Trees and Valuations

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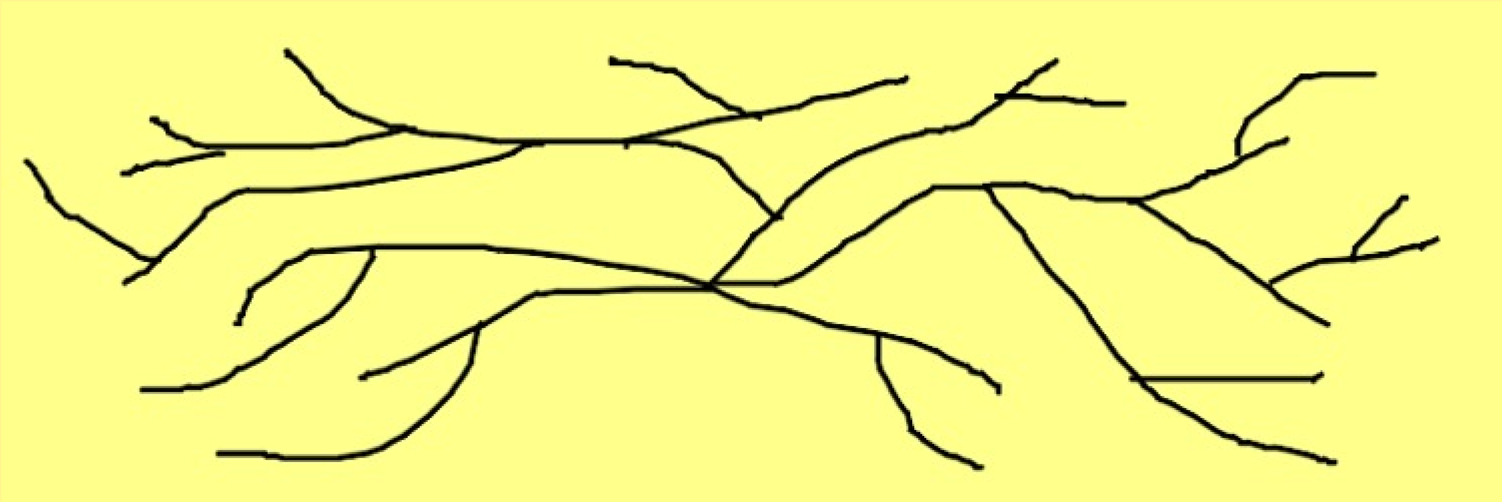
Executive Summary

- $\mathcal{V} := \{$normalized valuations on $\mathbf{C}[[x,y]]\}$
  (reg. local ring of dim 2 with alg. closed res. field)
- Main result: $\mathcal{V}$ has natural tree structure



Krull valuations

- $R = \mathbf{C}[[x, y]]$, $\mathfrak{m}$ = max. ideal, K = quotient field.
- Krull valuation: $\nu : K^* \to \Gamma$, Γ = ordered abelian group.
  - $\nu(\phi\psi) = \nu(\phi) + \nu(\psi)$;
  - $\nu(\phi + \psi) \geq \min(\nu(\phi), \nu(\psi))$;
  - $\nu(1) = 0$.
- Centered if $\nu|_R \geq 0$ and $\nu|_{\mathfrak{m}} > 0$.
- Numerical invariants:
  - $\mathrm{rk}(\nu) := \dim(R_\nu)$
  - $\mathrm{rat.rk}(\nu) := \dim_{\mathbf{Q}}(\nu(K) \otimes \mathbf{Q})$
  - $\mathrm{tr.deg}(\nu) := \mathrm{tr.deg}(k_\nu : k)$
- Abhyankar's inequalities:

$$\mathrm{rk}(\nu) + \mathrm{tr.deg}(\nu) \leq \mathrm{rat.rk}(\nu) + \mathrm{tr.deg}(\nu) \leq 2.$$

Examples of Krull valuations

(1) $\mathfrak{m}$-adic valuation $\nu_{\mathfrak{m}}(\phi) = \max\{j \mid \phi \in \mathfrak{m}^j\}$

(2) Divisorial valuations.

(3) Curve valuations.

(4) Exceptional curve valuations.

(5) Others.

Valuations

- Valuation: $\nu : R^* \to [0, \infty]$ such that
  - $\nu(\phi\psi) = \nu(\phi) + \nu(\psi)$;
  - $\nu(\phi + \psi) \geq \min(\nu(\phi), \nu(\psi))$;
- Centered if $\nu(\mathfrak{m}) := \min\{\nu(\phi) \mid \phi \in \mathfrak{m}\} > 0$.
- Proper if $\nu(\mathfrak{m}) < \infty$.
- Normalized if $\nu(\mathfrak{m}) = 1$.

Valuations vs Krull valuations

- From proper valuations to Krull valuations. Pick $\nu : R \to [0,\infty]$ and consider the prime ideal $I := \{\nu = \infty\}$:
  - If $I = 0$ take $\Gamma = \mathbf{R}$.
  - If $I = (\phi)$ take $\Gamma = \mathbf{Z} \times \mathbf{Q}$ and let $\tilde{\nu} : R^* \to \Gamma$ be minimal s.t.

$$\tilde{\nu}(\phi) = (1,0) \quad \text{and} \quad \tilde{\nu}(\mathfrak{m}) = (0,1).$$

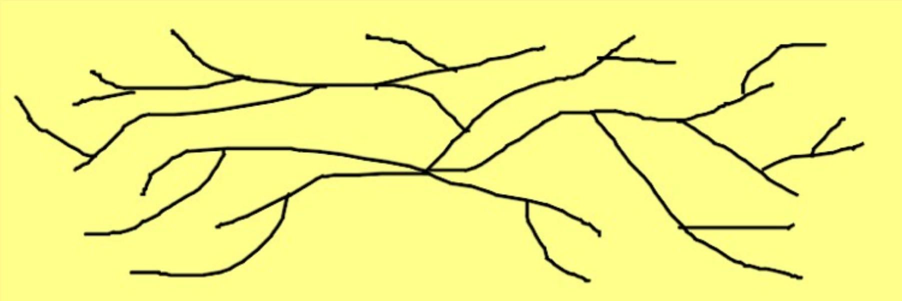
- All Krull valuations are of this form except exceptional curve valuations ($\Gamma = \mathbf{Z} \times \mathbf{Z}$ and $\nu(\mathfrak{m}) \geq (1,0)$).

Valuation space

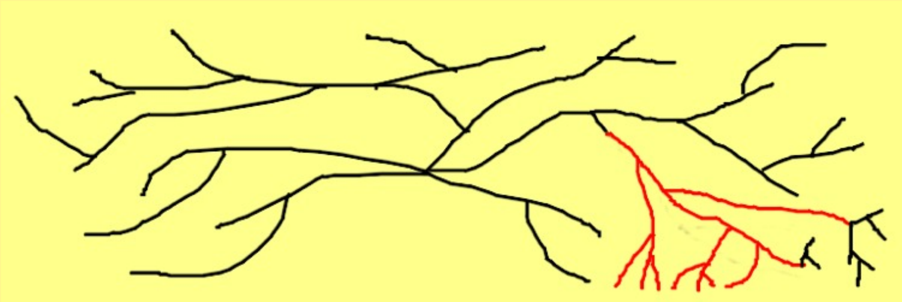
- $\mathcal{V} := \{$normalized valuations on $R\}$
- Partial ordering: $\nu \leq \mu$ iff $\nu(\phi) \leq \mu(\phi)\ \forall \phi \in R$
- Weak topology: $\nu_j \to \nu$ iff $\nu_j(\phi) \to \nu(\phi)\ \forall \phi \in R$.
- Strong topology: $\nu_j \to \nu$ iff $\sup_\phi |\frac{\nu_{\mathfrak{m}}(\phi)}{\nu_j(\phi)} - \frac{\nu_{\mathfrak{m}}(\phi)}{\nu(\phi)}| \to 0$.
- Skewness: $\alpha(\nu) := \sup_\phi \frac{\nu(\phi)}{\nu_{\mathfrak{m}}(\phi)} \in [1, \infty]$.

Trees

- Tree: partially ordered set $\mathcal{T}$ such that:
  - $\mathcal{T}$ has unique minimal element;
  - if $\sigma_1 < \sigma_2$, then $[\sigma_1, \sigma_2] := \{\sigma \mid \sigma_1 \leq \sigma \leq \sigma_2\}$ is totally ordered, isomorphic to $[0, 1] \subset \mathbf{R}$.

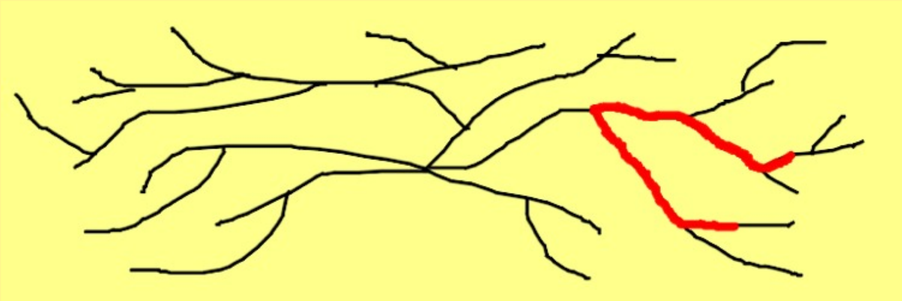


- Weak topology: generated by sets of the form below. Hausdorff. Compact if the tree is complete i.e. every increasing chain has a supremum.



Metric trees

- Metric tree: metric space with a unique path $[\sigma_1, \sigma_2]$ connecting every two points (and $[\sigma_1, \sigma_2] \simeq [0, 1]$).



- Every metric tree defines a tree and every tree can be given a (nonunique) metric under which it becomes a metric tree.
- Example: if (X, d) is an ultrametric space of diameter 1, then $\mathcal{T} := X \times [1, \infty]/\sim$ is a metric tree, where $(x, s) \sim (y, t)$ iff $d(x, y) \leq \frac{1}{s} = \frac{1}{t}$.

Main results

Theorem 1. *The valuation space $\mathcal{V}$ with its natural partial ordering is a tree.*

Theorem 2. *The weak topology on $\mathcal{V}$ coincides with its weak topology as a tree.*

Theorem 3. *There is a natural invariant tree metric on $\mathcal{V}$ which induces the strong topology on $\mathcal{V}$.*

Related work

- Serre and Morgan-Shalen associate a tree (+group action) to a *fixed* valuation ν.
- Zariski identified Krull valuations of K with sequences of blowups.
- Spivakovsky gave more detailed classification.
- Our main tool is inspired by work of S. MacLane (1936).

STKP's

- Main tool: identification of a valuation with a Sequence of Toroidal Key Polynomials.
- $\nu(x)$ determines ν on $\mathbf{C}[[x]]$.
- $\nu(y)$ then determines ν on "most" elements in $\mathbf{C}[[x,y]]$.
- Key idea $\exists !\ U_2 \in \mathbf{C}[[x,y]]$ of min y-deg s.t. $\nu(U_2)$ not determined by $\nu(x)$ and $\nu(y)$.
- Inductively: $\exists !$ "minimal" U_k s.t. $\bar\beta_k := \nu(U_k)$ not determined by $\bar\beta_0 := \nu(U_0), \ldots, \bar\beta_{k-1} := \nu(U_{k-1})$ ($U_0 = x$, $U_1 = y$).

STKP's

- Formally an STKP is given by $(U_j)_0^k$ and $(\bar{\beta}_j)_0^k$, $1 \leq k \leq \infty$ s.t.
  - $U_j \in \mathbf{C}[x, y]$ for all j, $\bar{\beta}_j \in \mathbf{Q}$ for $j < k$ and $\bar{\beta}_k \in [1, \infty]$;
  - $U_0 = x$, $U_1 = y$;
  - $U_{j+1} = U_j^{n_j} - \theta_j U_0^{m_{j,0}} \dots U_{j-1}^{m_{j,j-1}}$, $\theta_j \in \mathbf{C}^*$;
  - arithmetic properties of $\bar{\beta}_j$'s, n_j's, $m_{j,i}$'s.
- Main result: STKP's correspond 1-1 to normalized valuations.
- Proof of this is inductive (for both implications).
- Key ingredient: understanding of the (graded) ring $\mathbf{C}[[x, y]]/\sim$, where $\phi \sim \psi$ iff $\nu(\phi - \psi) > \nu(\phi)$.
- Ex: $k = 1$, $\bar{\beta}_0 = 1$, $\bar{\beta}_1 = s \in [1, \infty]$ gives

$$\nu\left(\sum a_{ij} x^i y^j\right) = \min\{i + sj \mid a_{ij} \neq 0\}.$$

Classification of valuations through STKP's

- Def: A valuation $\nu = \mathrm{val}[(U_j)_0^k; (\bar\beta_j)_0^k]$ is a
  - toroidal valuation if $k < \infty$ and $\bar\beta_k < \infty$;
    - ⋆ divisorial if $\bar\beta_k \in \mathbf{Q}$;
    - ⋆ irrational if $\bar\beta_k \notin \mathbf{Q}$;
  - curve valuation if $k < \infty$, $\bar\beta_k = \infty$ or $k = \infty$, $n_j = 1\ \forall j \gg 1$;
  - solenoidal valuation if $k = \infty$ and $n_j > 1$ for ∞ many $j \geq 1$;
- Classification can be expressed in terms of numerical invariants, hence does not depend on coordinate choices (STKP's do):

Divisorial	$\mathrm{rk} = 1$	$\mathrm{rat.rk} = 1$	$\mathrm{tr.deg} = 1$
Irrational	$\mathrm{rk} = 1$	$\mathrm{rat.rk} = 2$	$\mathrm{tr.deg} = 0$
Curve	$\mathrm{rk} = 2$	$\mathrm{rat.rk} = 2$	$\mathrm{tr.deg} = 0$
Solenoidal	$\mathrm{rk} = 1$	$\mathrm{rat.rk} = 1$	$\mathrm{tr.deg} = 0$

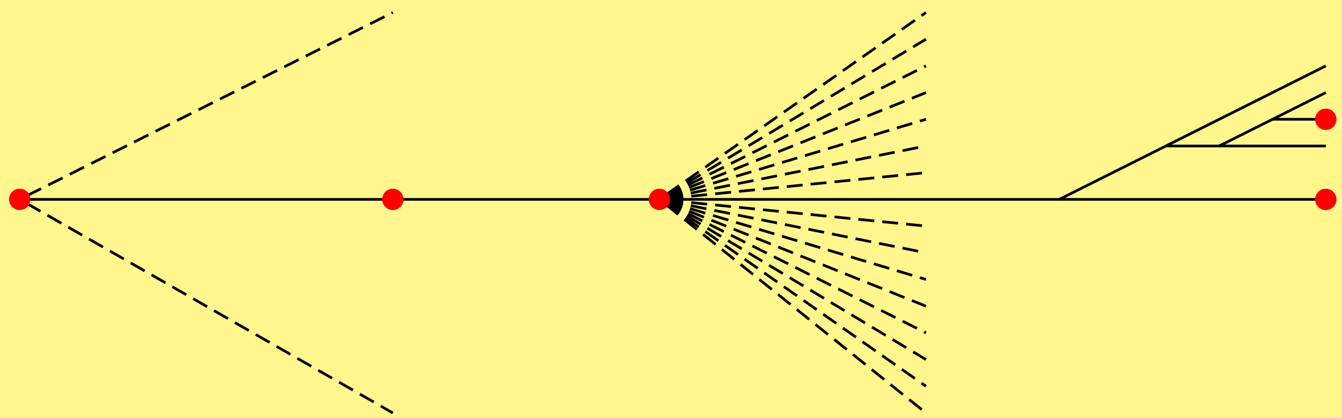
- Prop: A Krull valuation that is not associated to a valuation is an exceptional curve valuation.

Tree structure on $\mathcal{V}$

- Prop: The condition $\nu \leq \nu'$ in $\mathcal{V}$ can be formulated in terms of the STKP's for ν and ν'.
- Cor: $\mathcal{V}$ is a complete tree ($\bar{\beta}_k$ is a local parameter).
- Cor: Any family ν_i in $\mathcal{V}$ admits an infimum $\wedge_i \nu_i \in \mathcal{V}$.
- Prop: Weak convergence $\nu^n \to \nu$ in $\mathcal{V}$ can be formulated in terms of the STKP's for ν^n and ν.
- Cor: The weak topology on $\mathcal{V}$ coincides with the weak topology as a tree. It is compact.

Dendrology of $\mathcal{V}$

- Root (minimal element) of $\mathcal{V}$: $\nu_{\mathfrak{m}}$
- Ends (maximal elements) of $\mathcal{V}$:
  - curve valuations;
  - solenoidal valuations.
- Regular points of $\mathcal{V}$:
  - irrational valuations;
- Branch points of $\mathcal{V}$:
  - divisorial valuations;
  - tangent space isomorphic to $\mathbf{P}^1$;



Metric tree structure on $\mathcal{V}$

- Look for tree metrics on $\mathcal{V}$ invariant under autom. $f : R \to R$.
- Idea: find parameterization of segment $[\nu_{\mathfrak{m}}, \nu_\phi]$ in $\mathcal{V}$, $\phi \in \mathfrak{m}$ irr.
- Def/Prop: $\forall \alpha \in [1, \infty]$ $\exists!$ min $\nu = \nu_{\phi,\alpha} \in \mathcal{V}$ s.t. $\nu(\phi) \geq \alpha \nu_{\mathfrak{m}}(\phi)$. This gives parameterization

$$[1, \infty] \ni \alpha \mapsto \nu_{\phi,\alpha} \in [\nu_{\mathfrak{m}}, \nu_\phi].$$

- Prop: the skewness of $\nu_{\phi,\alpha}$ is α.
- We get an invariant tree metric d on $\mathcal{V}$ with diam 2 by declaring

$$d(\mu, \nu) := \left(\frac{1}{\alpha(\mu)} - \frac{1}{\alpha(\mu \wedge \nu)}\right) + \left(\frac{1}{\alpha(\nu)} - \frac{1}{\alpha(\mu \wedge \nu)}\right).$$

- Prop: This tree metric on $\mathcal{V}$ induces the strong topology on $\mathcal{V}$.

Valuations and intersection multiplicities

- A curve valuation ν_ϕ acts like $\nu_\phi(\psi) = \frac{\phi \cdot \psi}{m(\phi)}$, where "$\cdot$" denotes intersection multiplicity and "$m()$" multiplicity.
- Spivakovsky: if ν divisorial then $\nu(\psi) \sim \phi \cdot \psi$ for "ν-generic" ϕ (geometrically defined).
- In tree notation, for any toroidal $\nu \in \mathcal{V}$:

$$\nu(\psi) = \min \left\{ \frac{\phi \cdot \psi}{m(\phi)} \;\middle|\; \nu_\phi > \nu \right\}.$$

- Use this to define (intersection) multiplicities for valuations:

$$m(\nu) := \min\{m(\phi) \mid \nu_\phi \geq \nu\}$$

$$\mu \cdot \nu := \min \left\{ \frac{\phi \cdot \psi}{m(\phi) m(\psi)} \;\middle|\; \nu_\phi \geq \mu, \nu_\psi \geq \nu \right\}$$

- Prop: $\mu \cdot \nu = \alpha(\mu \wedge \nu)$.

