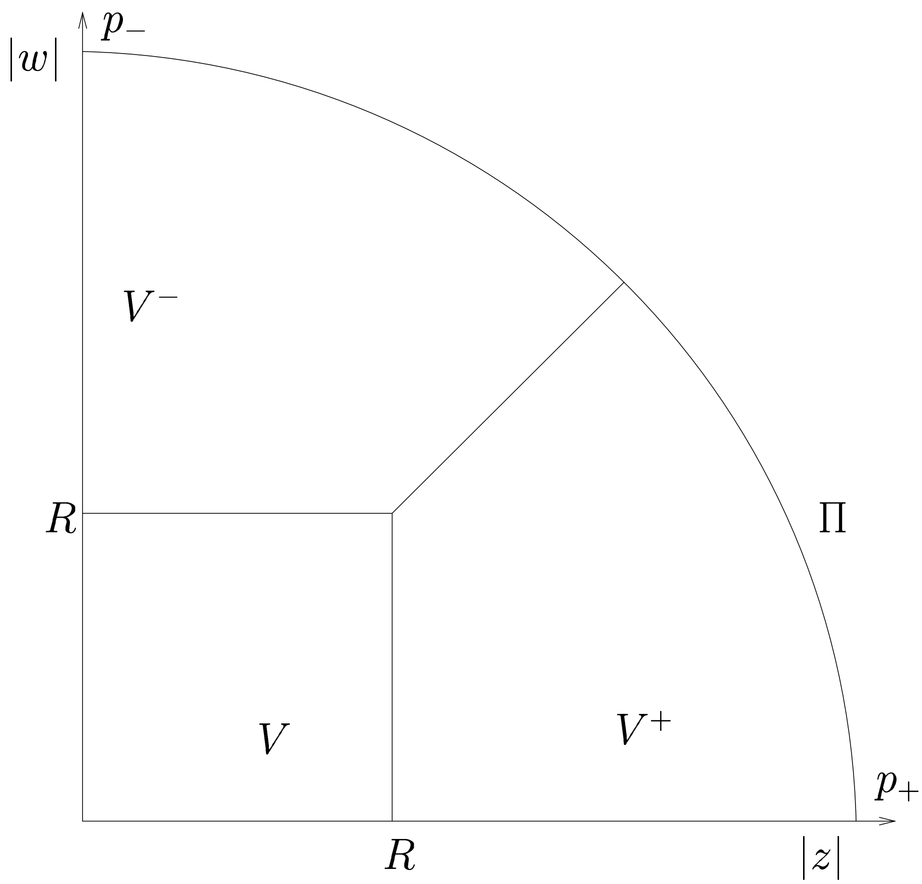
where $p(z) = z^d + \ldots$ and $b \neq 0$. For simplicity we only consider f of this form.

Note that

$$f^{-1}(z,w) = (w, \frac{1}{b}(z - p(w))).$$

is of the same type.

We have a filtration for $f = (p(z) + bw, z)$.



- $f(V^+) \subset V^+$.
- $f^{-1}(V^-) \subset V^-$.
- $f^n \to p_\pm$ on $V^\pm$ as $n \to \pm\infty$.
- If $|f^n x| \to \infty$ as $n \to \pm\infty$, then $f^n x \to p_\pm$.

These observations are trivial but crucial!

$K^{\pm} := \{x : f^n x$ bounded as $n \to \pm\infty\}$.
$K := K^{+} \cap K^{-}$.

Filtration $\Rightarrow$ $K^{\pm} \subset V \cup V^{\pm}$ is closed.
$K \subset V$ is compact.

$J^{\pm} := \partial K^{\pm}$ (Julia set for $\{f^{\pm n}\}_{n \geq 0}$).
$J := J^{+} \cap J^{-} = \partial K$ (Julia set for $\{f^n\}_{n \in \mathbf{Z}}$?)

What causes $\{f^n\}$ to be non-normal at $J^{\pm}$?

Cosider fixed points (or periodic points):
$fp = p$, λ_1, λ_2 eigenvalues of Df_p.

