Local Singularities of Planar Plurisubharmonic Functions

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Introduction

Setup: u psh function near origin in $\mathbf{C}^2$, $u(0) = -\infty$.

Goal: study (logarithmic) singularity of u.

Examples:

1. $u(x, y) = \log |(x, y)| = \log \max\{|x|, |y|\}$
2. $u(x, y) = \log |y|$
3. $u(x, y) = \log |\log |(x, y)||$

Invariants:

1. Lelong number.
2. Kiselman numbers.
3. Demailly-Lelong numbers.
4. Arnold multiplicity.
5. Multiplier ideal.

Connections between invariants?

Invariants

1. Lelong number

$$\nu^{\mathrm{L}}(u) = \lim_{r\to 0} \frac{1}{\log r} \sup_{\Delta_r \times \Delta_r} u$$

2. Kiselman numbers

$$\nu^{\mathrm{K}}_{1,t}(u) = \lim_{r\to 0} \frac{1}{\log r} \sup_{\Delta(r)\times\Delta(r^t)} u$$

where $t \geq 1$.

3. Demailly-Lelong numbers

$$\nu^{\mathrm{D}}_{\varphi}(u) = (dd^c u \wedge dd^c \varphi)\{0\},$$

for φ psh, $\varphi^{-1}\{-\infty\} = \{0\}$, e^{φ} Hölder cont, i.e. a Hölder weight.

4. Arnold multiplicity

$$\lambda(u) = \inf\{\lambda > 0 \mid e^{-u/\lambda} \in L^2_{\mathrm{loc}}\}.$$

5. Multiplier ideal

$$J(u) = \{\phi \text{ analytic germ} \mid |\phi| e^{-u} \in L^2_{\mathrm{loc}}\}.$$

Main Results

Thm A. (Criterion for equisingularity) If u, v psh, then

$$J(tu) = J(tv)\ \forall t > 0 \quad \Leftrightarrow \quad \nu^{\mathrm{D}}_{\varphi}(u) = \nu^{\mathrm{D}}_{\varphi}(v)\ \forall \varphi.$$

Thm B. If u psh, then

$$\lambda(u) = \sup \left\{ \frac{\nu^{\mathrm{K}}_{1,t}(h^* u)}{1+t} \;\middle|\; t \geq 1, h \text{ local biholomorphism} \right\}.$$

Cor (Skoda). $\nu^{\mathrm{L}}(u) < 1 \implies \lambda(u) < 1 \implies e^{-u} \in L^2_{\mathrm{loc}}$.

Thm (Mimouni). If $\nu^{\mathrm{L}}(u) = 1$, then $e^{-u} \in L^2_{\mathrm{loc}}$ unless $dd^c u$ puts mass on a curve.

Thm C. If $\nu^{\mathrm{L}}(u) = 1$, then exactly one of the following holds:

- $e^{-u} \in L^2_{\mathrm{loc}}$;
- $u = \log|\phi| + v$, where $(\phi = 0)$ is smooth and $\nu^{\mathrm{L}}(v) = 0$.

Thm D. (Openness conjecture) If $\lambda = \lambda(u)$, then $e^{-u/\lambda} \notin L^2_{\mathrm{loc}}$. Thus the interval

$$\{\lambda > 0 \mid e^{-u/\lambda} \in L^2_{\mathrm{loc}}\}$$

is open.

Main Tools

Valuations.

– Give information on singularities of curves and ideals.
– Action extends to psh functions.
– Generalize Kiselman numbers.
– The set of all valuations has a nice (tree) structure.

Demailly approximation.

– Approximate a general psh function by log-singular psh functions.
– Well adapted to valuations.
– Construction uses multiplier ideals!