Balls of curves

- Define an ultrametric d on $X = \{\text{local irr. curves}\}$ by
$$d(\phi,\psi) = \frac{m(\phi)m(\psi)}{\phi\cdot\psi}.$$
- Cor: $d(\phi,\psi) = 1/\alpha(\nu_\phi \wedge \nu_\psi)$.
- Cor: This metric on X is the one induced from $\mathcal{V}$ and $\mathcal{V}$ is isomorphic to the tree induced by the ultrametric space (X,d).
- Cor: If $\phi \in X$ and $r \in [0,1]$ then
$$\{\psi \in X \mid d(\phi,\psi) \leq r\} = \{\psi \in X \mid \nu_\psi \geq \nu_{\phi,\frac{1}{r}}\}$$
- Hence toroidal valuations can be identified with balls of curves.
- Spivakovsky: "classification of vals and curve sings are the same".

The lost valuations

- What about the Krull valuations that are not valuations (the exceptional curve valuations)?
- They correspond to tangent vectors on $\mathcal{V}$ at divisorial valuations.

Visualization of toroidal valuations

- Any toroidal valuation ν is of the form $\nu = \nu_{\phi,\alpha}$ with $\phi \in X$ and $\alpha \in [1, \infty)$. May assume ϕ polynomial of minimal multiplicity.
- The condition $\nu(\phi) = \alpha \nu_{\mathfrak{m}}(\phi)$ can be interpreted in $\mathbf{C}^2$ as

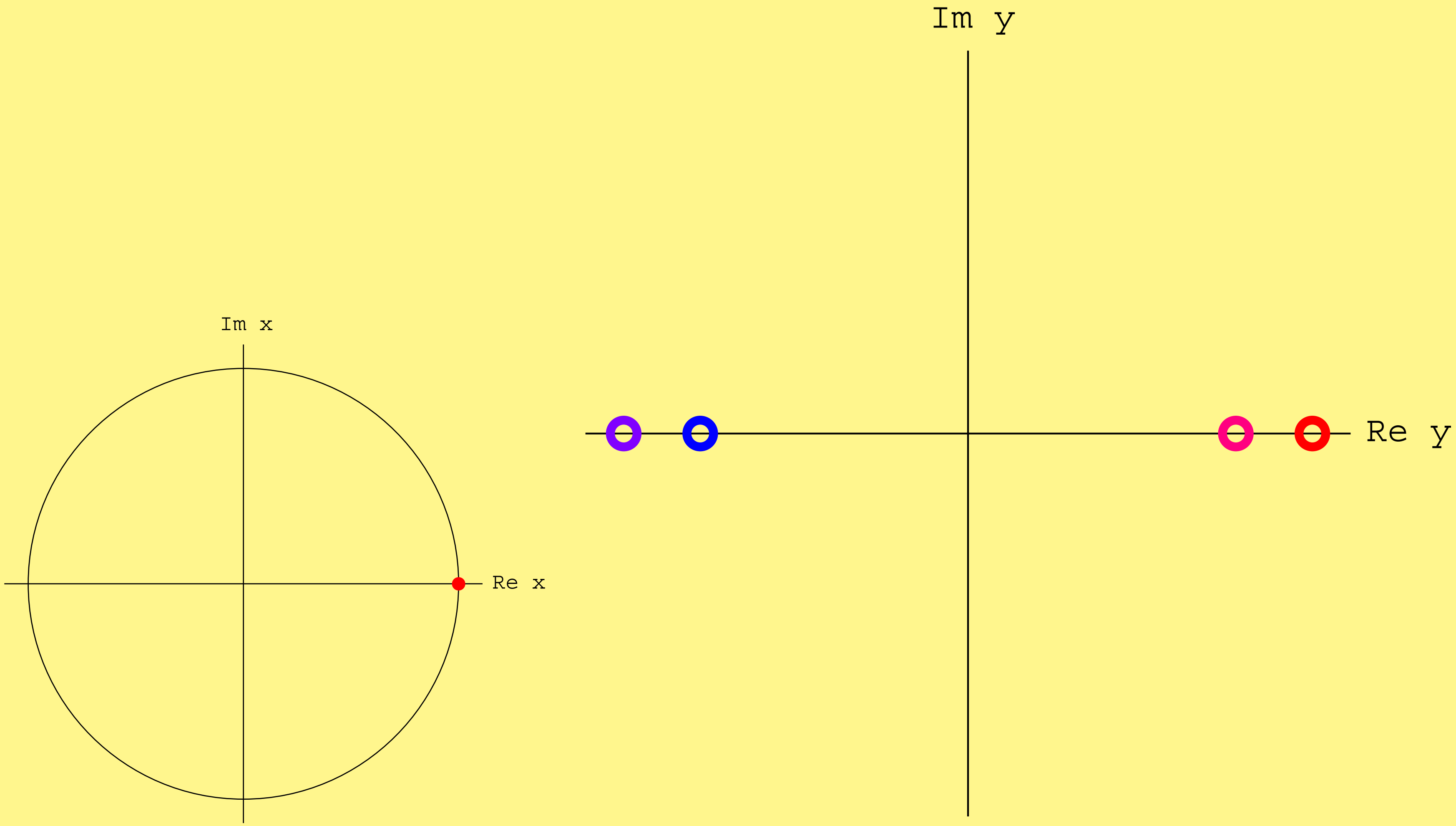
$$|\phi(x, y)| \sim |(x, y)|^{\alpha m(\phi)}$$

- Define (roughly) the domain in $\mathbf{C}^2$:

$$\Omega_{\nu,r} := \{|(x, y)| \sim r,\ |\phi(x, y)| \sim r^{\alpha m(\phi)}\}$$

  for $r \ll 1$. If $\psi \in \mathbf{C}[x, y]$, then $\nu(\psi) = \beta$, where $|\psi| \sim r^{\beta}$ on $\Omega_{\nu,r}$.
- The set $\Omega_{\nu,r}$ has a “toroidal structure” as illustrated in the next few slides.

Visualization of toroidal valuations



$$|x| \sim r, \qquad |(y^2 - x^3)^2 - x^9| \sim r^{10}$$

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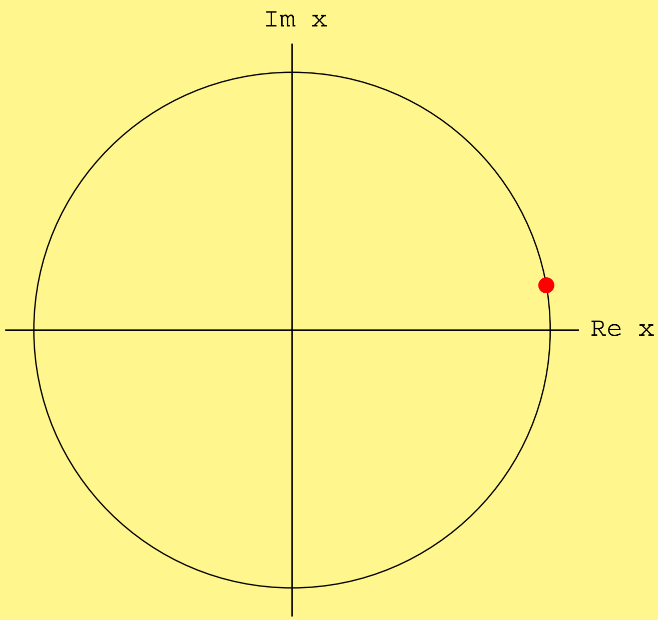

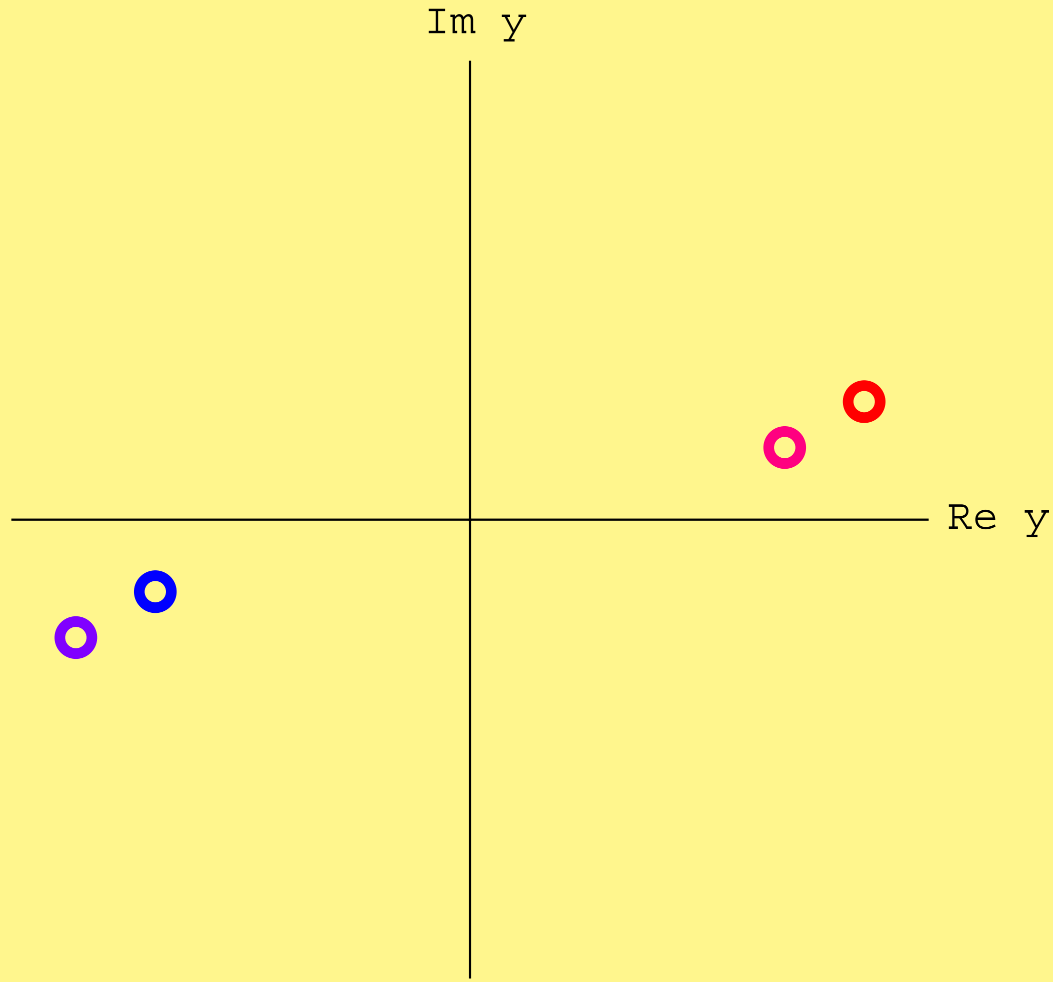

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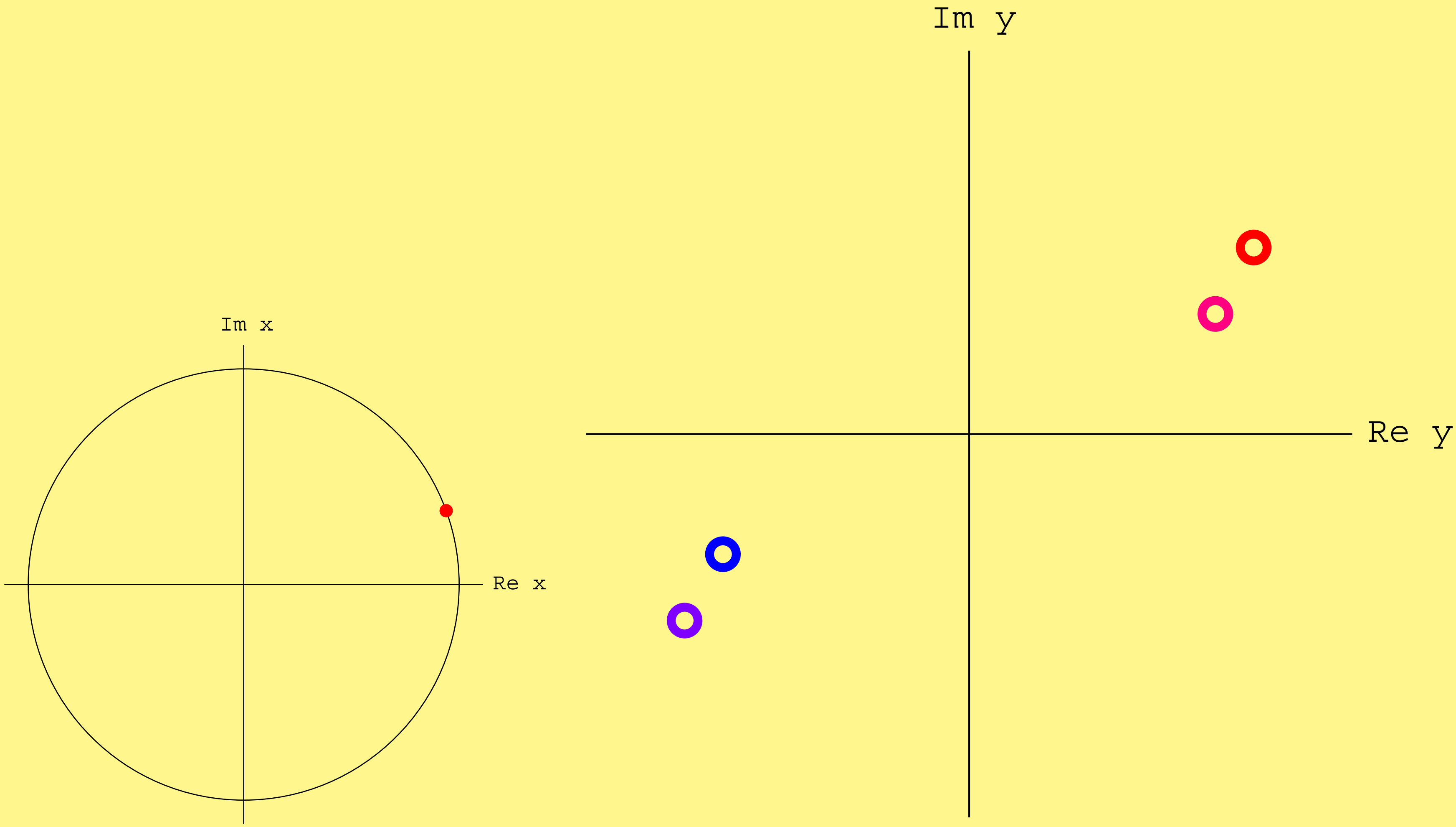

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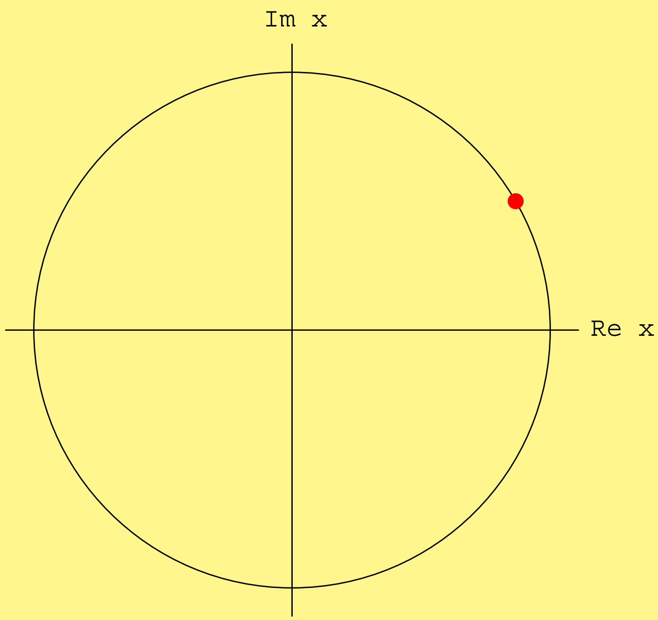