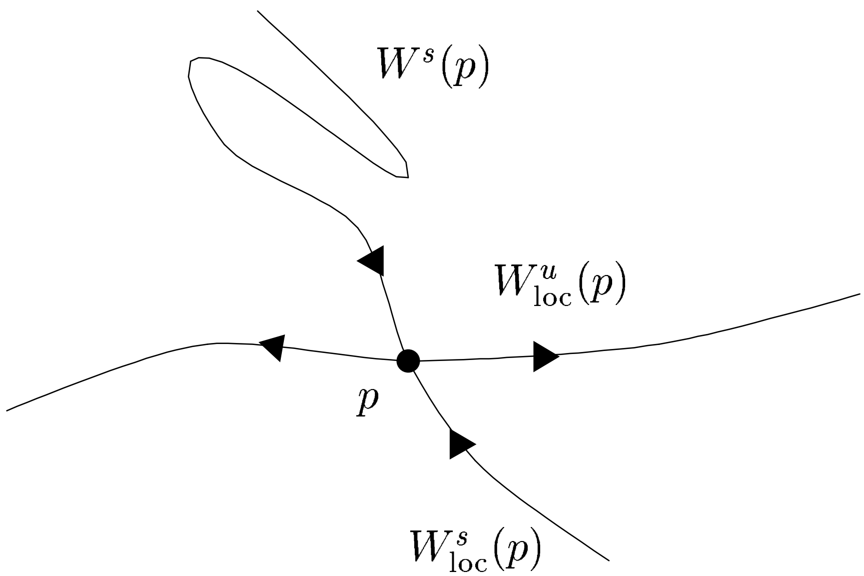
p *attracting* $(|\lambda_1|, |\lambda_2| < 1) \Rightarrow p \in J^{-} \cap \mathrm{int}(K^{+})$.
p *repelling* $(|\lambda_1|, |\lambda_2| > 1) \Rightarrow p \in J^{+} \cap \mathrm{int}(K^{-})$.
p *saddle* $(|\lambda_1| < 1 < |\lambda_2|) \Rightarrow p \in J^{+} \cap J^{-} = J$.

A saddle point has *local stable and unstable manifolds*.

$$W^s_{\text{loc}}(p) := \{x \in \mathbf{C}^2 : d(f^n x, p) < \delta \text{ for all } n \geq 0\}$$
$$W^u_{\text{loc}}(p) := \{x \in \mathbf{C}^2 : d(f^n x, p) < \delta \text{ for all } n \leq 0\}$$

for small $\delta > 0$.

By the *Stable Manifold Theorem* these are complex disks in $\mathbf{C}^2$.



We also have *global* stable/unstable manifolds.

$$W^s(p) := \{x \in \mathbf{C}^2 : f^n x \to p \text{ as } n \to +\infty\}$$
$$W^u(p) := \{x \in \mathbf{C}^2 : f^n x \to p \text{ as } n \to -\infty\}$$

These are immersed copies of $\mathbf{C}$ in $\mathbf{C}^2$.

$$W^s_{\text{loc}}(p) \subset W^s(p) \subset J^+$$
$$W^u_{\text{loc}}(p) \subset W^u(p) \subset J^-$$

Thm. (Bedford-Smillie)

$$\overline{W^s(p)} = J^+ \text{ and } \overline{W^u(p)} = J^-.$$

Compare with the horseshoe.

Proof uses pluripotential theory (currents).

Define Green function as in $\mathbf{C}$.

$$G^+ := \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n|.$$

- G^+ is continuous and plurisubharmonic.
- $G^+ \geq 0$, $\{G^+ = 0\} = K^+$.
- $G^+ \circ f = d\, G^+$.

Define positive closed current by

$$\mu^+ := dd^c G^+,$$

i.e.

$$\langle \mu^+, \phi \rangle = \frac{i}{\pi} \int G^+ \partial\overline{\partial}\phi$$

for a $(1,1)$ test form ϕ.

Prop. $\mathrm{supp}(\mu^+) = J^+$.
Proof. Show $x \in \mathrm{supp}(\mu^+) \Leftrightarrow G^+$ ph at x.
$x \in \mathrm{int}K^+ \Rightarrow G^+ = 0$ at x.
$x \in J^+ \Rightarrow G^+$ not ph at x (max. princ.).
$x \notin K^+ \Rightarrow G^+ \approx d^{-n} \log |z_n|$ ph at x (filtr.) □

Remark Similarly G^-, μ^-, ...