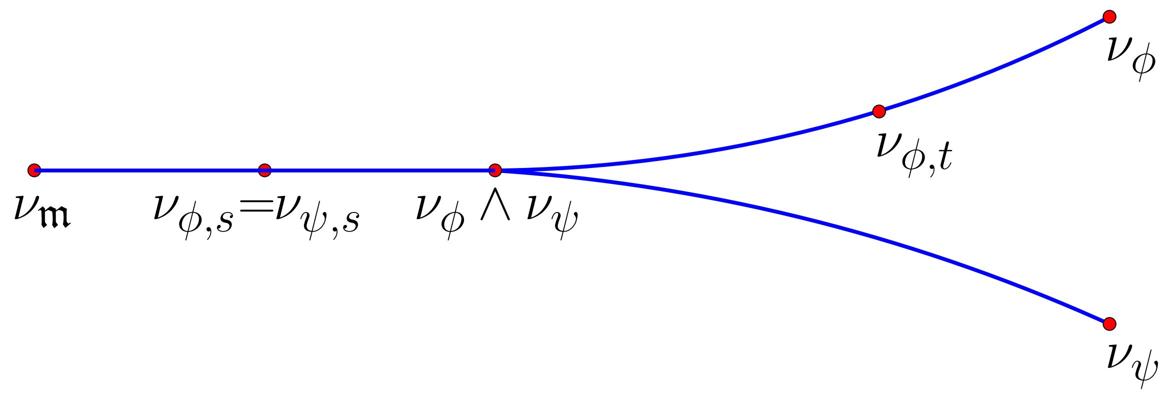
The Valuative Tree

A subtree:

$$\mathcal{V}_{\rm tor} = \{\text{"toroidal" normalized valuations on } \mathbf{C}[x,y]\}$$
$$= \{(\phi,t) \mid (\phi=0) \text{ local irreducible curve, } t \geq 1\}/\sim,$$

where $(\phi,t)\sim(\psi,t)$ iff ϕ,ψ "sufficiently tangent" (depending on t).

Natural tree structure:
- partial ordering.
- metric
- topologies



The full tree:

$$\mathcal{V} = \text{"completion" of } \mathcal{V}_{\rm tor}$$
$$= \mathcal{V}_{\rm tor} \cup \{\text{endpoints}\}$$
$$= \{\text{normalized valuations on } \mathbf{C}[x,y]\}.$$

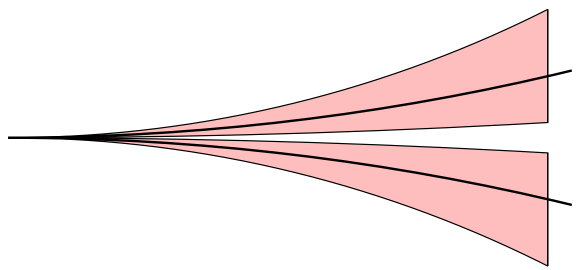
Special elements:
- Root: $\nu_{\mathfrak{m}} = \nu_{\phi,1}$. Acts on $\mathbf{C}[x,y]$ by $\nu_{\mathfrak{m}}(\psi) = m(\psi) =$ multiplicity of ψ.
- Curves: $\nu_\phi = \nu_{\phi,\infty}$. Act on $\mathbf{C}[x,y]$ by $\nu_\phi(\psi) = \phi\cdot\psi/m(\phi)$.

Characteristic Regions

Def. A toroidal valuation $\nu = \nu_{\phi,t}$ defines a characteristic region in $\mathbf{C}^2$:

$$\Omega_\nu(r) = \{|(x,y)| \le r, |\phi(x,y)| \le |(x,y)|^{m(\phi)t}\}$$

Essentially independent of representative ϕ.



Def. Thinness $A(\nu)$ of ν defined by $|\Omega_\nu(r)| \sim r^{2A(\nu)}$.

Def. Action of ν on psh functions: $\nu(u) := \lim_{r\to 0} \frac{1}{\log r} \sup_{\Omega_\nu(r)} u$.

Prop. This is well-defined.

Proof. Consider two cases.
Case 1: ϕ smooth. WLOG $\phi = y$. Then $\nu(u) = \nu^{\mathrm{K}}_{1,t}(u)$.
Case 2: ϕ singular. Desingularize by birational map π. Reduce to Case 1.
Then $\nu(u) = \text{const} \cdot \nu^{\mathrm{K}}_{1,s}(\pi^* u)$, $s > 0$. □

Rem. Consistent with alg. def. if we identify $\phi \in \mathbf{C}[x,y]$ with $\log|\phi|$ psh.