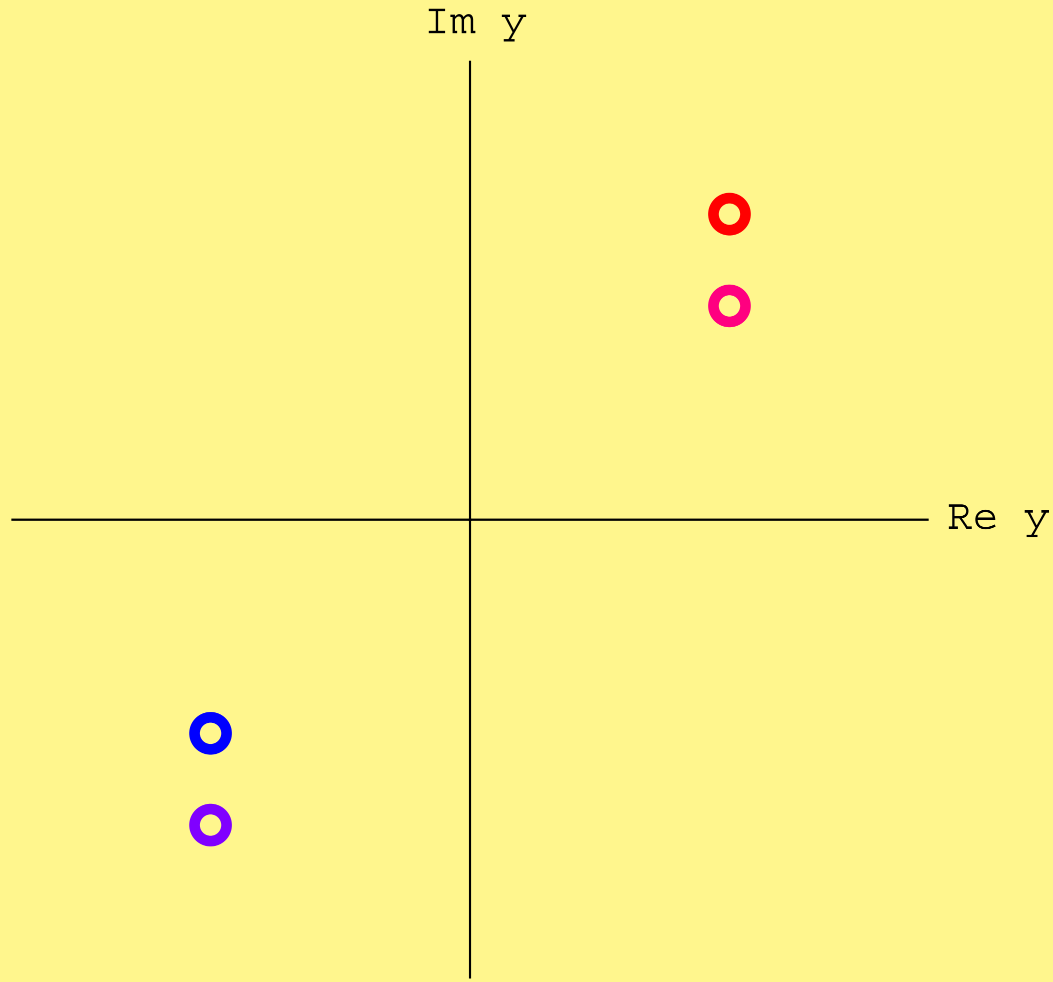

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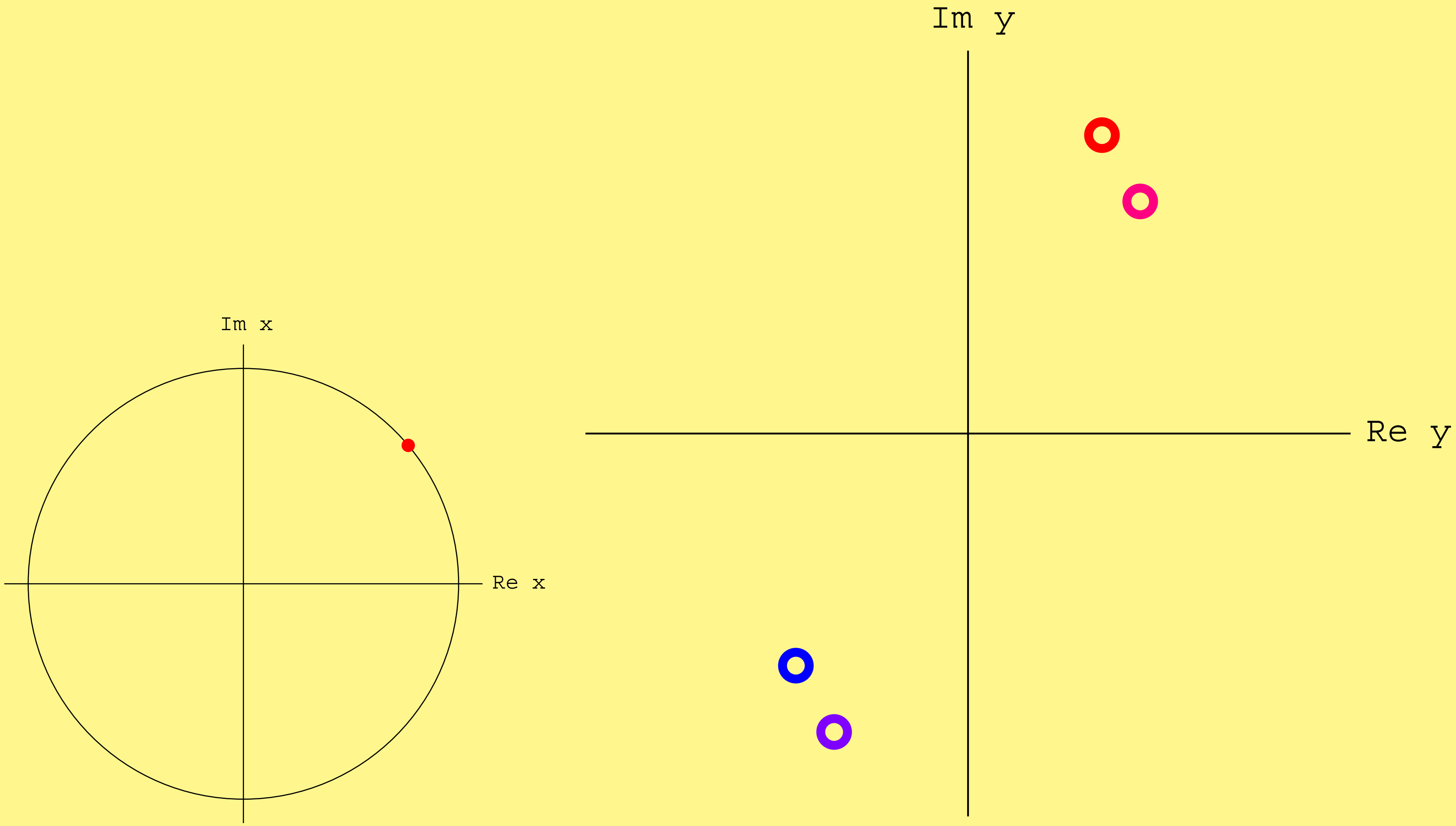

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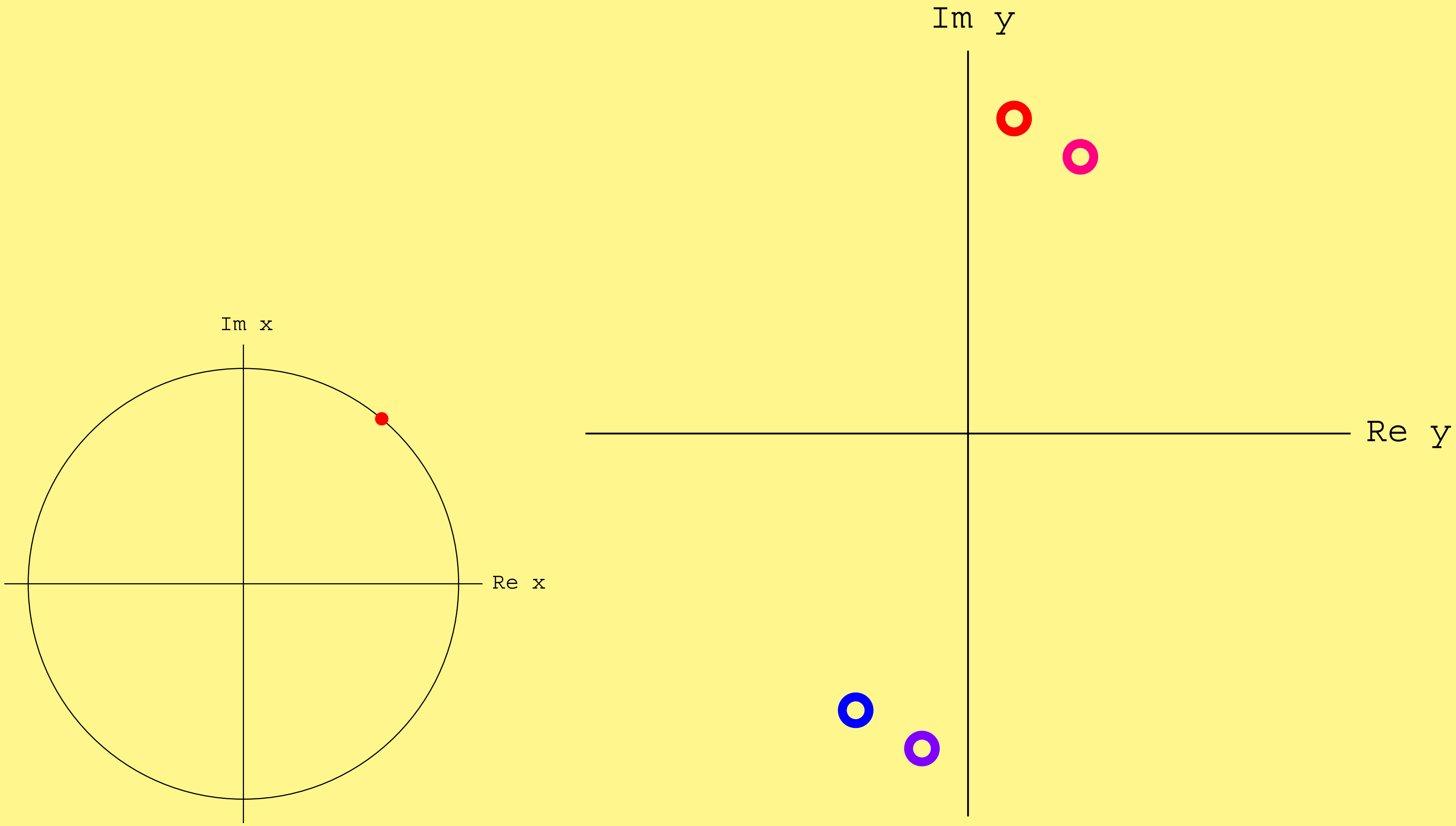

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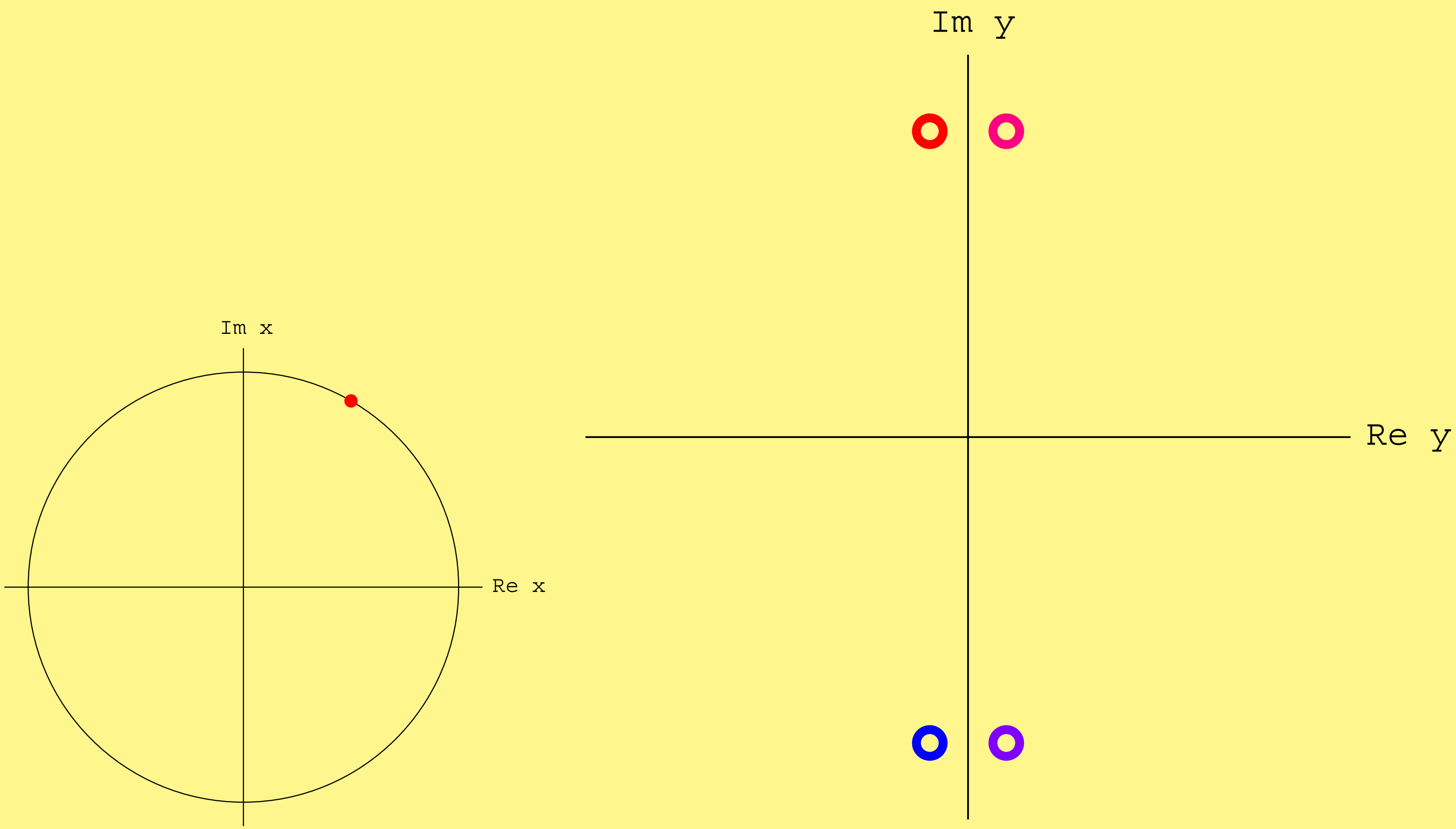

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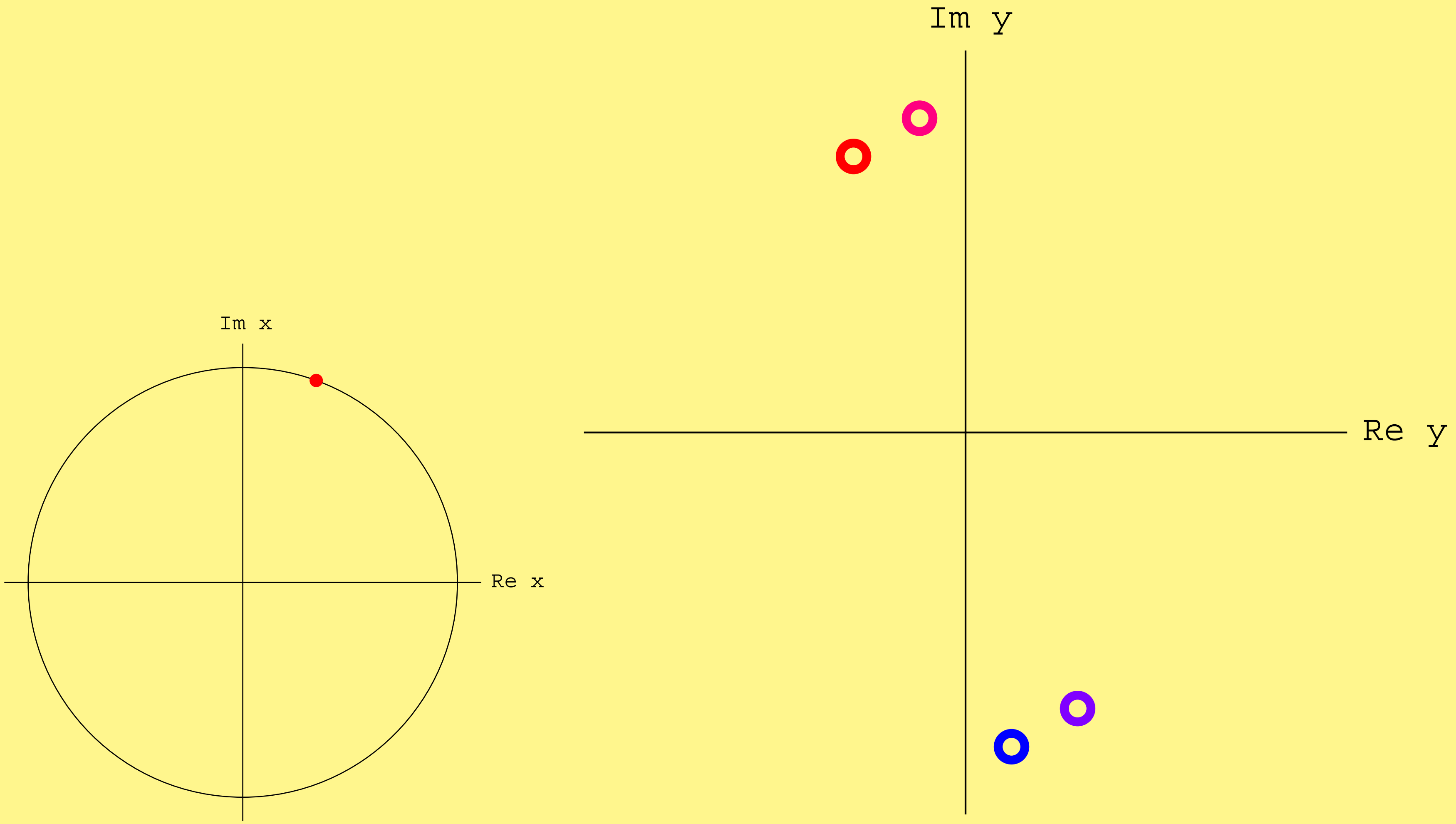

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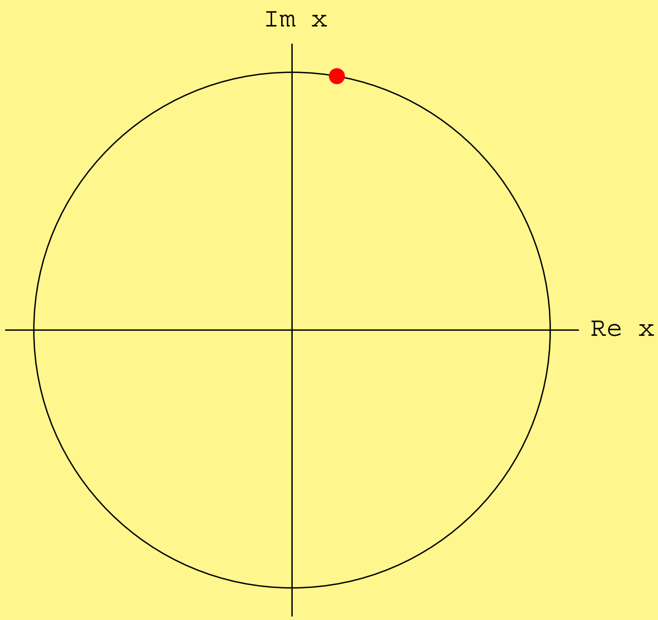