M submanifold of $\mathbf{C}^2$. *Current of integration*:

$$\langle [M], \phi \rangle := \int_M \phi.$$

Push-forward of current: $\langle f_* S, \phi \rangle := \langle S, f^* \phi \rangle$.
Pull-back of current: $f^* S := (f^{-1})_* S$.
Example: $f^*[M] = [f^{-1} M]$.

Thm [BS]. For any saddle point p:

$$\frac{1}{d^n} f^{n*}[W^s_{\text{loc}}(p)] \to c\mu^+$$

as $n \to \infty$, where $c > 0$.

Cor. $\overline{W^s(p)} = J^+$.

Proof of Thm (Fornæss-Sibony) in 3 steps.

$$\sigma_n := \frac{1}{d^n} f^{n*}[W^s_{\text{loc}}(p)].$$

1. σ_n has uniformly bounded mass in $\mathbf{C}^2$.
2. Any limit point of $\{\sigma_n\}$ is closed.
3. If S is a positive closed current on $\mathbf{C}^2$ with supp$(S) \subset J^+$, then $S = c\mu^+$, $c > 0$.

Sketch of 1. Let $\omega = \frac{1}{2} dd^c \log(1 + |(z,w)|^2)$ Kähler form on $\mathbf{P}^2$. The mass of σ_n is

$$\|\sigma_n\| := \langle \sigma_n, \omega \rangle = \langle [W^s_{\text{loc}}(p)], \frac{1}{d^n} f^n_* \omega \rangle.$$

Now $\frac{1}{d^n} f^n_* \omega = \frac{1}{2d^n} dd^c \log(1 + |f^{-n}|^2)$ converges to $dd^c G^- = \mu^-$. Hence $\|\sigma_n\|$ is uniformly bounded and converges to

$$\langle [W^s_{\text{loc}}(p)], \mu^- \rangle = \int_{W^s_{\text{loc}}(p)} dd^c (G^-|_{W^s_{\text{loc}}(p)}).$$

Sketch of 2. Suffices to show $\partial \sigma_n \to 0$, i.e. $\langle \sigma_n, \partial \phi \rangle \to 0$ for any $(0,1)$ test form in $\mathbf{C}^2$. Use integration by parts and Cauchy-Schwarz: if S is a positive closed current of bidegree $(1,1)$ and α, β are $(1,0)$ test forms, then

$$(\alpha, \beta) \to i \langle S, \alpha \wedge \bar{\beta} \rangle$$

defines an inner product.

Sketch of 3. Let $\mathcal{S}$ be the set of positive closed (1,1)-currents supported on K^+. Show that $\frac{1}{d^n} f^{n*} S \to \mu^+$ uniformly for $S \in \mathcal{S}$.

Let $S \in \mathcal{S}$. Then $S = dd^c u$, u psh on $\mathbf{C}^2$, $u \leq \log|(z,w)| + O(1)$. Define $u_n = d^{-n} u \circ f^n$. Must show $u_n \to G^+$ in L^1_{loc}.

If not, then $\exists\ n_k$, $\delta > 0$ and $\Omega \subset \text{int}(K)$ s.t. $u_{n_k} \leq -\delta$ on Ω, i.e.

$$f^{n_k}\Omega \subset \{u \leq -\delta d^{n_k}\}.$$

By pluripotential theory, RHS is "small". By dynamics, LHS is "not too small". Contradiction. □

FURTHER RESULTS ON HENON MAPPINGS

By pluripotential theory

$$\mu := \mu^{+} \wedge \mu^{-}$$

is a well-defined, invariant measure. Recall $\text{supp}(\mu^{\pm}) = J^{\pm}$. Hence $\text{supp}(\mu) \subset J$.

Open problem: is $\text{supp}(\mu) = J$?