The Tree Transform I

- Any u psh has a tree transform $\hat{u} : \mathcal{V}_{\mathrm{tor}} \to [0, \infty)$ given by

$$\hat{u}(\nu) := \nu(u)$$

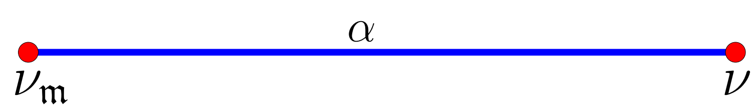
- What properties does $\hat{u}$ have?
- Kiselman: $t \mapsto \nu^{\mathrm{K}}_{1,t}(u)$ is concave.
- Must understand concavity on the tree $\mathcal{V}_{\mathrm{tor}}$.
- Parameterizations of $\mathcal{V}_{\mathrm{tor}}$ (or $\mathcal{V}$):
  - Thinness $A : \mathcal{V} \to [2, \infty]$.
  - Skewness $\alpha : \mathcal{V} \to [1, \infty]$;
    - $\alpha(\nu_{\phi,t}) = t$.
    - Intrinsically: $\alpha(\nu) = \sup \frac{\nu(\phi)}{m(\phi)}$.
    - For ϕ irreducible: $\nu(\phi) = m(\phi)\alpha(\nu \wedge \nu_\phi)$
  - Relation: $A(\nu) = 2 + \int_{\nu_{\mathfrak{m}}}^{\nu} m(\mu)\, d\alpha(\mu)$, where $m(\mu) = \min\{m(\psi) \mid \nu_\psi \geq \mu\}$
- Use skewness to define concavity.

Potential Theory

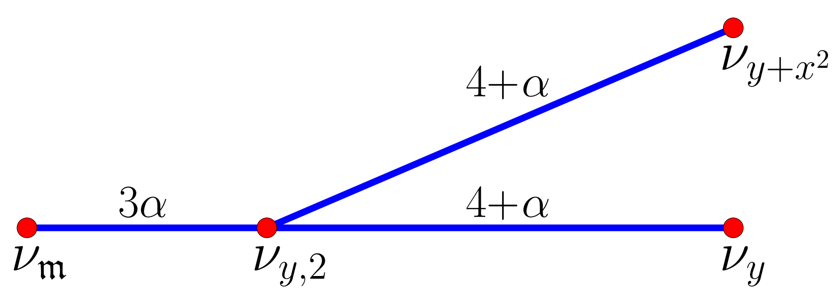
- On $\mathbf{R}$: (normalized) concave fcns $\leftrightarrow$ positive Borel measures. Identification given by $\Delta = -\partial^2/\partial x^2$.
- Do something similar on the valuative tree $\mathcal{V}$.
- $\Phi : \mathcal{V} \to [0, \infty]$ is a tree potential if
  - Φ is increasing;
  - Φ has directional derivatives everywhere;
  - At $\nu \neq \nu_{\mathfrak{m}}$: $\sum$ outgoing derivatives $\leq$ incoming derivative.
  - At $\nu_{\mathfrak{m}}$: $\sum$ outgoing derivatives $\leq \Phi(\nu_{\mathfrak{m}})$.

Thm. $\exists$ homeomorphism $\Delta : \{$tree potentials$\} \to \{$Borel measures$\}$.

Ex. $\Phi := \alpha(\nu \wedge \cdot)$ has $\Delta\Phi = \delta_\nu$.



Ex. If Φ is as in the example below, then $\Delta\Phi = \delta_{\nu_{y,2}} + \delta_{\nu_y} + \delta_{\nu_{y+x^2}}$



Demailly Approximation

Consider u psh on $B = B(0,1)$ and $m \geq 1$. Define Hilbert space $\mathcal{H}_m$ by

$$\mathcal{H}_m := \left\{ \phi \in \mathcal{O}(B) \;\middle|\; \int_B |\phi|^2 e^{-2mu} < \infty \right\}.$$

Define psh function u_m by

$$u_m(p) = \frac{1}{2m} \log \sum_{j=1}^{\infty} |h_{mj}(p)|^2 \quad \text{where } (h_{mj})_j \text{ is an ON basis for } \mathcal{H}_m$$

$$= \frac{1}{m} \sup \left\{ \log |\phi(p)| \;\middle|\; \|u\|_m \leq 1 \right\}.$$