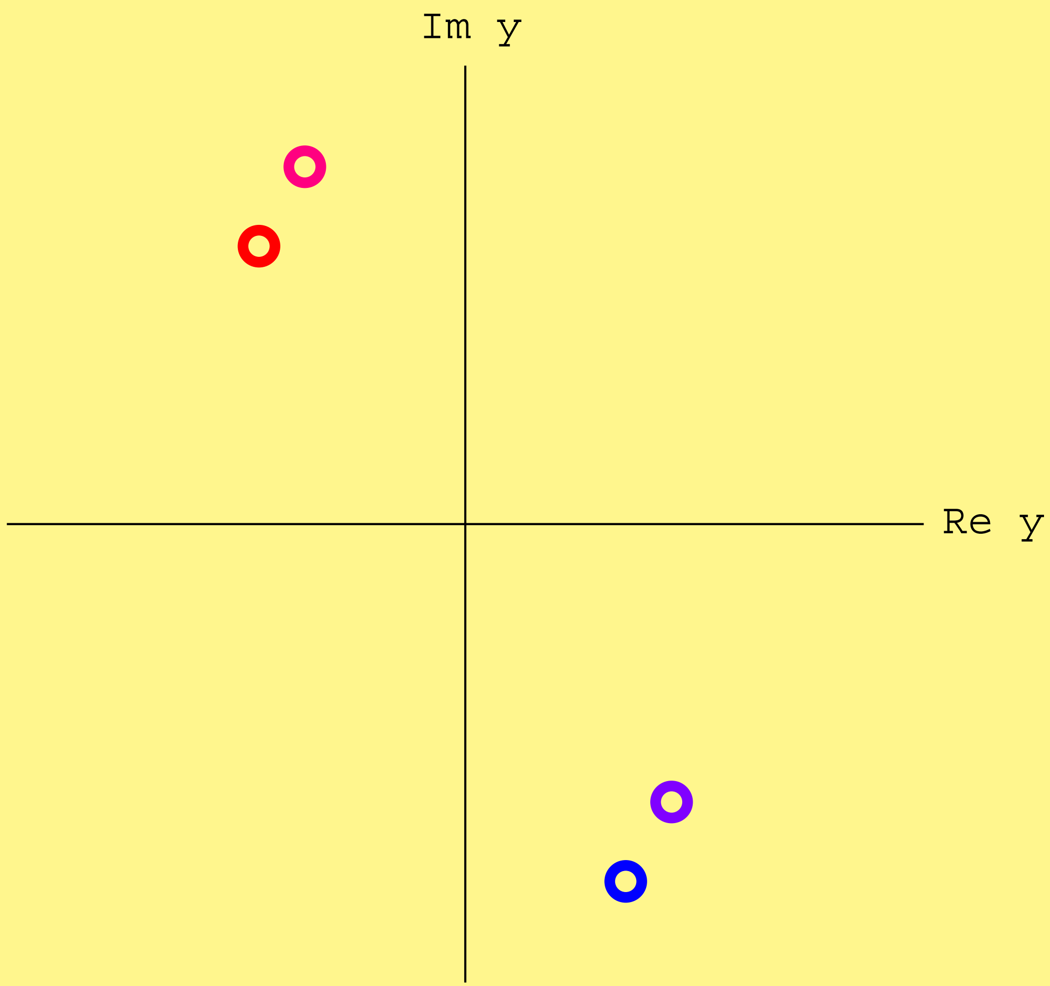

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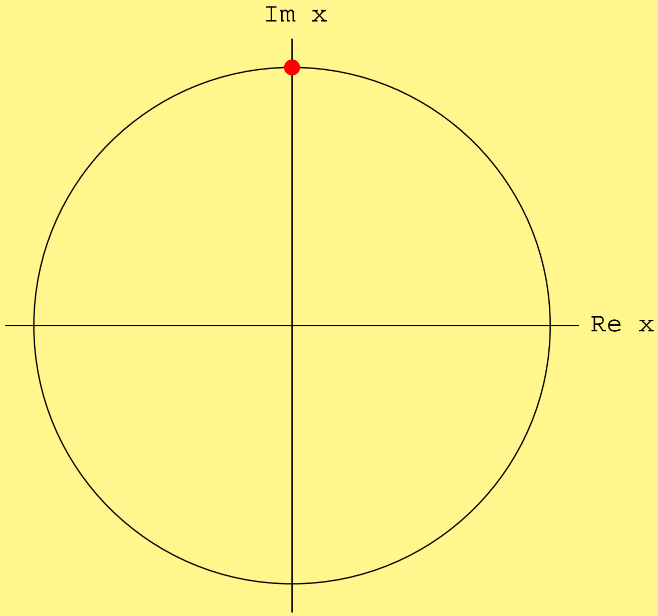

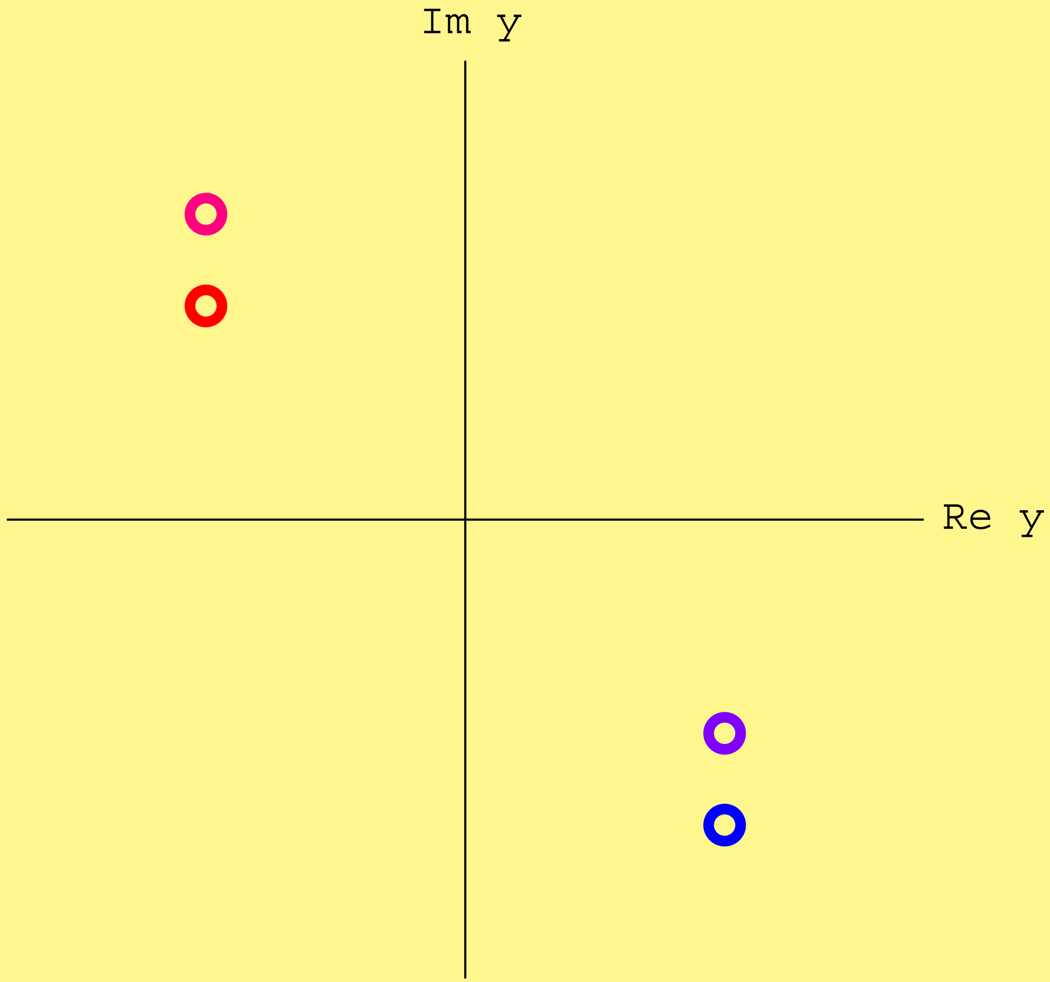

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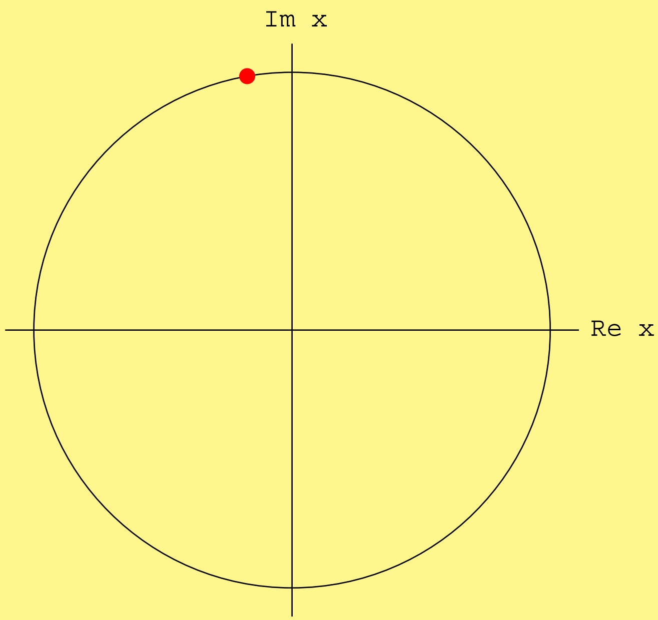

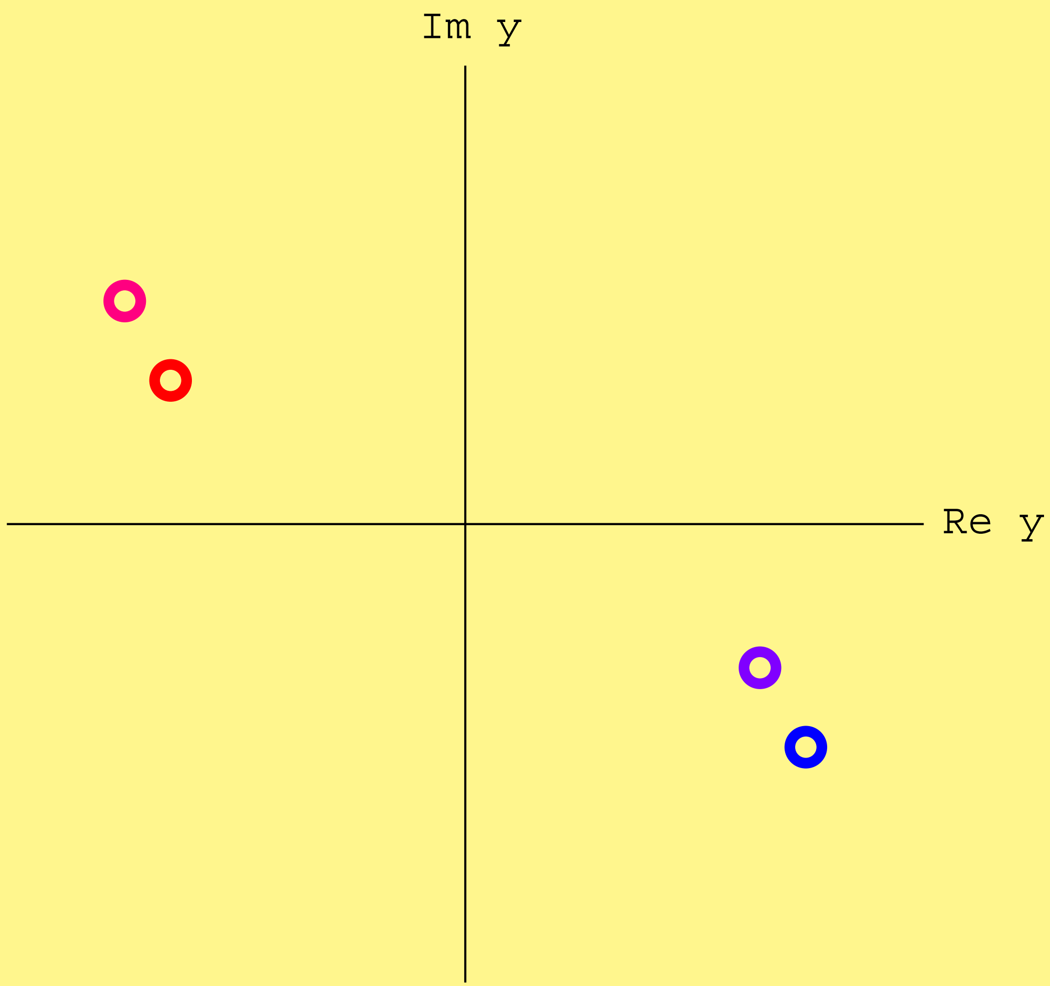

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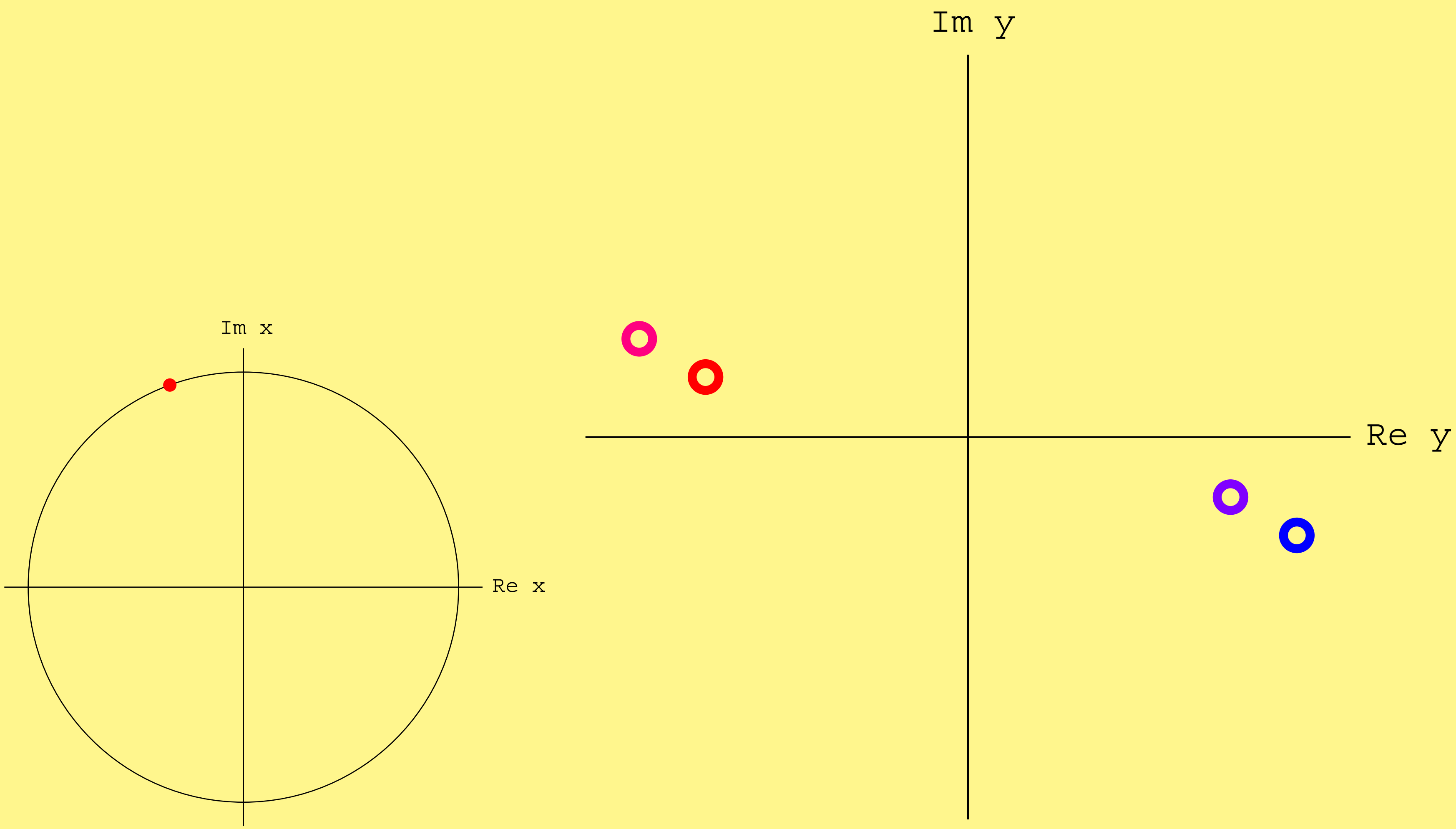

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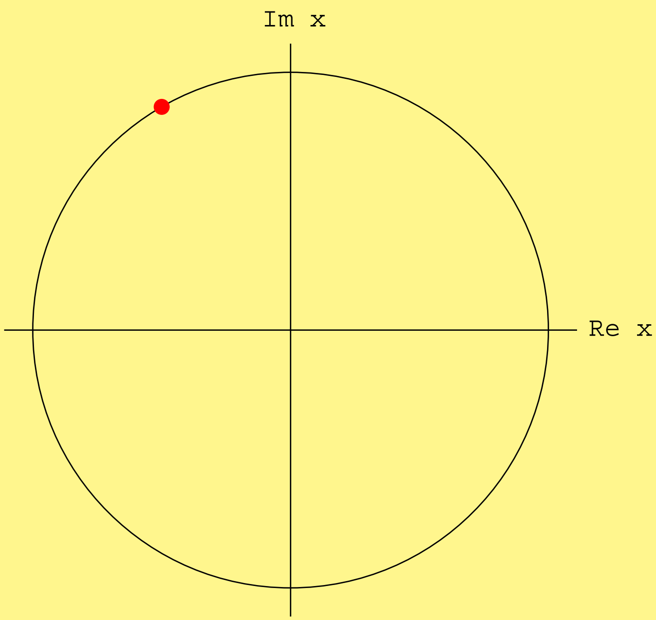