Thm (Bedford-Lyubich-Smillie)
1. (f, μ) is measurably conj. to 2-sided shift.
2. μ describes (saddle) periodic points:

$$\lim_{n \to \infty} \frac{1}{d^n} \sum_{f^n p = p} \delta_p = \mu.$$

3. $\mu^{\pm}$ are *laminar* currents.

The proof uses *Pesin Theory* (non-uniform hyperbolicity).

Misc. Entropy, Fatou components, ...

REGULAR POLYN. MAPPINGS OF $\mathbf{C}^2$.

(with E. Bedford).

Def. A polynomial map $f : \mathbf{C}^2 \to \mathbf{C}^2$ of deg. $d \geq 2$ is *regular* if any of the following holds.
R1 $|f(x)| \geq c|x|^d$ as $|x| \to \infty$, $c > 0$.
R2 f extends continuously (holom.) to $\mathbf{P}^2$.
R3 The homogeneous part f_h of degree d satisfies $f_h^{-1}(0) = 0$.

Idea: $\mathbf{P}^2$ good compactification of $\mathbf{C}^2$.

Examples
1. $f(z, w) = (p(z), q(w))$, $\deg p = \deg q = d$.
2. Hénon mappings are *not* regular.
3. Rational maps on $\widehat{\mathbf{C}} \simeq \mathbf{P}^1$ correspond to homogeneous regular polynomial maps of $\mathbf{C}^2$:

$$\begin{array}{ccc} \mathbf{C}^2_* & \longrightarrow & \mathbf{C}^2_* \\ \downarrow \pi & & \downarrow \pi \\ \mathbf{P}^1 & \longrightarrow & \mathbf{P}^1 \end{array}$$

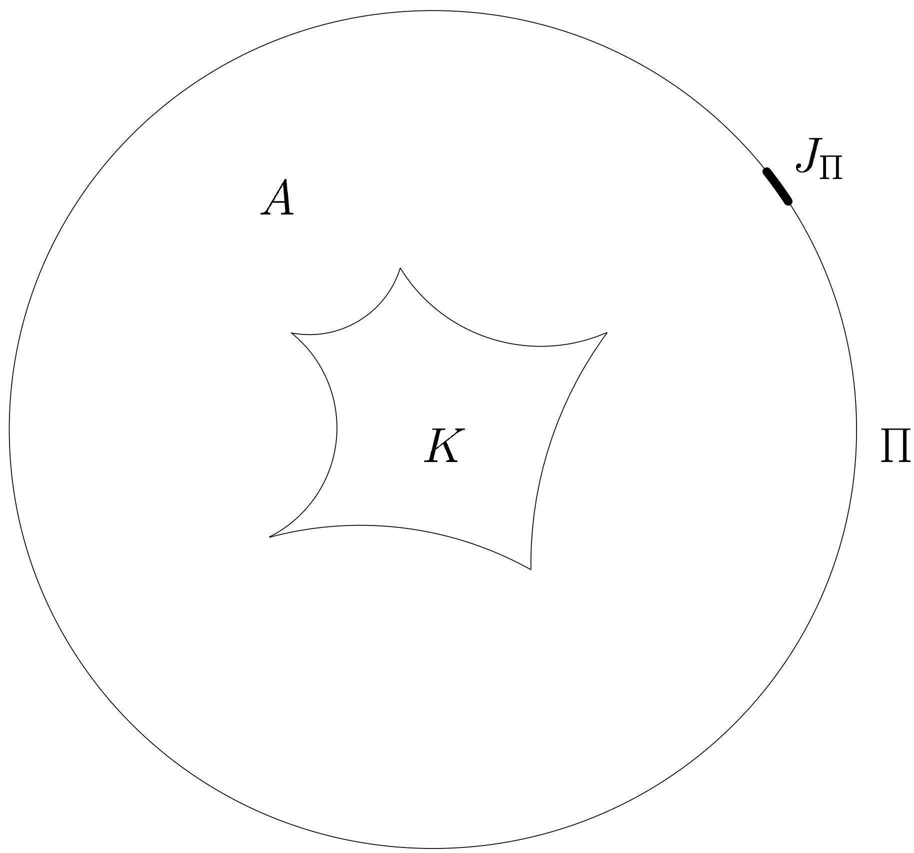
$\Pi := \mathbf{P}^2 - \mathbf{C}^2 \simeq \mathbf{P}^1$ line at infinity.