Then u_m is log-singular and approximates u well:

Prop. For any u psh, $\nu \in \mathcal{V}_{\mathrm{tor}}$, $m \geq 1$ we have

$$|\hat{u}(\nu) - \hat{u}_m(\nu)| = |\nu(u) - \nu(u_m)| \leq \frac{A(\nu)}{m}.$$

Proof. Adaptation of the case $\nu = \nu_{\mathfrak{m}} = \nu^{\mathrm{L}}$. □

Prop. If u is a Hölder weight, then the Demailly approximation is better:

$$|\hat{u}(\nu) - \hat{u}_m(\nu)| = |\nu(u) - \nu(u_m)| \leq \frac{C}{m}.$$

The Tree Transform II

Thm. If u is psh, then its tree transform $\hat{u}$ defines a tree potential.

Proof. Use that $\{$tree pot's$\}$ closed under sums, minima and pointwise limits.
Step 1: $u = \log|\phi|$, ϕ irreducible. Then $\Delta\hat{u} = m(\phi)\delta_{\nu_\phi}$ using the formula

$$\hat{u}(\nu) = \nu(\phi) = m(\phi)\alpha(\nu \wedge \nu_\phi).$$

Step 2: $u = \log|\phi|$, $\phi = \prod \phi_i$ reducible. Then $\hat{u} = \sum \hat{u}_i$.

Step 3: $u = \log \frac{1}{2} \sum |\phi_j|^2$ log-singular. Then $\hat{u} = \min_j \hat{u}_j$.

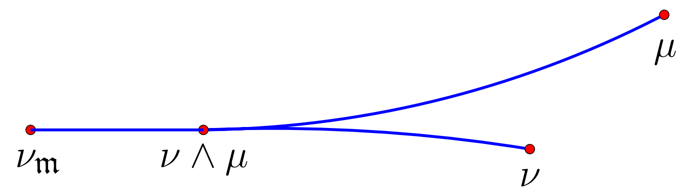
Step 4: u general psh. Then $\hat{u} = \lim \hat{u}_m$ pointwise. $\square$

Question. Which measures on $\mathcal{V}$ appear as $\Delta\hat{u}$ for u psh? (Partial answers.)

Intersection Formula I

- Have natural intersection pairing on $\mathcal{V}$:

$$\nu\cdot\mu := \alpha(\nu\wedge\mu) \in [1,\infty].$$



- Extend to Borel measures on $\mathcal{V}$ by linearity:

$$\rho\cdot\rho' := \iint_{\mathcal{V}\times\mathcal{V}} \nu\cdot\nu'\, d\rho(\nu)d\rho'(\nu')$$

- Could hope that for u, v psh

$$(dd^c u \wedge dd^c v)\{0\} = \rho_u\cdot\rho_v = \iint_{\mathcal{V}\times\mathcal{V}} \mu\cdot\nu\, d\rho_u(\mu)d\rho_v(\nu), \qquad (\star)$$

  where $\rho_u = \Delta\hat{u}$ and $\rho_v = \Delta\hat{v}$.

- However, $(\star)$ fails for $u = \log\max\{|x|, \log|y|\}$, $v = \log|y|$:

$$\nu_{\mathfrak{m}}(u) = \nu^{\mathrm{L}}(u) = 0 \implies \hat{u}\equiv 0 \implies \rho_u = 0 \implies \iint = 0$$

  but $dd^c u\wedge dd^c v = dd^c(u|_{y=0}) = \delta_0$.

- Still, $(\star)$ holds in many cases, e.g. if $u = \log|\phi|$, $v = \log|\psi|$.

Intersection Formula II

Thm. If u is psh and v is a Hölder weight then ($\star$) holds:

$$(dd^c u \wedge dd^c v)\{0\} = \rho_u \cdot \rho_v = \iint_{\mathcal{V}\times\mathcal{V}} \mu \cdot \nu \, d\rho_u(\mu) d\rho_v(\nu). \qquad (\star)$$

Proof. Four steps.