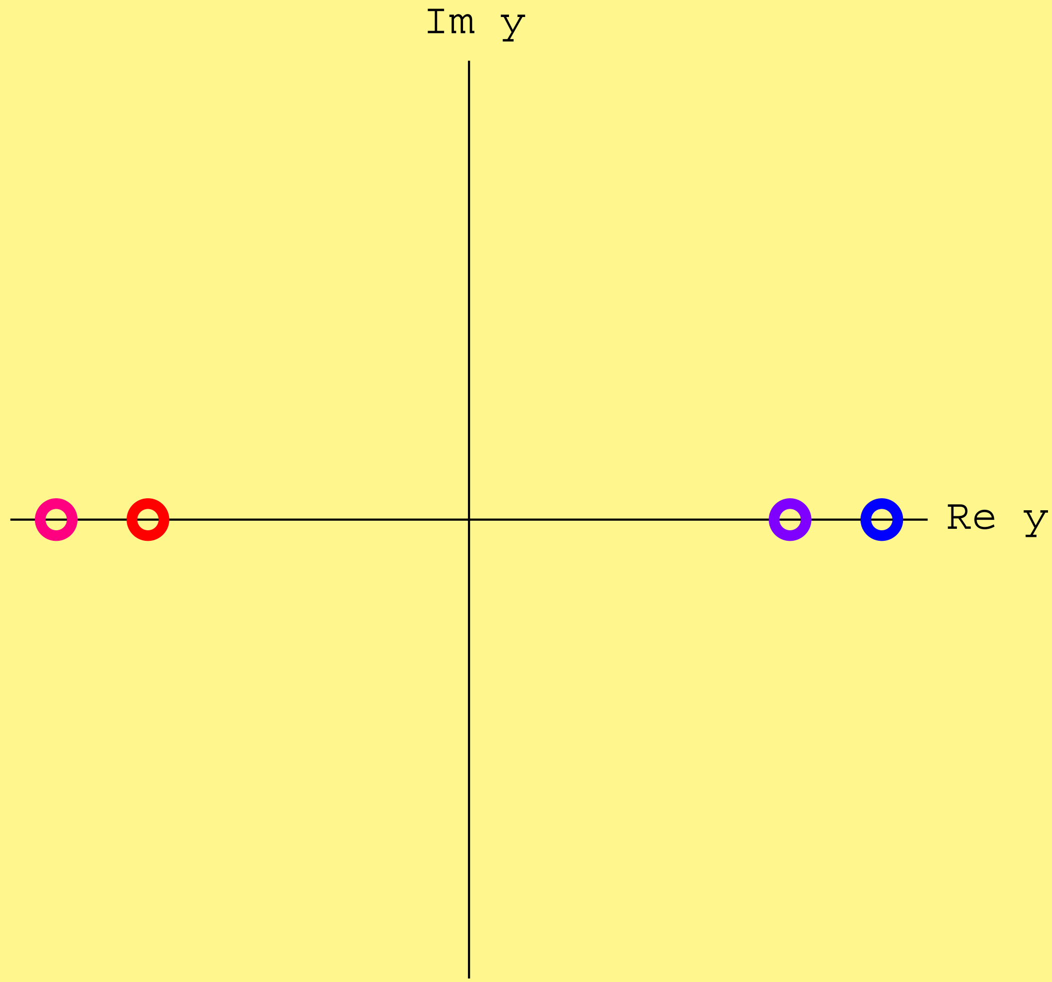

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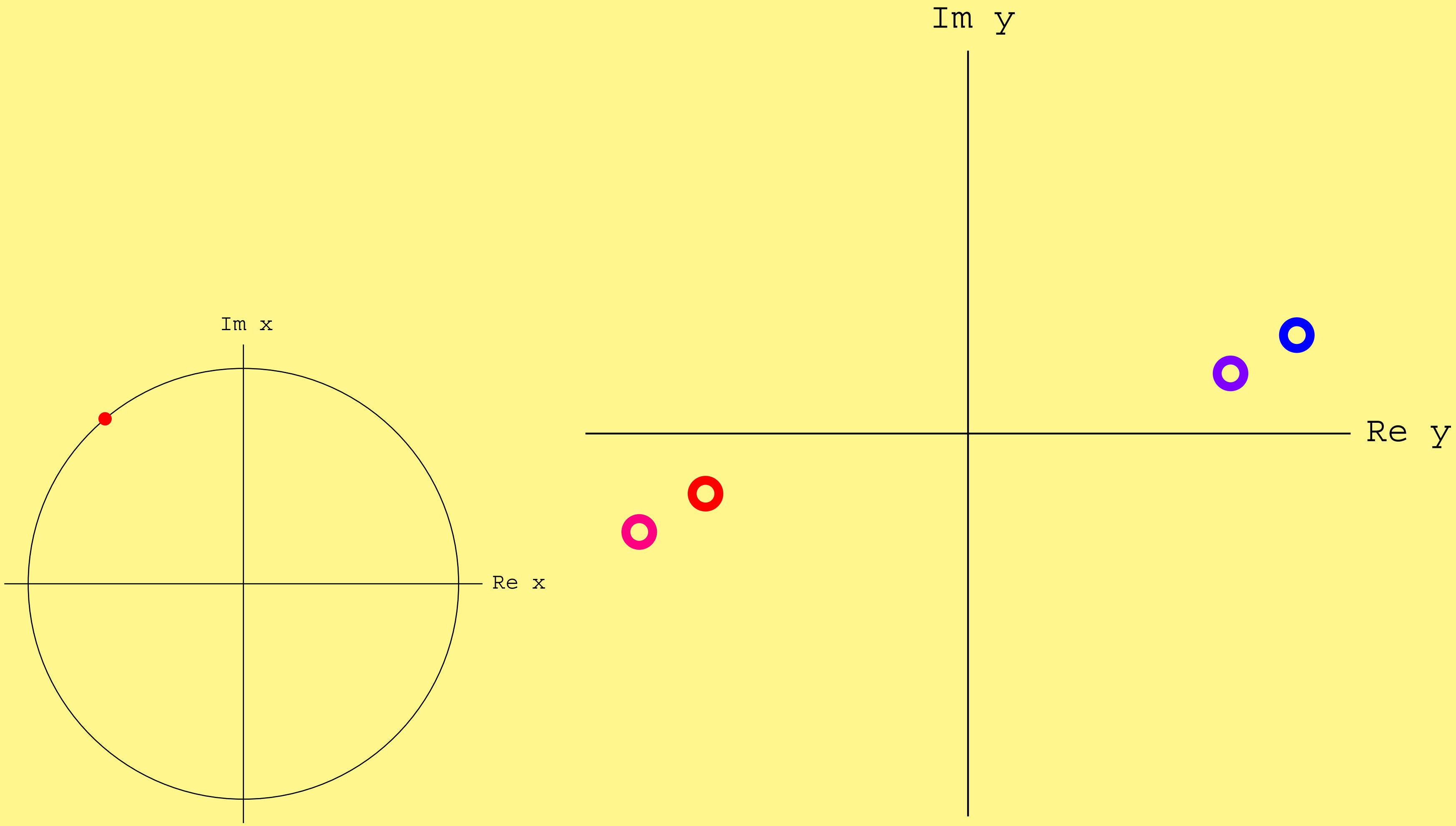

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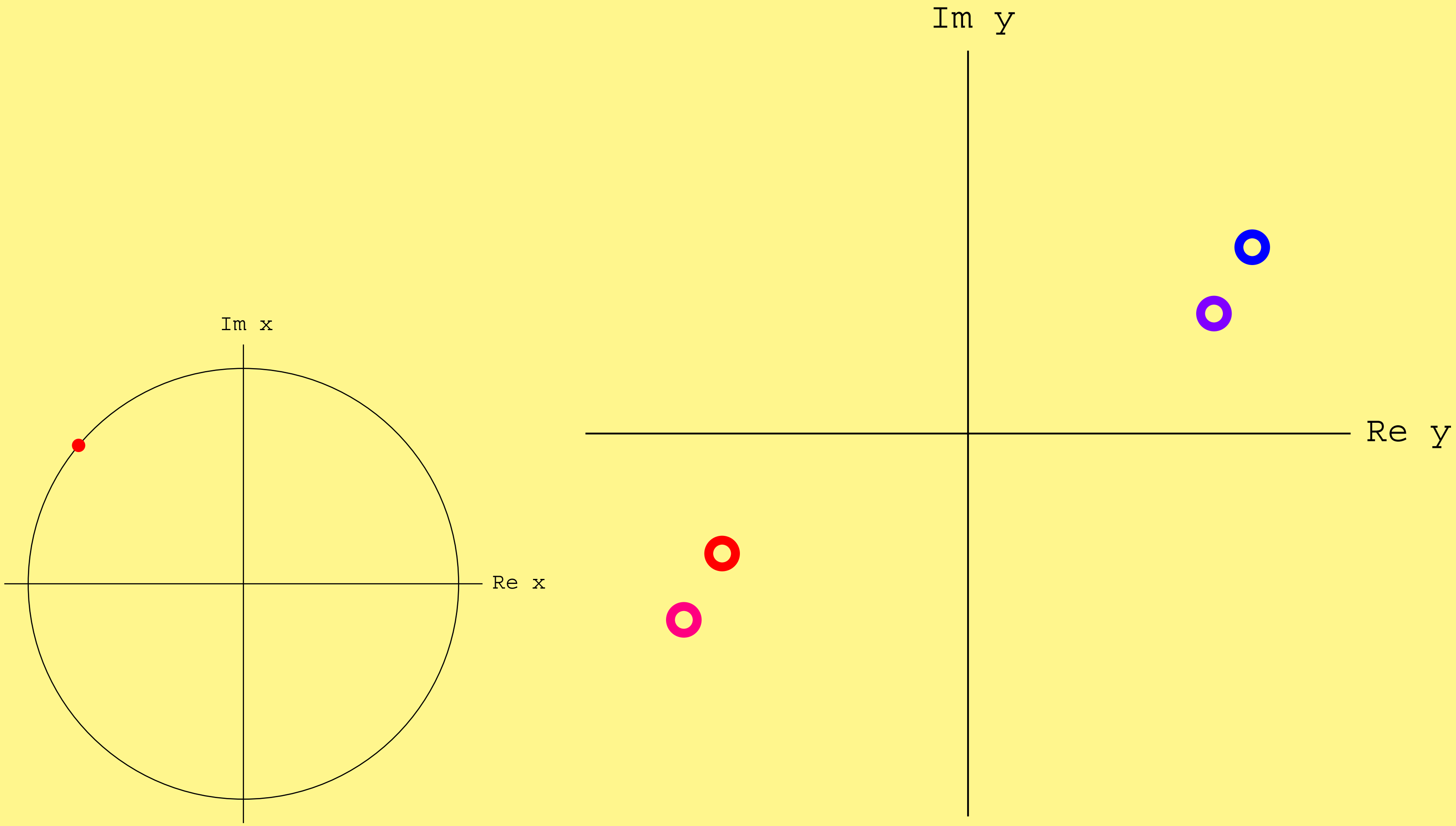

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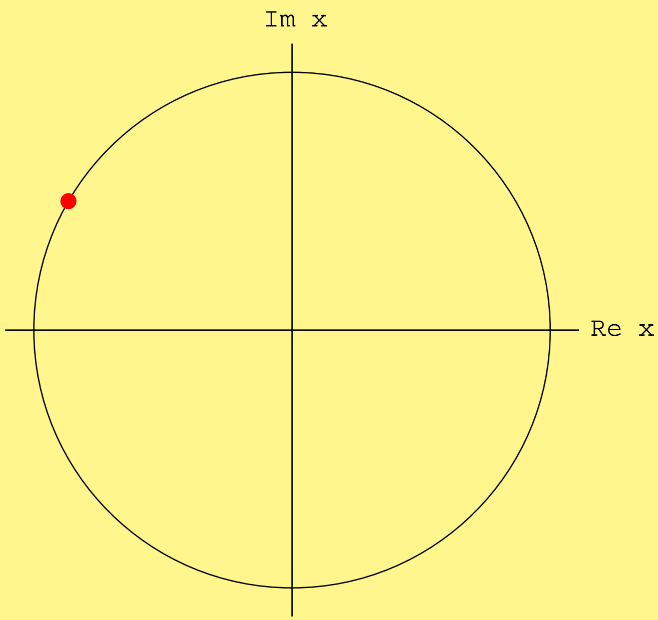

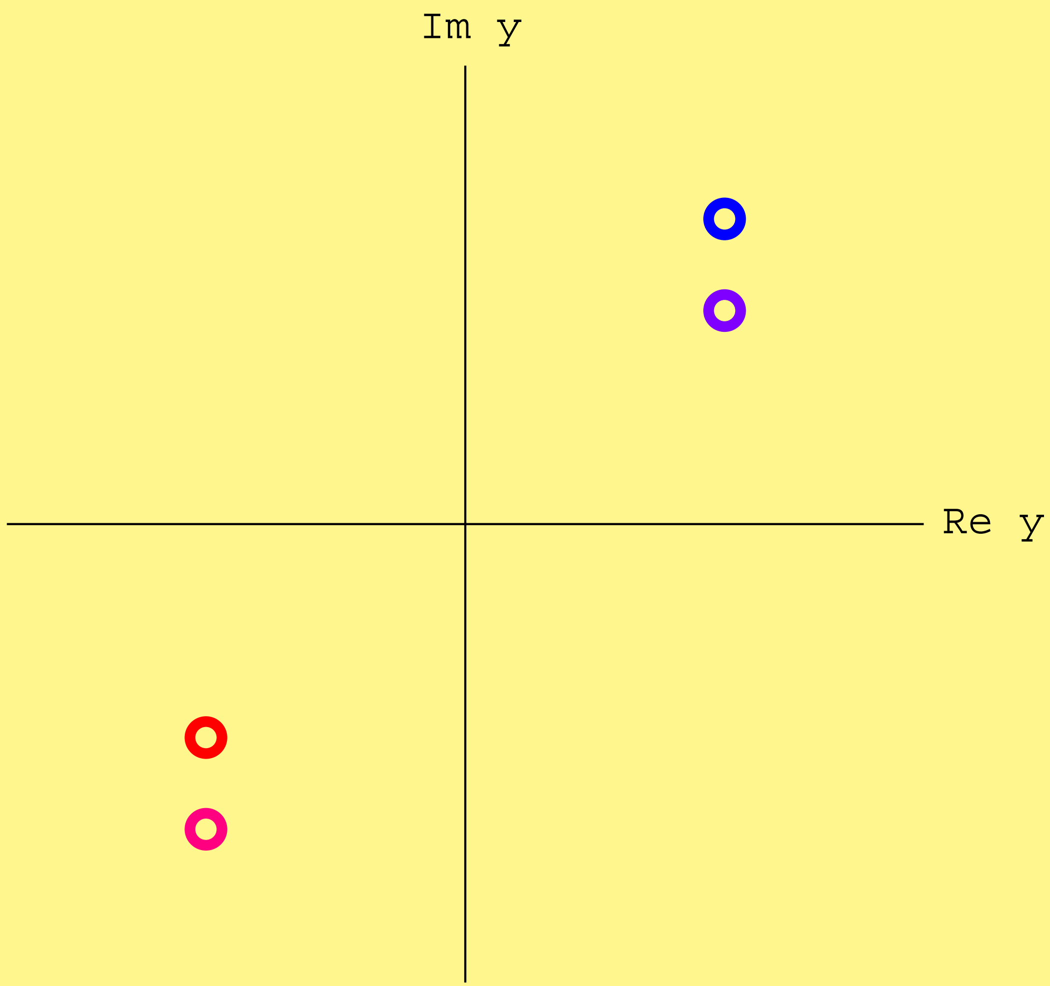

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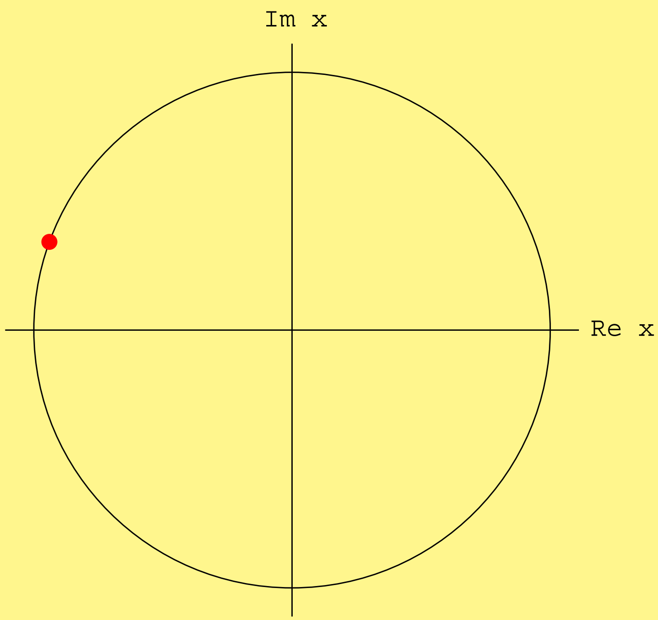

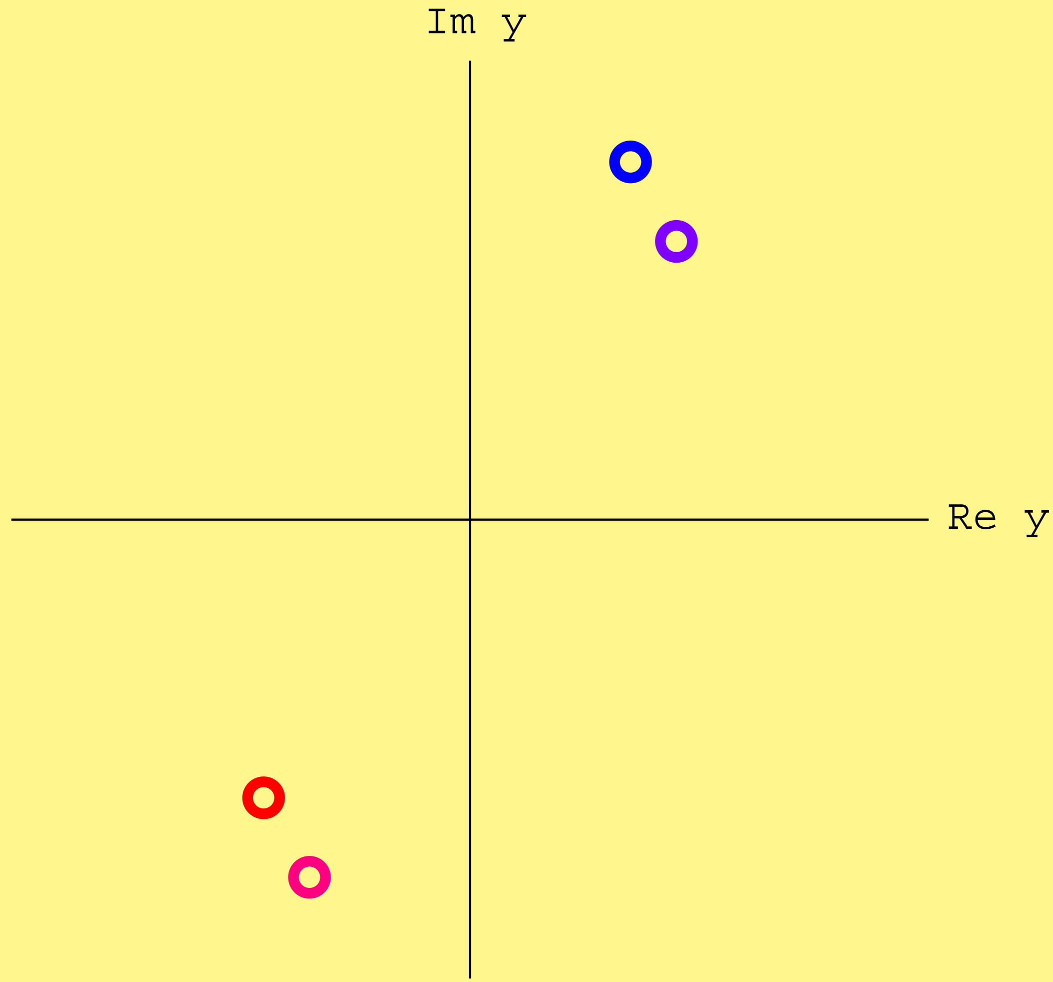

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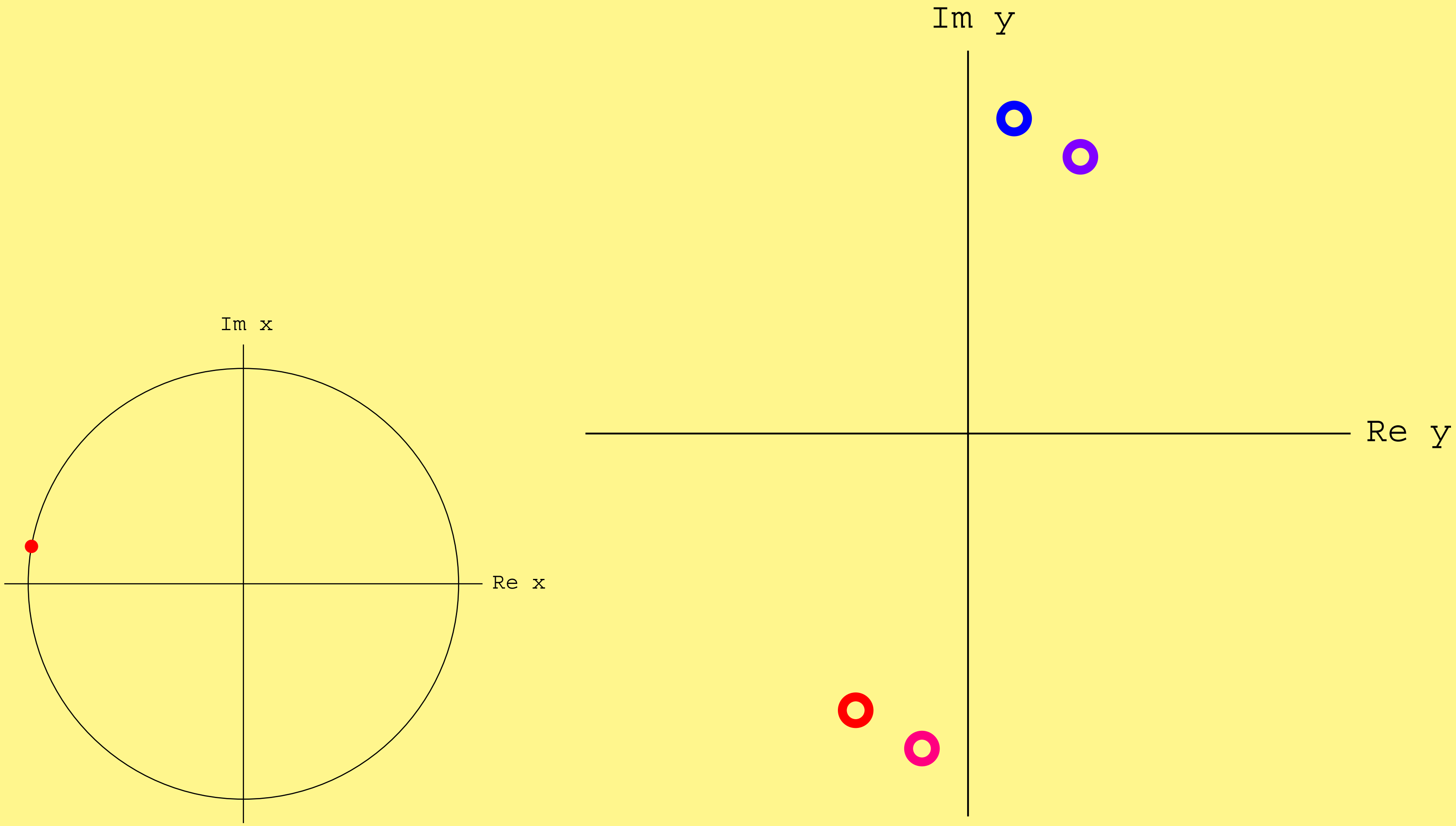