$f_\Pi := f|_\Pi$ rational map of degree d.

$K := \{x \in \mathbf{C}^2 : f^n x \text{ bounded}\}$.

$A :=$ Basin of attraction of Π.

$\mathbf{P}^2 = K \cup A$ completely invariant partition.



Idea: Approach dynamics on K from Π.

In $\mathbf{C}^1$ all polynomial mappings are regular.

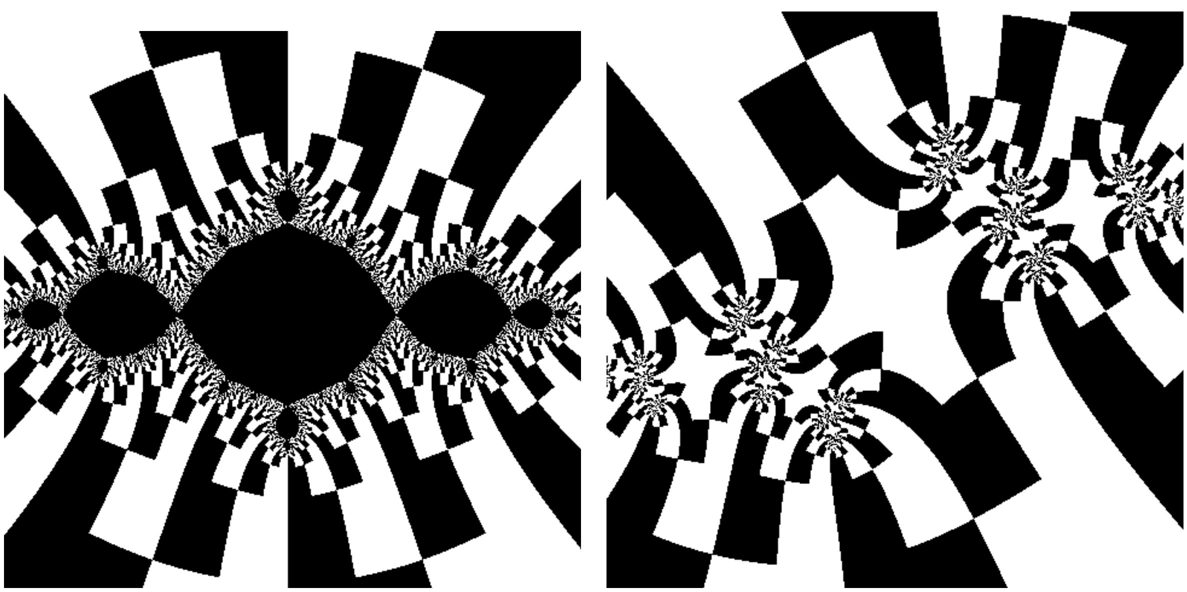
$$p(z) = z^d + a_{d-1} z^{d-1} + \dots$$

Dynamics near $\Pi = \infty$ is described by *Böttcher coordinate* $\phi(z) = z + O(1)$:

$$\phi(p(z)) = \phi(z)^d.$$

The relation between ϕ and G is $\log|\phi| = G$.

External rays := gradient lines of G.



Gradient lines for $p(z) = z^2 - 1$ and $z^2 - 1 + i$.

ϕ induces $\mathcal{E} := \{\text{external rays}\} \simeq S^1$.

Let $\nu \simeq \frac{d\theta}{2\pi}$ be Lebesgue measure on $\mathcal{E}$.

Thm There is a.e. defined landing map $e : \mathcal{E} \to J$ and $e_*\nu = \mu$.

Thus μ is a quotient of ν.

Thm If J is connected and p is uniformly expanding on J, then $e : \mathcal{E} \to J$ is continuous.

Thus J is a topological quotient of S^1.

Goal: Prove corresponding result in $\mathbf{C}^2$.

Problem: What are external rays.