1. $\rho_v = \delta_\nu$ for $\nu \in \mathcal{V}_{\mathrm{tor}}$, $m(\nu) = 1$. Then WLOG $\nu = \nu_{y,t}$
   By Demailly/Kiselman: $(dd^c u \wedge dd^c v)\{0\} = \nu^{\mathrm{K}}_{1,t}(u) = \nu_{y,t}(u)$.
   (a) $u = \log|\phi|$, ϕ irr. Then $\nu_{y,t}(u) = m(\phi)\alpha(\nu \wedge \nu_{y,t}) = \rho_u \cdot \rho_v$.
   (b) $u = \log|\phi|$, ϕ reducible. Linearity and (a) gives ($\star$).
   (c) u log-singular. Then ρ_u is atomic so linearity and (b) gives ($\star$).
   (d) u arbitrary psh. Demailly approximation gives ($\star$).
2. $\rho_v = \delta_\nu$ for $\nu \in \mathcal{V}_{\mathrm{tor}}$, $m(\nu) > 1$. Reduce to Step 1 by blowing up.
3. v log-singular. Then ρ_v is atomic. Apply Steps 1-2 and linearity.
4. v Hölder weight. Use Demailly approximation. □

Cor. If φ is a Hölder weight then

$$\nu^{\mathrm{D}}_\varphi = \int_{\mathcal{V}} \nu \, d\rho_\varphi(\nu)$$

i.e. *a Demailly-Lelong number is an average of valuations*.

Thm. For any u, v psh one inequality in ($\star$) holds: $(dd^c u \wedge dd^c v)\{0\} \geq \rho_u \cdot \rho_v$.

A Valuative Criterion of Integrability

Recall that $J(u) := \{\psi \mid |\psi| e^{-u} \in L^2_{\mathrm{loc}}\}$.

Def. $\lambda(u;\psi) := \inf\{\lambda > 0 \mid |\psi| e^{-u/\lambda} \in L^2_{\mathrm{loc}}\}$.

Thm. For any u psh and any analytic germ ψ we have

$$\lambda(u;\psi) = \sup_{\nu \in \mathcal{V}} \chi(\nu; u, \psi) := \sup_{\nu \in \mathcal{V}} \frac{\nu(u)}{\nu(\psi) + A(\nu)}.$$

Proof. Three steps.

1. $\lambda = \chi(\nu; u, \psi)$ for $\nu \in \mathcal{V}_{\mathrm{tor}}$ implies $|\psi| e^{-u/\lambda} \notin L^2_{\mathrm{loc}}$.
   Indeed, $|\Omega_\nu(r)| \sim r^{2A(\nu)}$ and in $\Omega_\nu(r)$ we have (roughly)
   $$|\psi|^2 e^{-2u/\lambda} \sim r^{2(\nu(\psi) - \lambda^{-1}\nu(u))} = r^{-2A(\nu)}.$$
2. u log-singular, $\lambda > \sup_\nu \chi(\nu; u, \psi)$ implies $|\psi| e^{-u/\lambda} \in L^2_{\mathrm{loc}}$.
   "Standard": resolve sings of u, ψ and compute in local coords.
3. u general. Use Demailly approximation and Hölder's ineq. Note that:
   $$|\chi(\nu; u, \psi) - \chi(\nu; v, \psi)| \le \frac{1}{m} \quad \text{for all } \nu. \quad \square$$

Proof of Main Results I

Thm D. (Openness conjecture) If $\lambda = \lambda(u)$, then $e^{-u/\lambda} \notin L^2_{\mathrm{loc}}$. Thus the interval $\{\lambda > 0 \mid e^{-u/\lambda} \in L^2_{\mathrm{loc}}\}$ is open.