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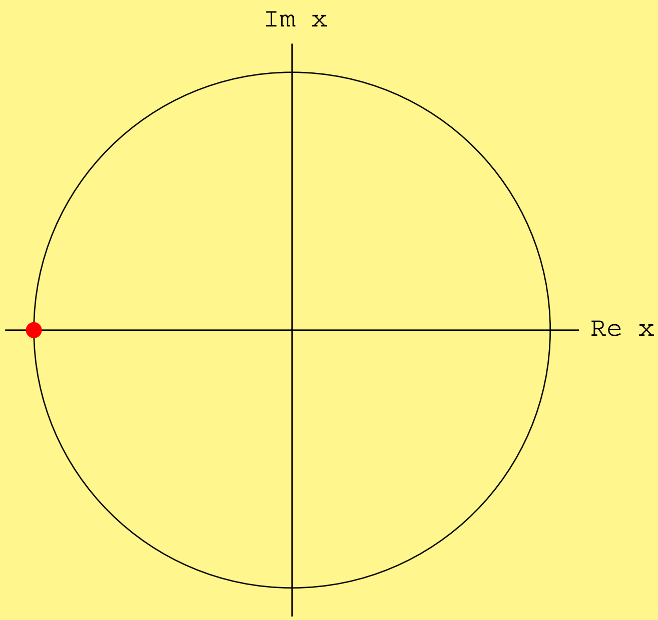

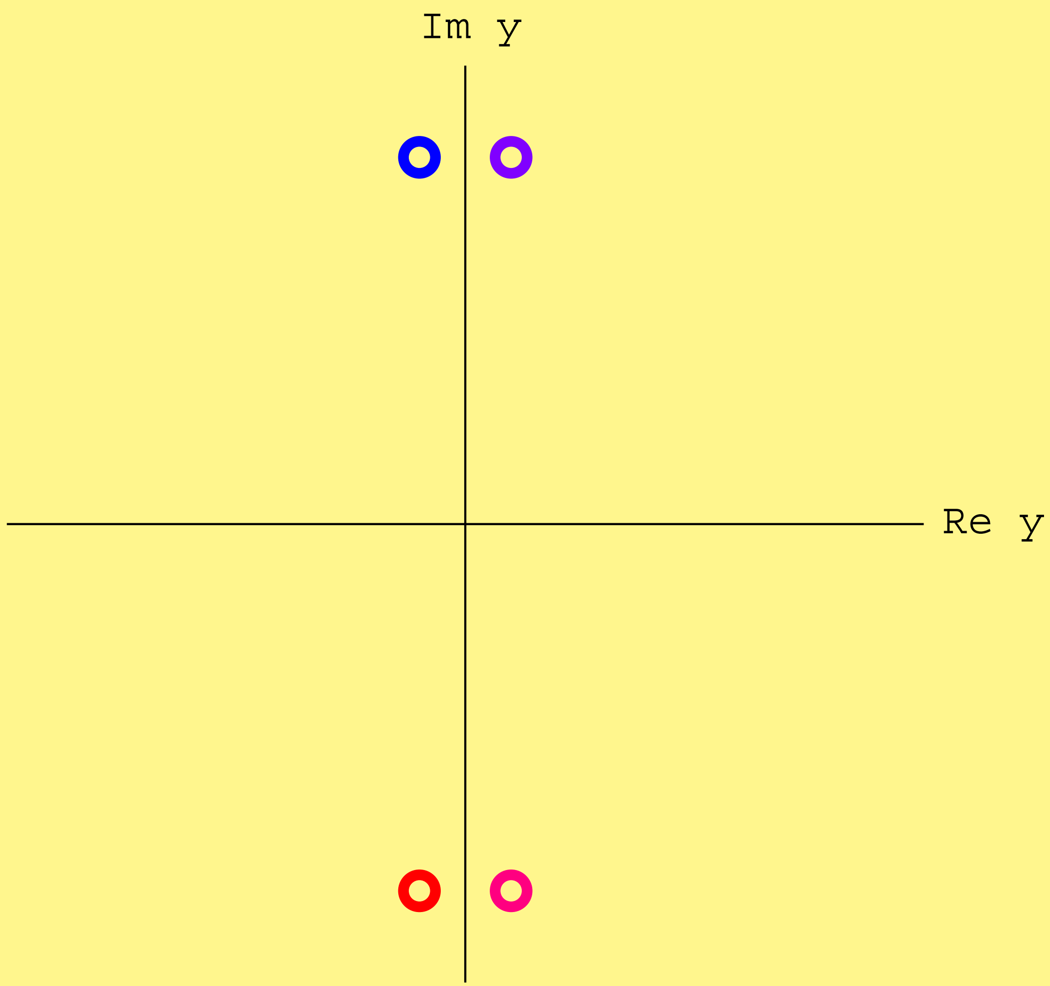

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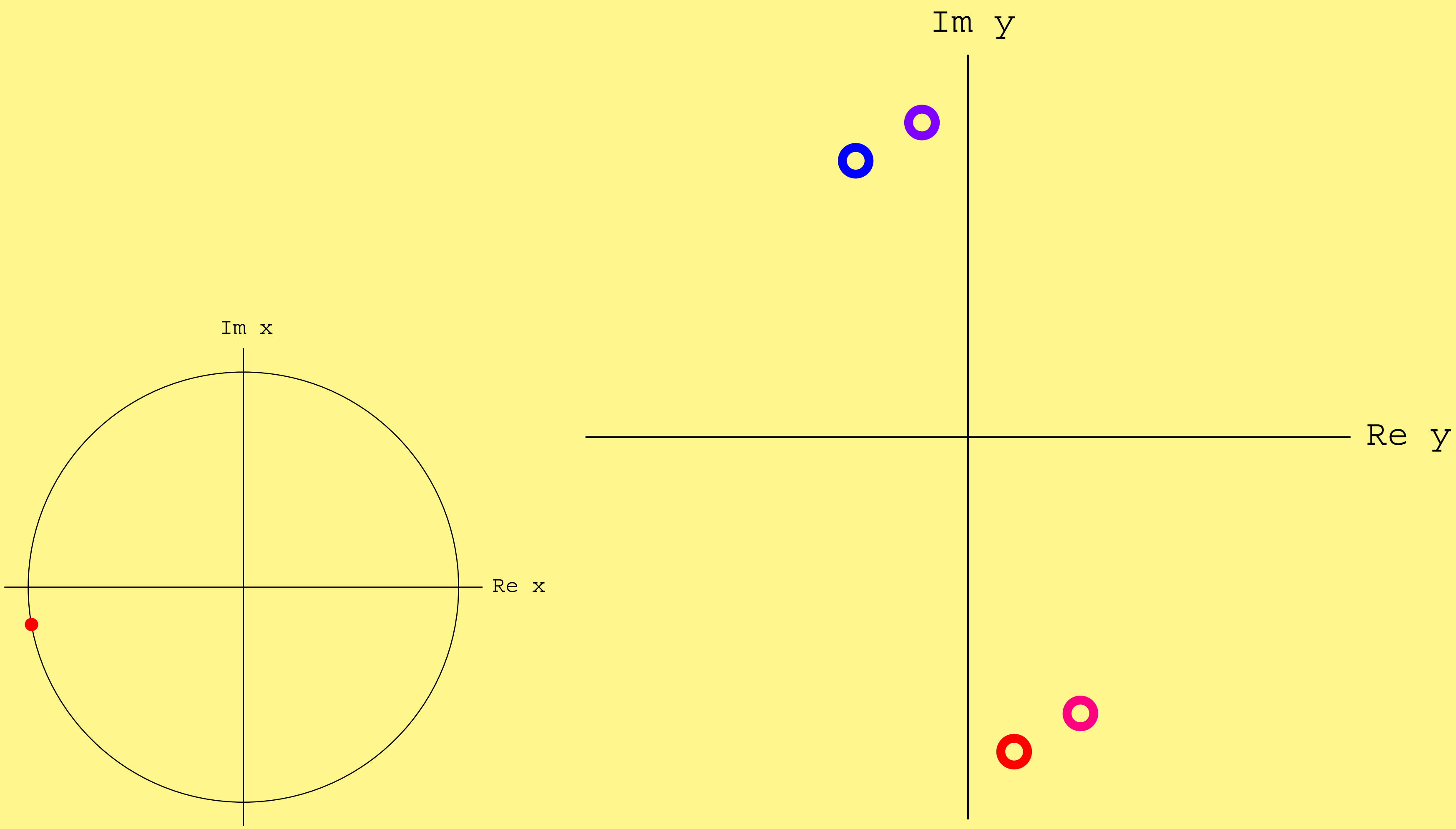
Visualization of toroidal valuations



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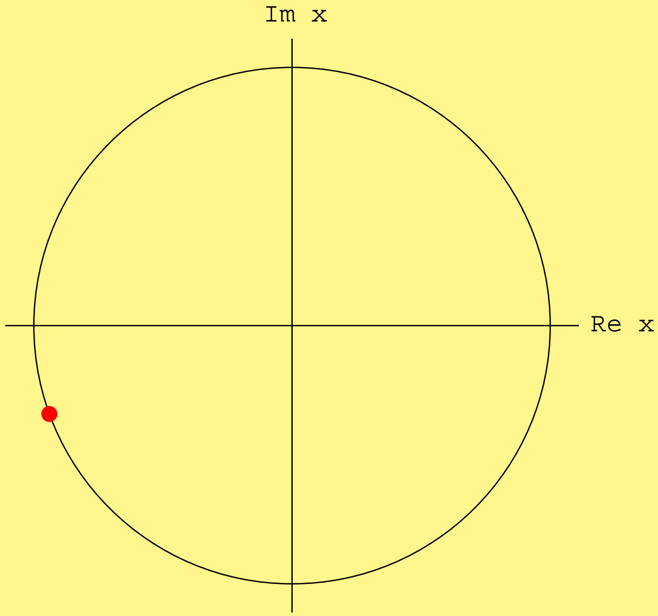
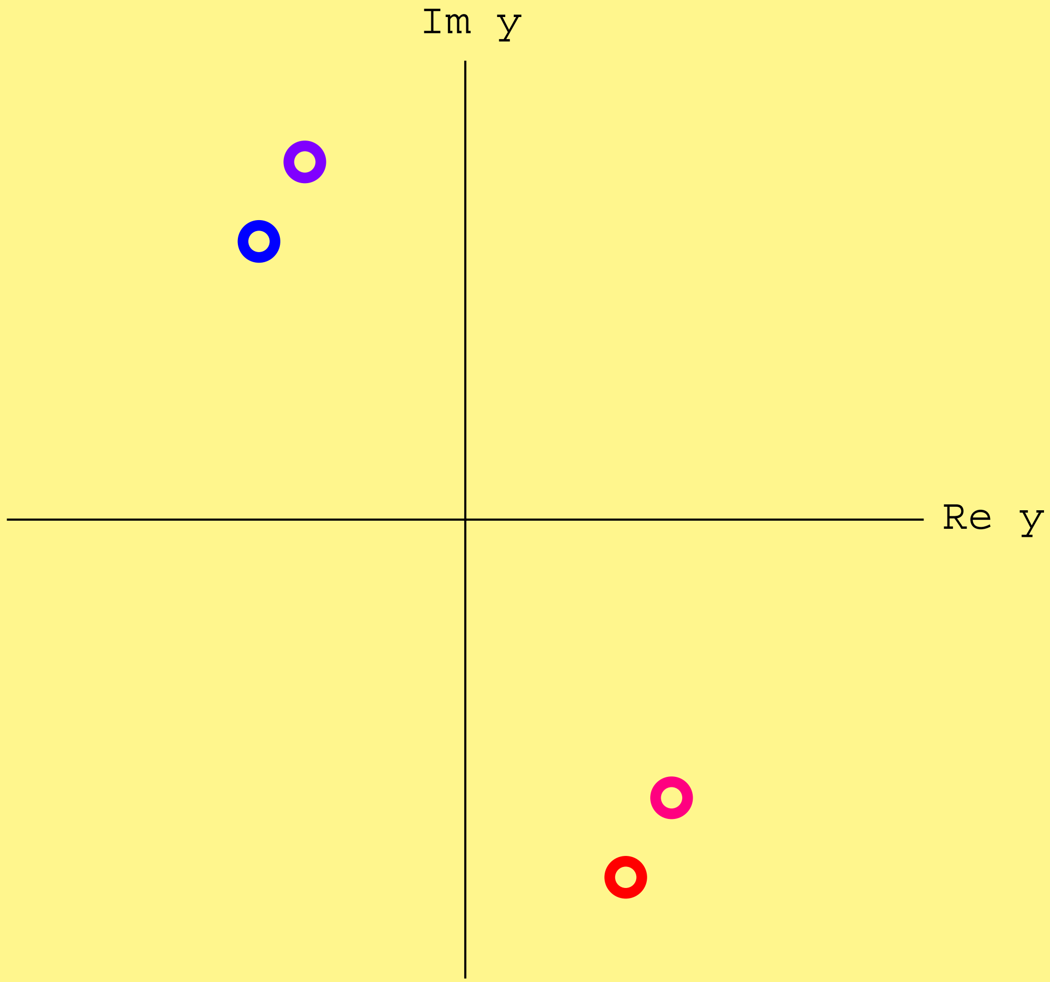
Visualization of toroidal valuations





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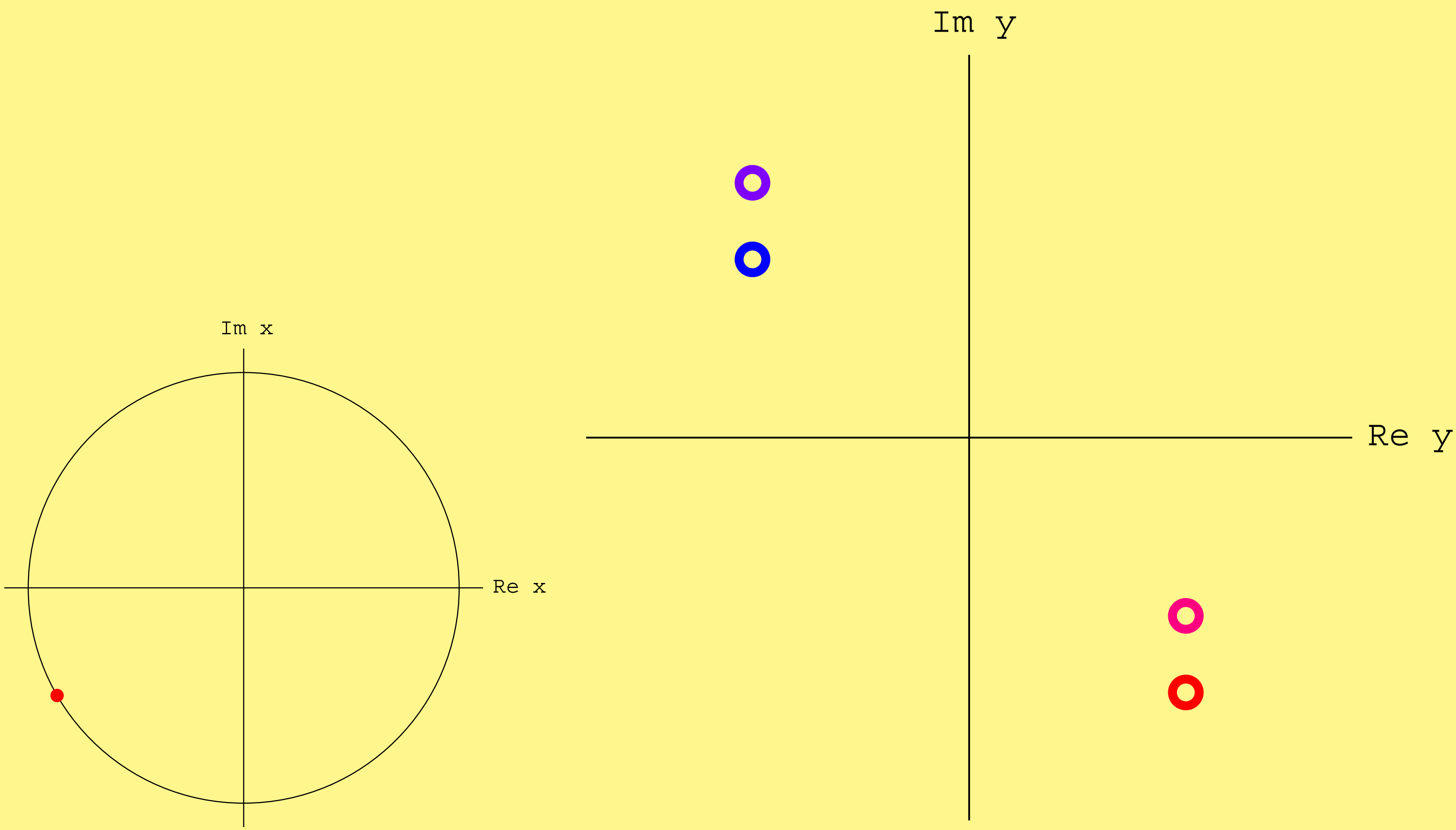
Visualization of toroidal valuations