Introduce pluripotential theory.

$$G := \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n|.$$

- G is continuous and psh on $\mathbf{C}^2$.
- $G \geq 0$, $\{G = 0\} = K$.

$$T := dd^c G$$

positive closed current on $\mathbf{C}^2$ (and $\mathbf{P}^2$).

$$\mu := T \wedge T = dd^c(GT)$$

is an invariant probability measure on $\mathbf{C}^2$, "harmonic measure", supported on $J := \partial_{Sh} K$.

Thm (F-S, Briend). μ is invariant, ergodic ... and describes (repelling) periodic points:

$$\lim_{n\to\infty} \frac{1}{d^{2n}} \sum_{f^n p = p} \delta_p = \mu.$$

There is a measure μ_Π (Lyubich measure) supported on J_Π with similar properties for f_Π. In fact μ_Π is determined by T:

$$\mu_\Pi = T|_\Pi.$$

Model case: $f = f_h$ homogeneous. Then

$$G(\lambda x) = G(x) + \log|\lambda|$$

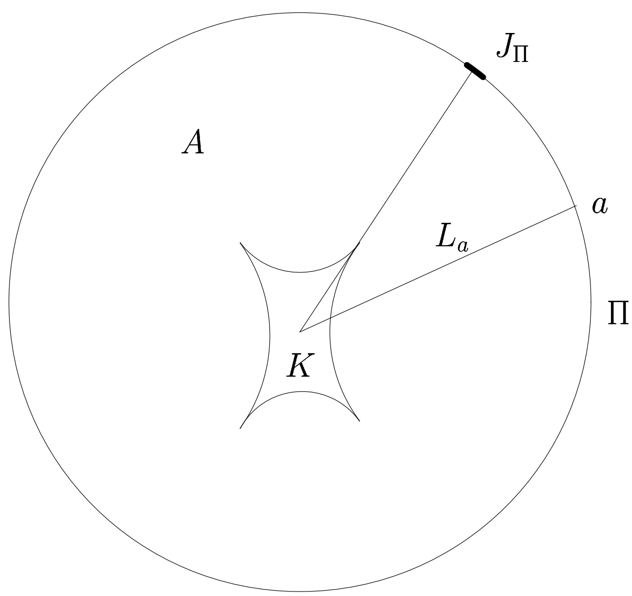
for $x \in A$, $|\lambda| \geq 1$. This implies

Lemma. If $f = f_h$, then on A

$$T = \int_{J_\Pi} [L_a]\, \mu_\Pi(a),$$

$$\mu = \int_{J_\Pi} Leb(L_a \cap \partial A)\, \mu_\Pi(a), \qquad (*)$$

where L_a = line through $a \in \Pi$ and 0.



External rays are rays in L_a for $a \in J_\Pi$ and (*) gives a landing result for external rays.

Idea in general case: Stable manifolds ⇒ laminarity of T on A ⇒ external rays.

First prove T is laminar on A. Simplifications:
1. Show laminarity near Π.
2. Assume f_Π unif. expanding on J_Π, i.e.

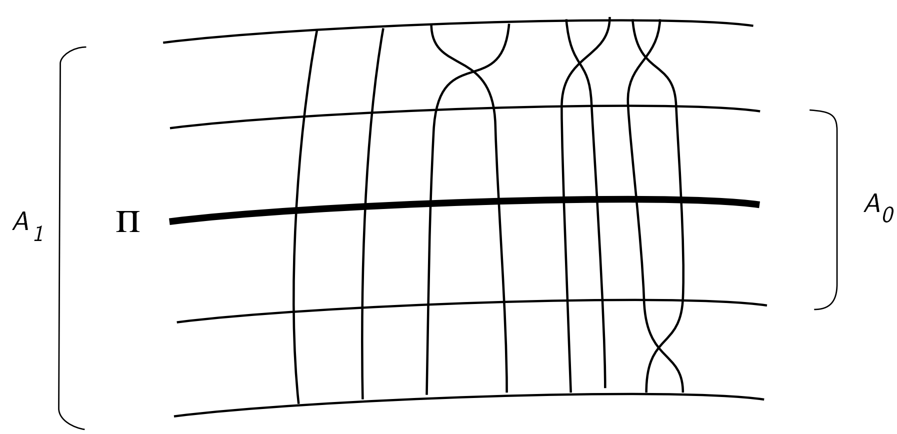
$$|Df_\Pi^n a| \geq c\lambda^n, \quad c > 0, \lambda > 1, a \in J_\Pi, n \geq 1.$$

Assume 2. If not, use Pesin Theory (non-uniform expansion).