Proof. The supremum in

$$\lambda(u;1) = \sup_{\nu \in \mathcal{V}} \chi(\nu; u, 1)$$

is attained at some $\nu \in \mathcal{V}$ (essentially usc fcn on compact set). If $\nu \in \mathcal{V}_{\mathrm{tor}}$, then Step 1 in previous proof gives $e^{-u/\lambda} \notin L^2_{\mathrm{loc}}$. The other cases are easier. $\square$

Proof of Main Results II

Thm A. (Criterion for equisingularity) If u, v psh, then

$$J(tu) = J(tv)\ \forall t > 0 \quad \Leftrightarrow \quad \nu^{\mathrm{D}}_{\varphi}(u) = \nu^{\mathrm{D}}_{\varphi}(v)\ \forall \varphi.$$

Proof. One direction follows immediately:

$$\begin{aligned} \nu^{\mathrm{D}}_{\varphi}(u) = \nu^{\mathrm{D}}_{\varphi}(v) \text{ for all } \varphi &\implies \nu(u) = \nu(v) \text{ for all } \nu \in \mathcal{V} \\ &\implies \lambda(u;\psi) = \lambda(v;\psi) \text{ for all } \psi \\ &\implies J(tu) = J(tv) \text{ for all } t. \end{aligned}$$

The other direction follows from

Prop. For u psh and $t > 0$ set

$$\Phi_t(\nu; u) := \min\{\nu(\psi) \mid \psi \in J(tu)\}$$

Then Φ_t is a tree potential and $t^{-1}\Phi_t \to \hat{u}$ pointwise as $t \to 0$.

Proof. Demailly approximation. $\square$

Proof of Main Results III

Thm B. If u psh, then

$$\lambda(u) = \sup \left\{ \frac{\nu^{\mathrm{K}}_{1,t}(h^* u)}{1+t} \;\middle|\; t \geq 1, h \text{ local biholomorphism} \right\}.$$

Proof. We have

$$\lambda(u) = \lambda(u;1) = \sup_{\nu} \frac{\nu(u)}{A(\nu)}.$$

If $\nu = \nu_{\phi,t}$ has $m(\nu) = 1$ (i.e. $(\phi = 0)$ smooth). then

$$A(\nu) = 1 + t \quad \text{and} \quad \nu(u) = \nu^{\mathrm{K}}_{1,t}(h^* u)$$

where h is local biholomorphism sending $(\phi = 0)$ to $(y = 0)$.

Using relation between skewness and thinness, can show that

$$\nu \mapsto \frac{\nu(u)}{A(\nu)}$$

is decreasing (and piecewise Möbius) when $m(\nu) > 1$. □

Proof of Main Results IV

Thm C. If $\nu^{\mathrm{L}}(u) = 1$, then exactly one of the following holds:

- $e^{-u} \in L^2_{\mathrm{loc}}$;
- $u = \log|\phi| + v$, where $(\phi = 0)$ is smooth and $\nu^{\mathrm{L}}(v) = 0$.

Proof. Suppose $u = \log|\phi| + v$ with ϕ smooth, v psh. WLOG $\phi = y$. Then

$$e^{-u} \geq |y|^{-1} \notin L^2_{\mathrm{loc}}.$$

Now suppose $\nu_{\mathfrak{m}}(u) = \nu^{\mathrm{L}}(u) = 1$ but $e^{-u} \notin L^2_{\mathrm{loc}}$. Then $\lambda(u) = 1$. But

$$\lambda(u) = \lambda(u; 1) = \sup_{\nu} \frac{\nu(u)}{A(\nu)}.$$

Consider the tree potential $\hat{u} : \mathcal{V}_{\mathrm{tor}} \to [1, \infty]$. The measure $\rho_u = \Delta\hat{u}$ has mass $\nu_{\mathfrak{m}}(u) = 1$. This implies

$$\nu(u) = \hat{u}(\nu) = 1 + \int_{\nu_m}^{\nu} \rho_u\{\mu' \geq \mu\}\, d\alpha(\mu) \leq \alpha(\nu).$$

Also $A(\nu) \geq 1 + \alpha(\nu)$ with equality iff $m(\nu) = 1$. Thus if $\lambda(u) = 1$, then there exists $\nu_k \in \mathcal{V}_{\mathrm{tor}}$ with

$$m(\nu_k) = 1, \quad \alpha(\nu_k) \to \infty \quad \text{and} \quad \rho_u\{\mu \geq \nu_k\} \to 1.$$

This implies $\nu_k \to \nu_\phi$ and $\rho_u = \delta_{\nu_\phi}$ where $(\phi = 0)$ is a smooth curve. Demailly approximation gives $u = \log|\phi| + v$. □