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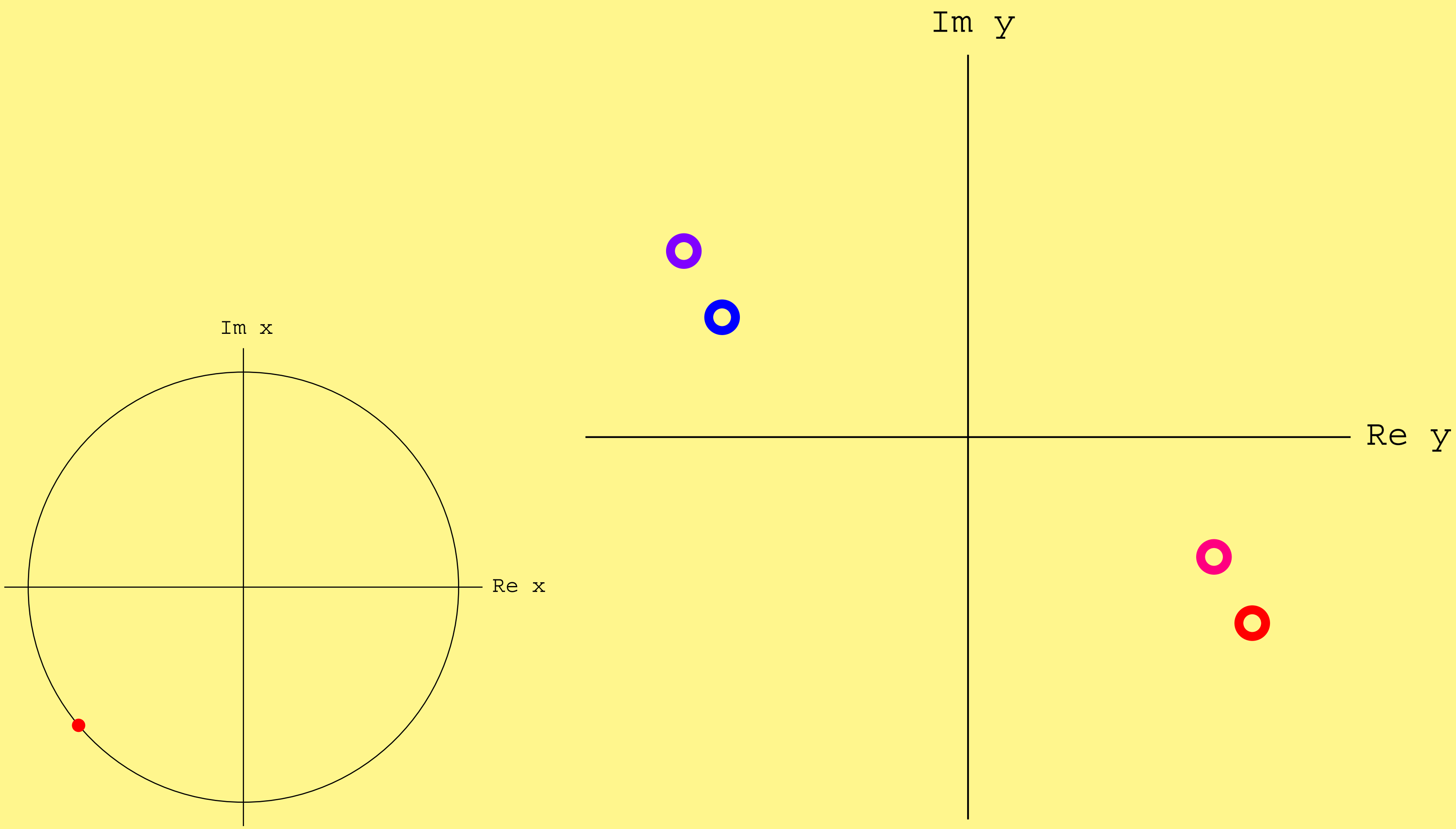
Visualization of toroidal valuations



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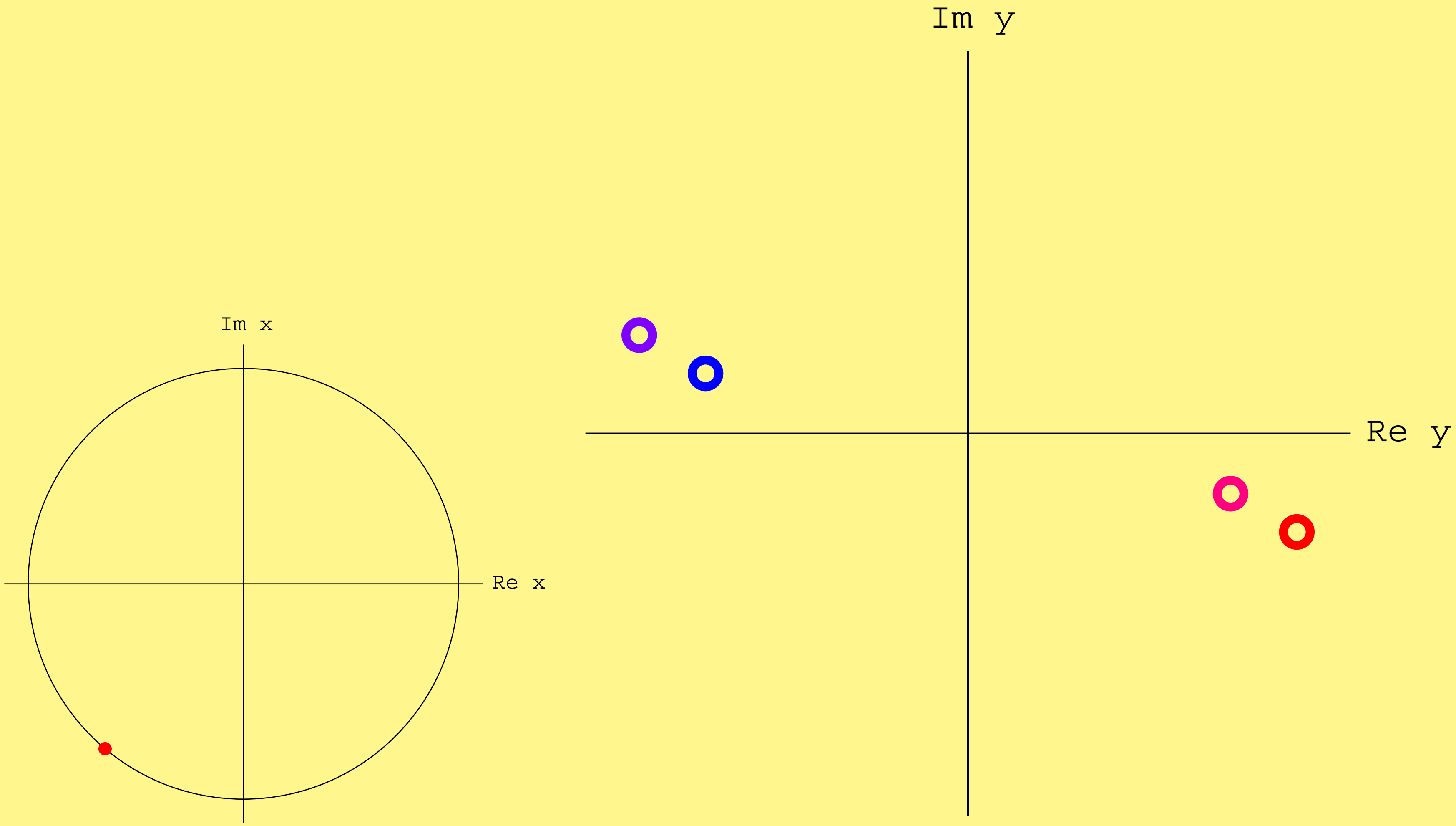
Visualization of toroidal valuations



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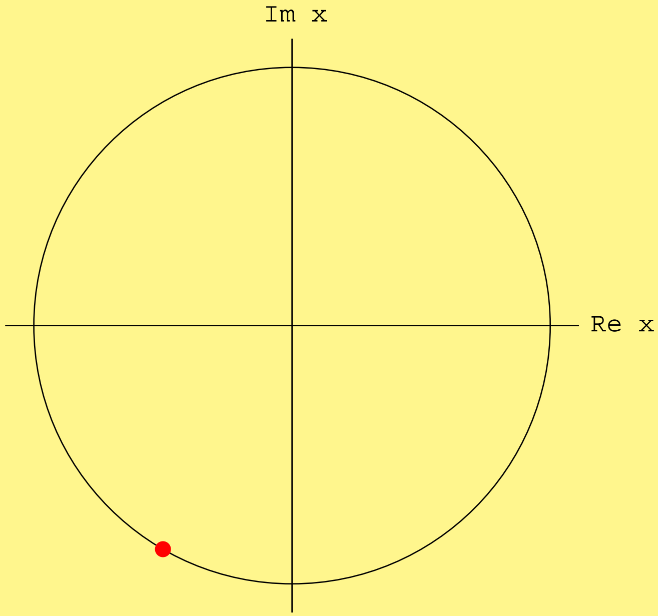
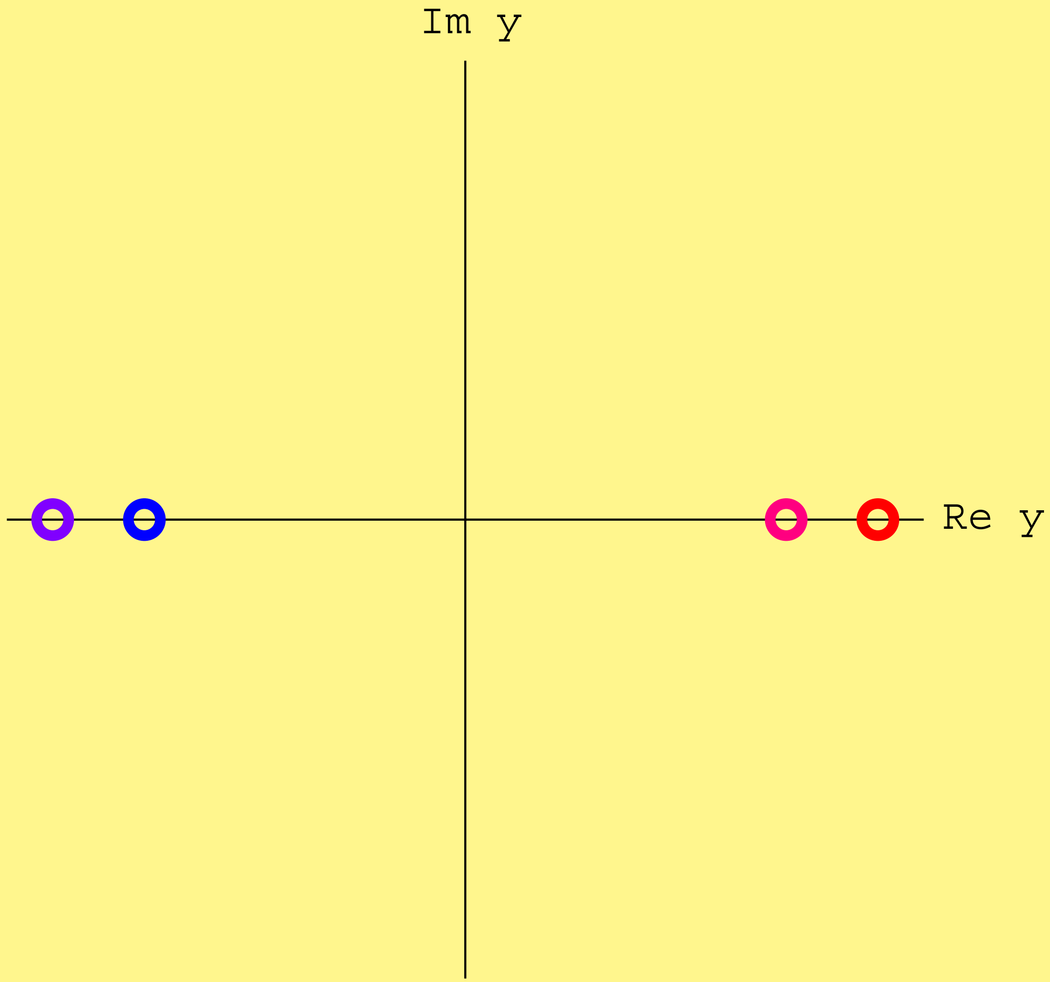
Visualization of toroidal valuations





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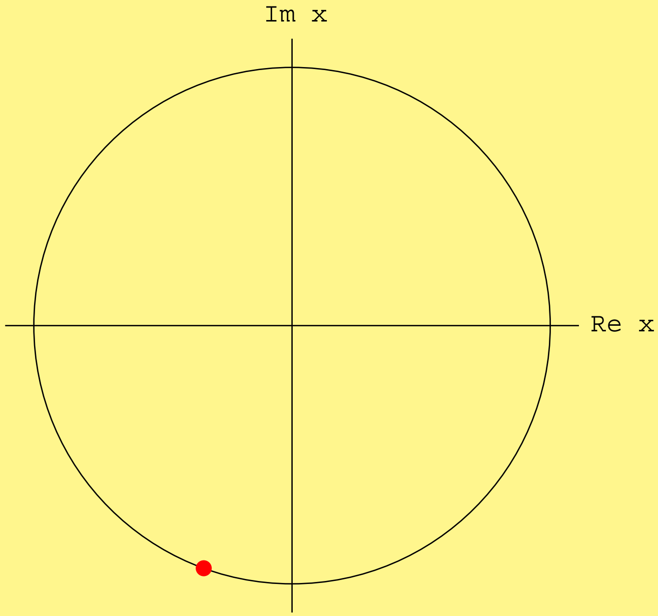
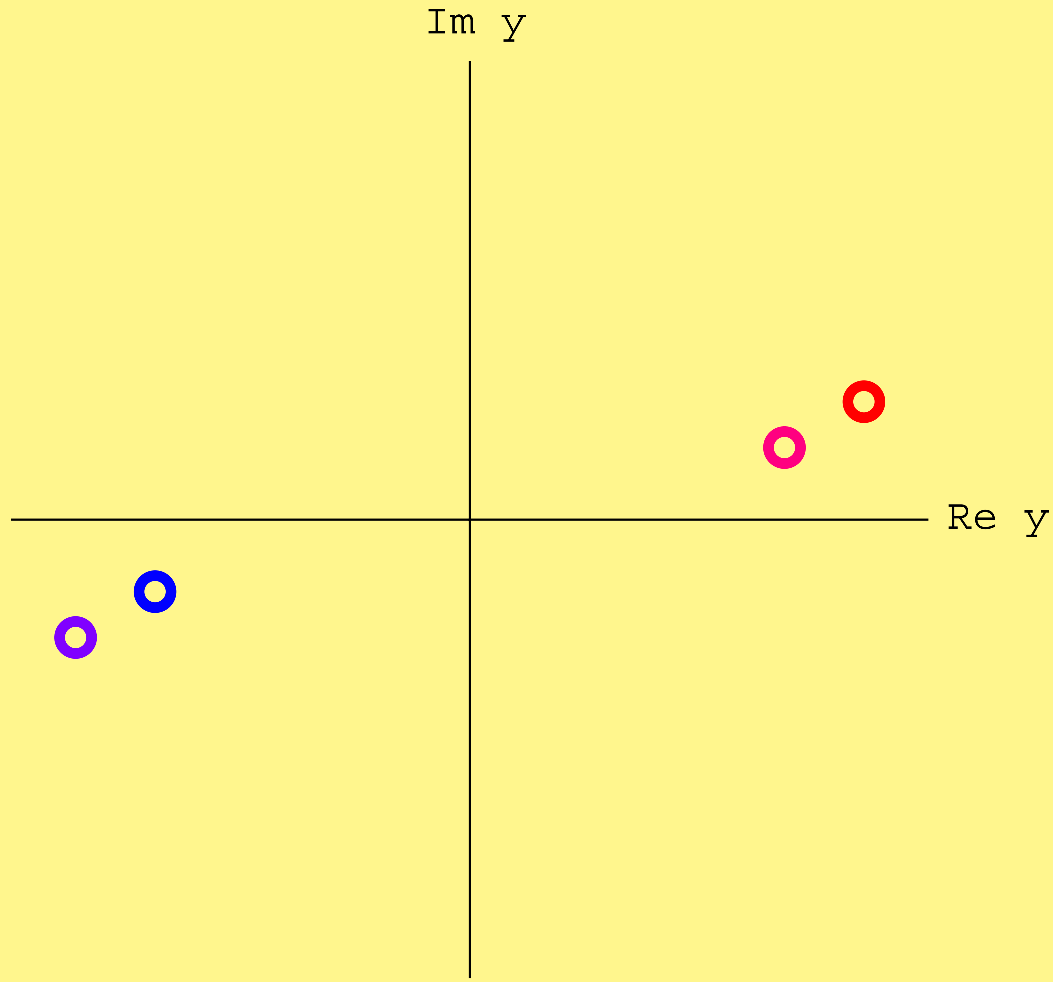
Visualization of toroidal valuations





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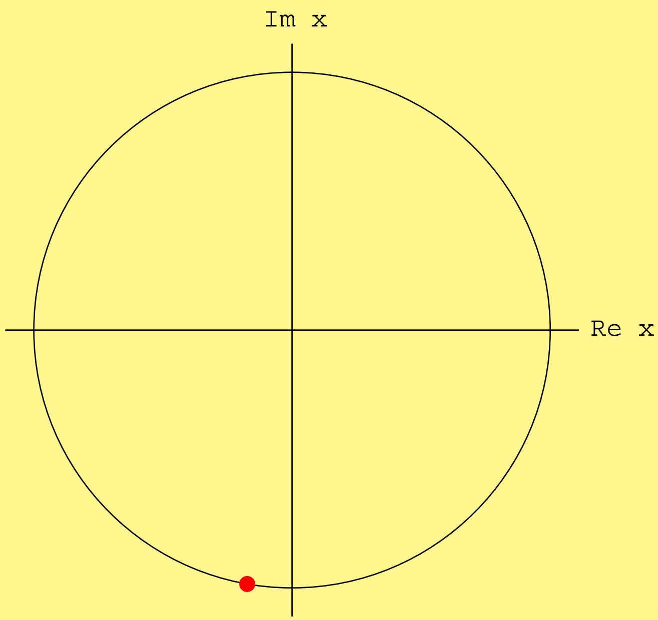
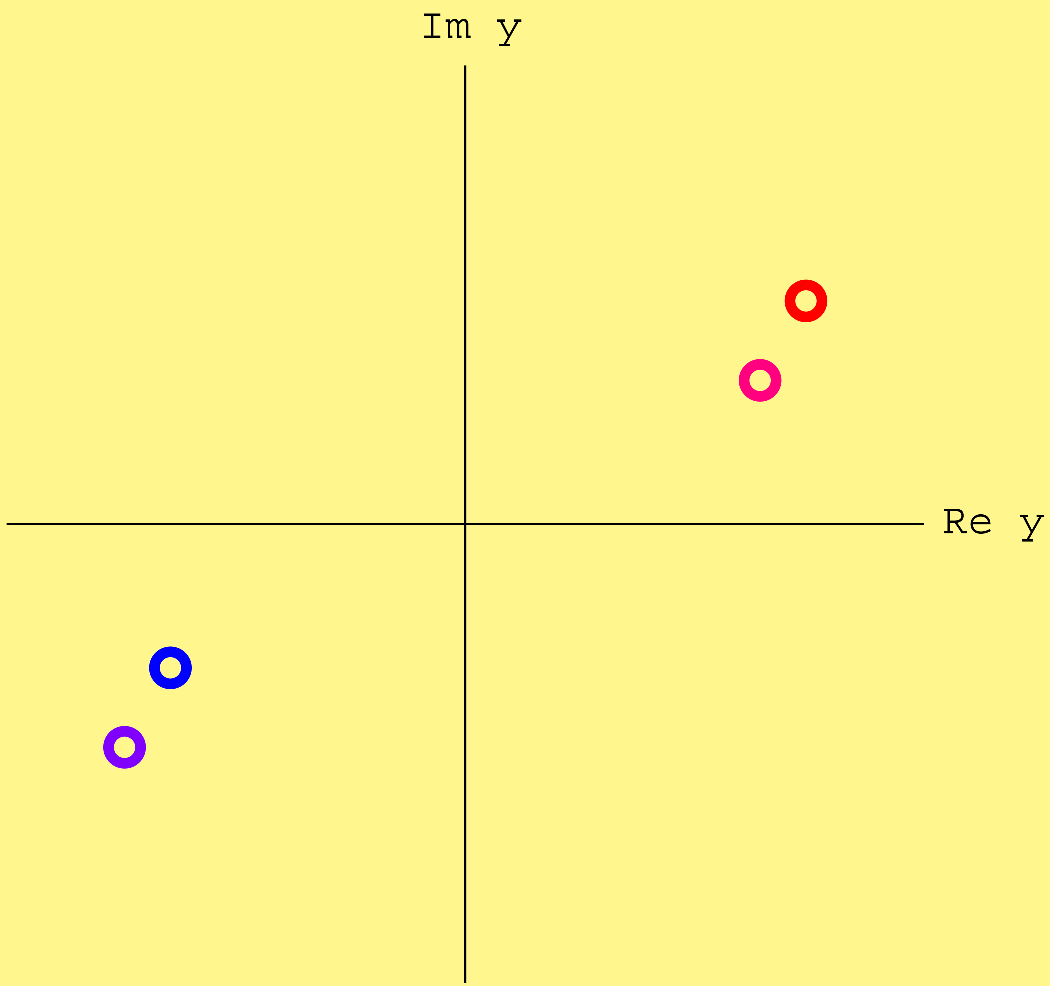
Visualization of toroidal valuations