Uniform expansion ⇒ stable manifolds:

$W^s_{\text{loc}}(a) := \{x \in \mathbf{P}^2 : d(f^n x, f^n a) < \delta \text{ for } n \geq 0\}$

is a complex disk for $a \in J_\Pi$ (and $\delta > 0$ small).



$A_0 := \{G > R_0\}$, $W_0^s(a) := W_{\text{loc}}^s(a) \cap A_0$.

Thm. If $R_0 \gg 0$, then on A_0

$$T = \int_{J_\Pi} [W_0^s(a)]\, \mu_\Pi(a). \qquad (1)$$

Proof. The formula holds for f_h:

$$T_h = \int_{J_\Pi} [L_a]\, \mu_\Pi(a). \qquad (2)$$

Apply $d^{-n} f^{n*}$ to (2) and restrict to A_0.

LHS $\to T$ because $d^{-n} G_h \circ f^n \to G$.

RHS $\to$ RHS(1) by the Stable Manifold Theorem and invariance of μ_Π. □

Want laminar formula for T on all of A.

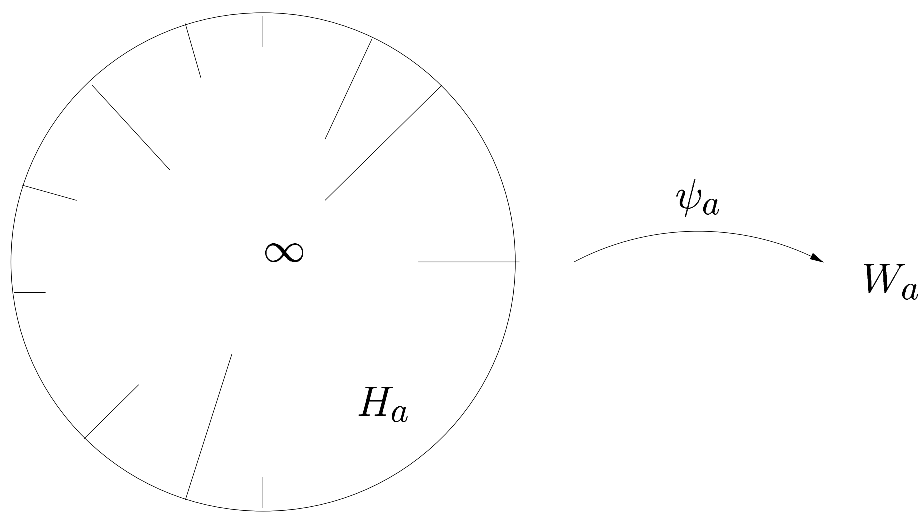
Problem: Analytic continuations of $W_0^s(a)$ can have locally infinite area in A.

Solution: divide into smaller pieces by cutting along gradient lines.

Thm. For $a \in J_\Pi$ there exists a complex disk W_a in A of finite area such that

$$T = \int_{J_\Pi} [W_a]\, \mu_\Pi(a)$$

on A. There is a "hedgehog domain" H_a and a conformal equivalence $\psi_a : H_a \to W_a$ with $G(\psi_a(\zeta)) = \log|\zeta|$.



External rays $= \psi_a$(rays in H_a) for $a \in J_\Pi$.
$\mathcal{E} :=$ { external rays } $\simeq J_\Pi \times S^1$
$\nu := \mu_\Pi \otimes \frac{d\theta}{2\pi}$ measure on $\mathcal{E}$.
$e : \mathcal{E} \to \partial K$ endpoint map.

Thm e is a.e. well defined and $e_*\nu = \mu$.

Ingredients in proof:

1. Laminarity of T on A.
2. $dd^c \max(G, r) \wedge dd^c G \to \mu$ as $r \to 0$.

Further results:

1. Conjugacy $f \simeq f_h$ (Böttcher coordinate).
2. Continuity of $e : \mathcal{E} \to J$.