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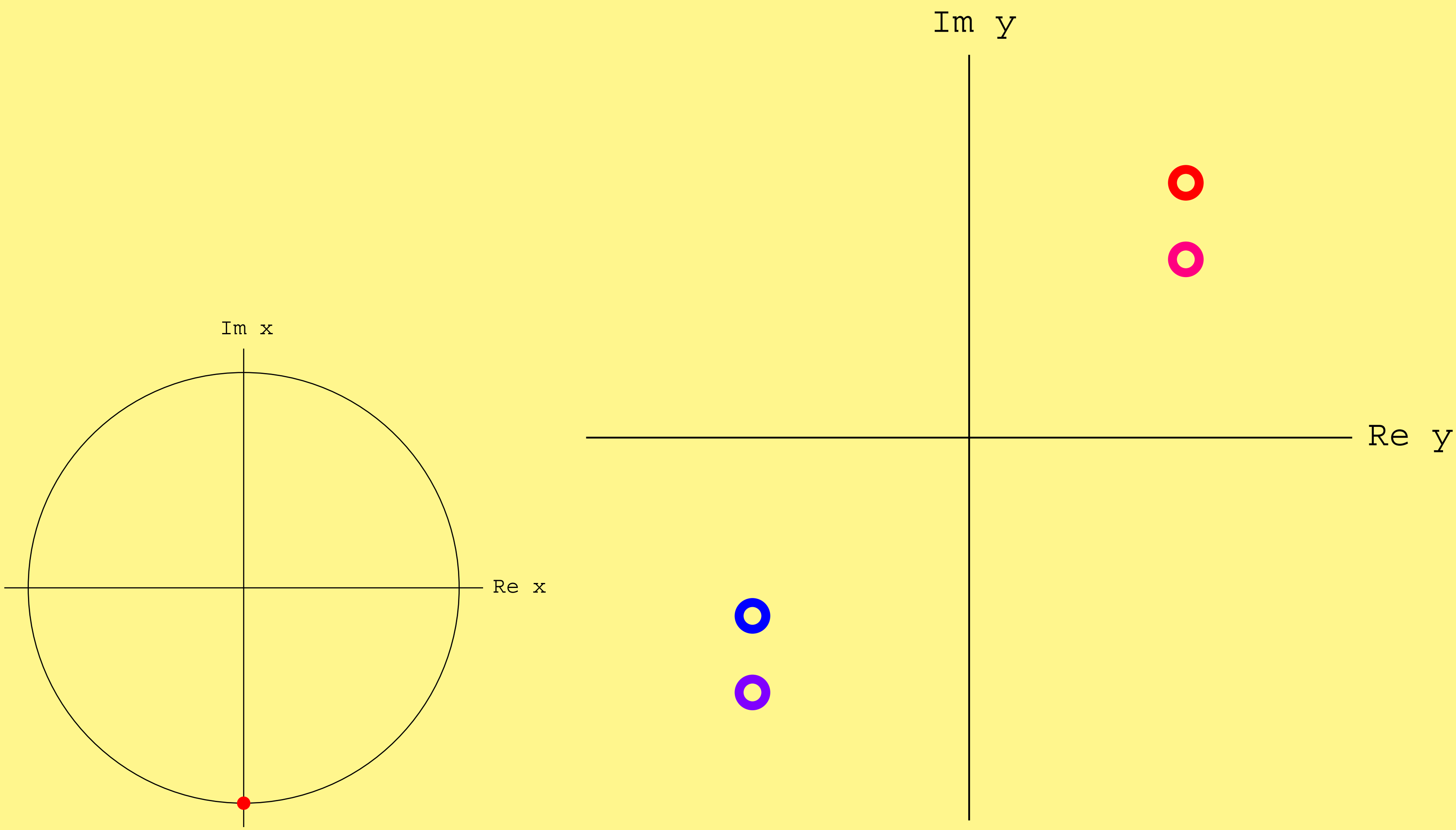
Visualization of toroidal valuations



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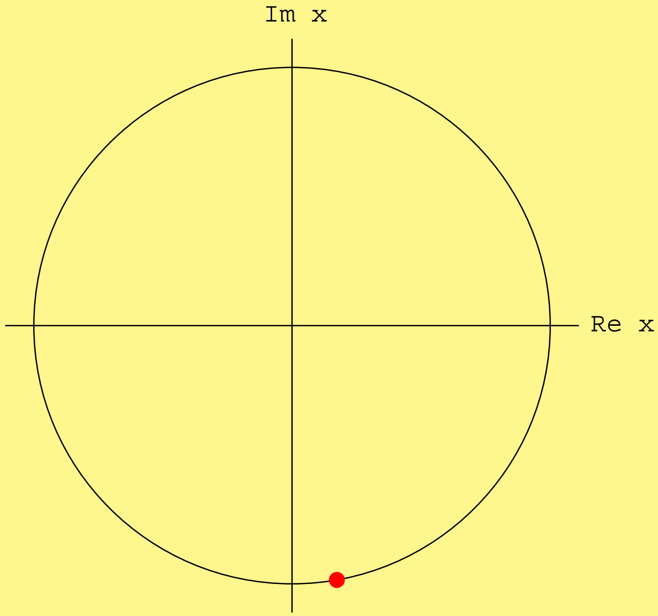

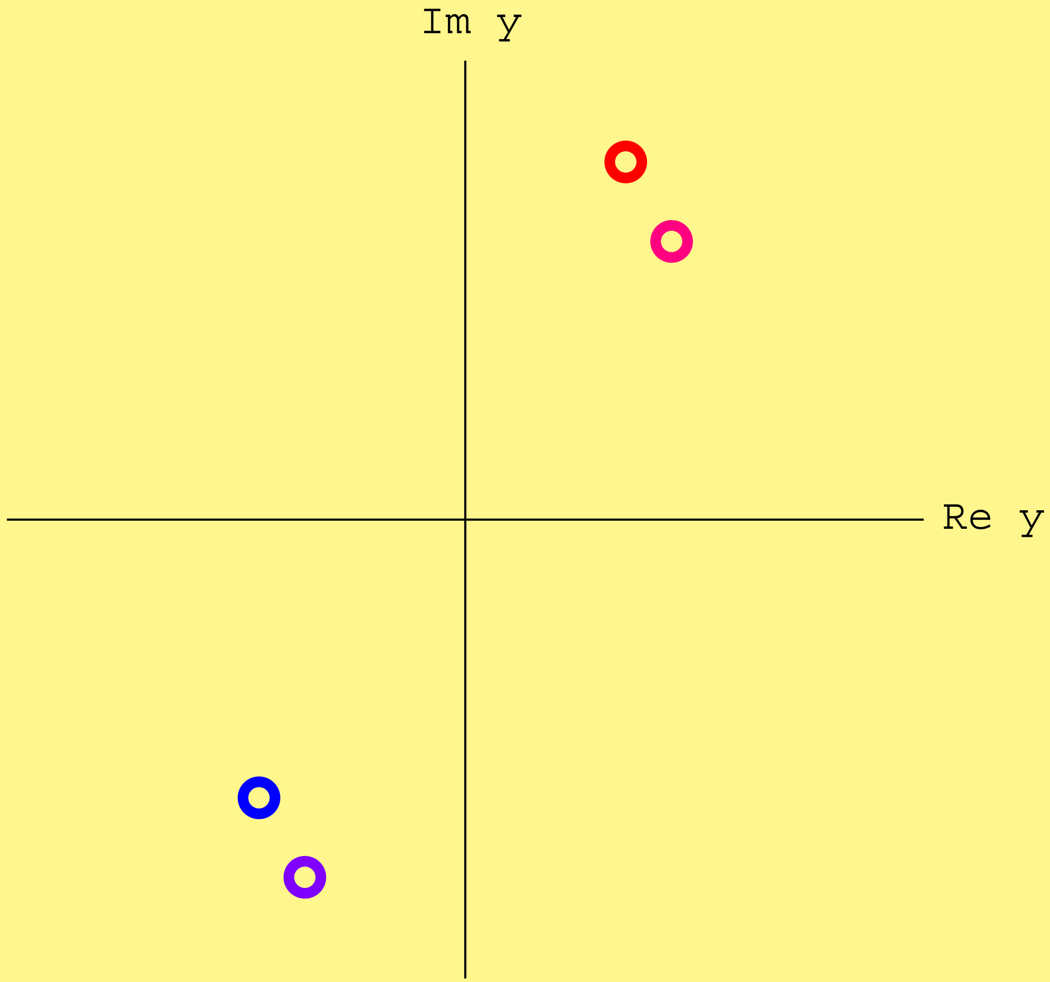

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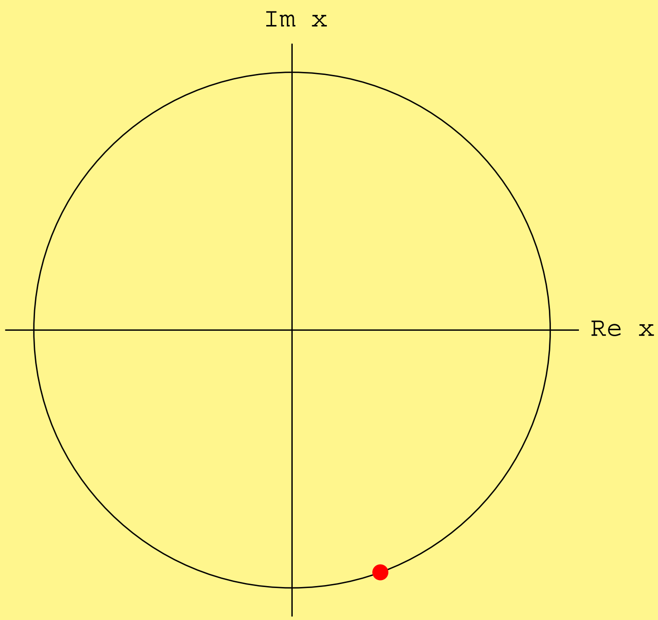
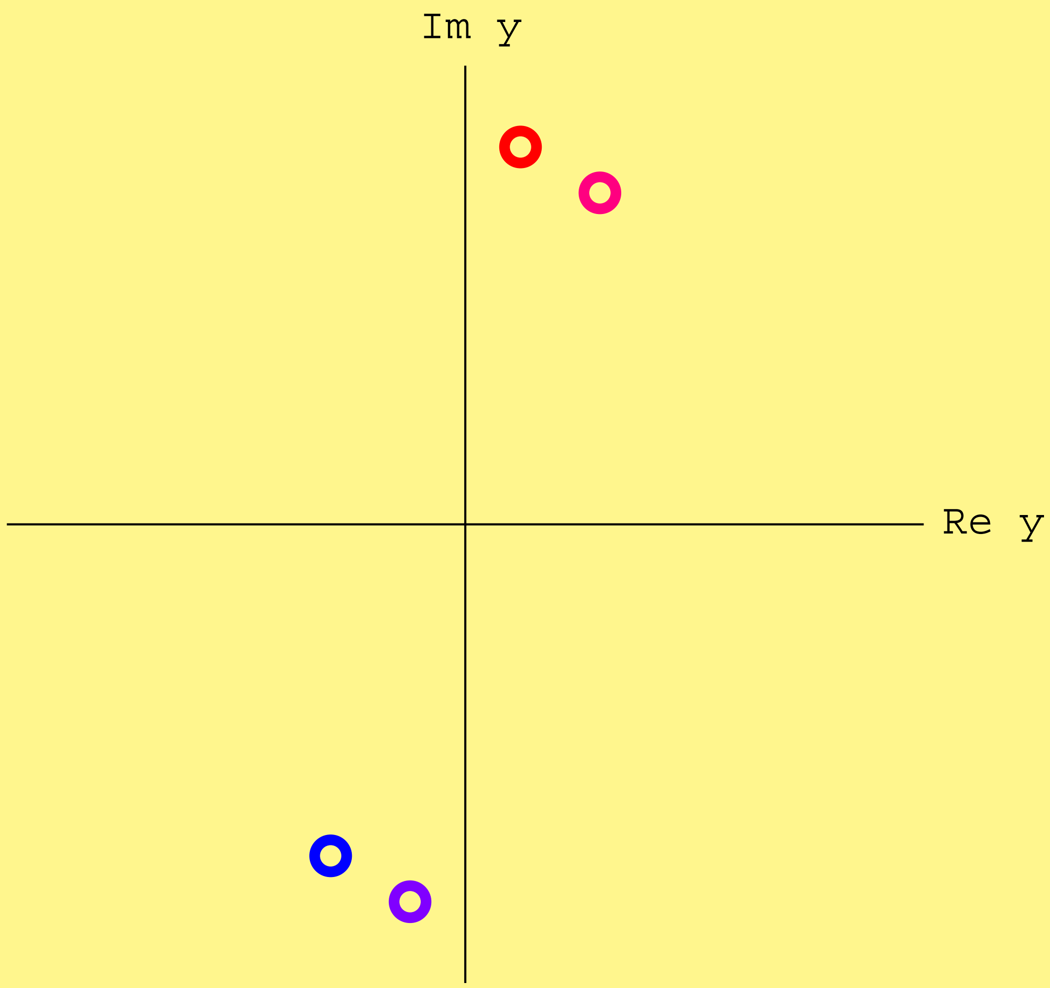
Visualization of toroidal valuations





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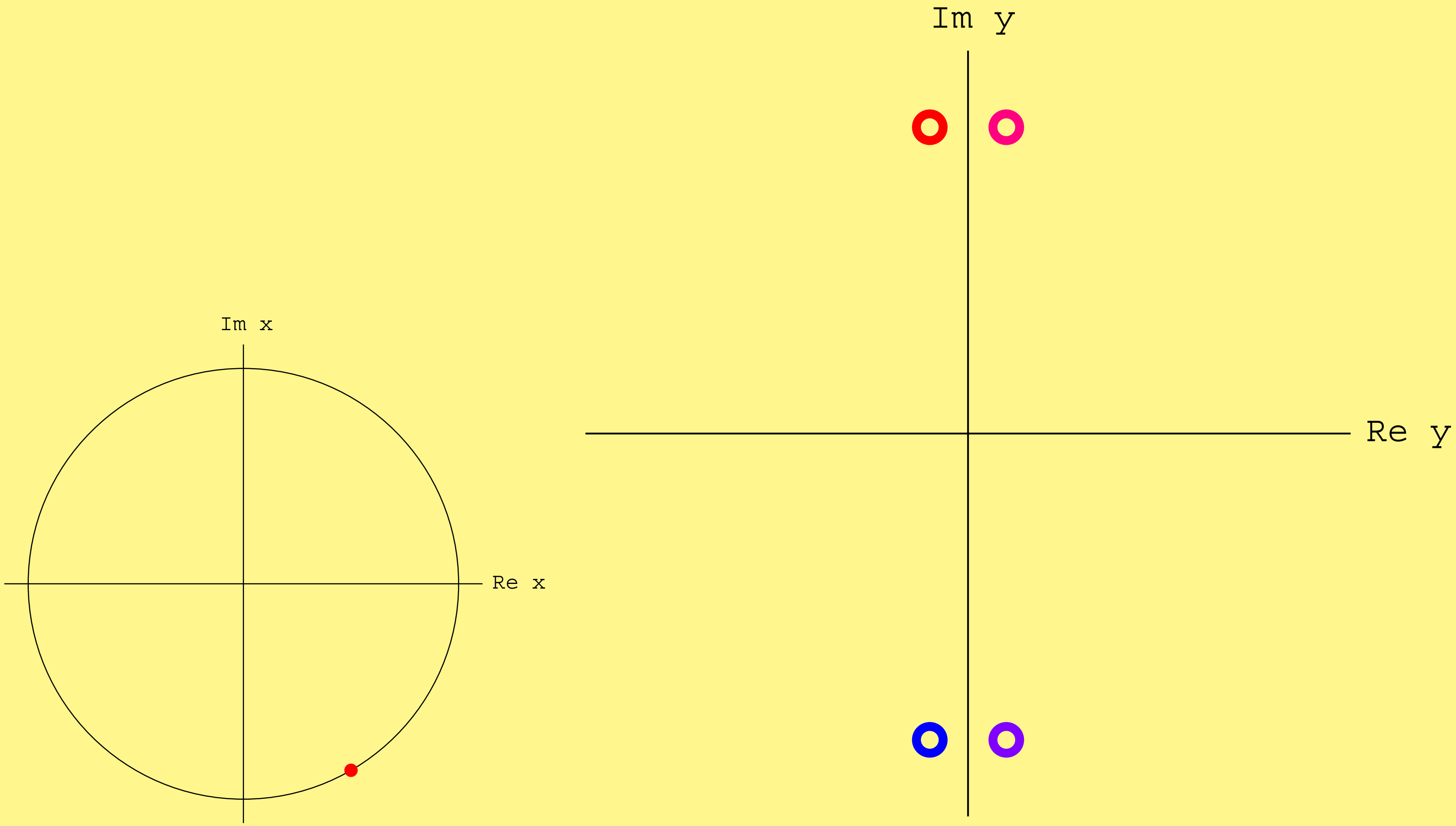
Visualization of toroidal valuations



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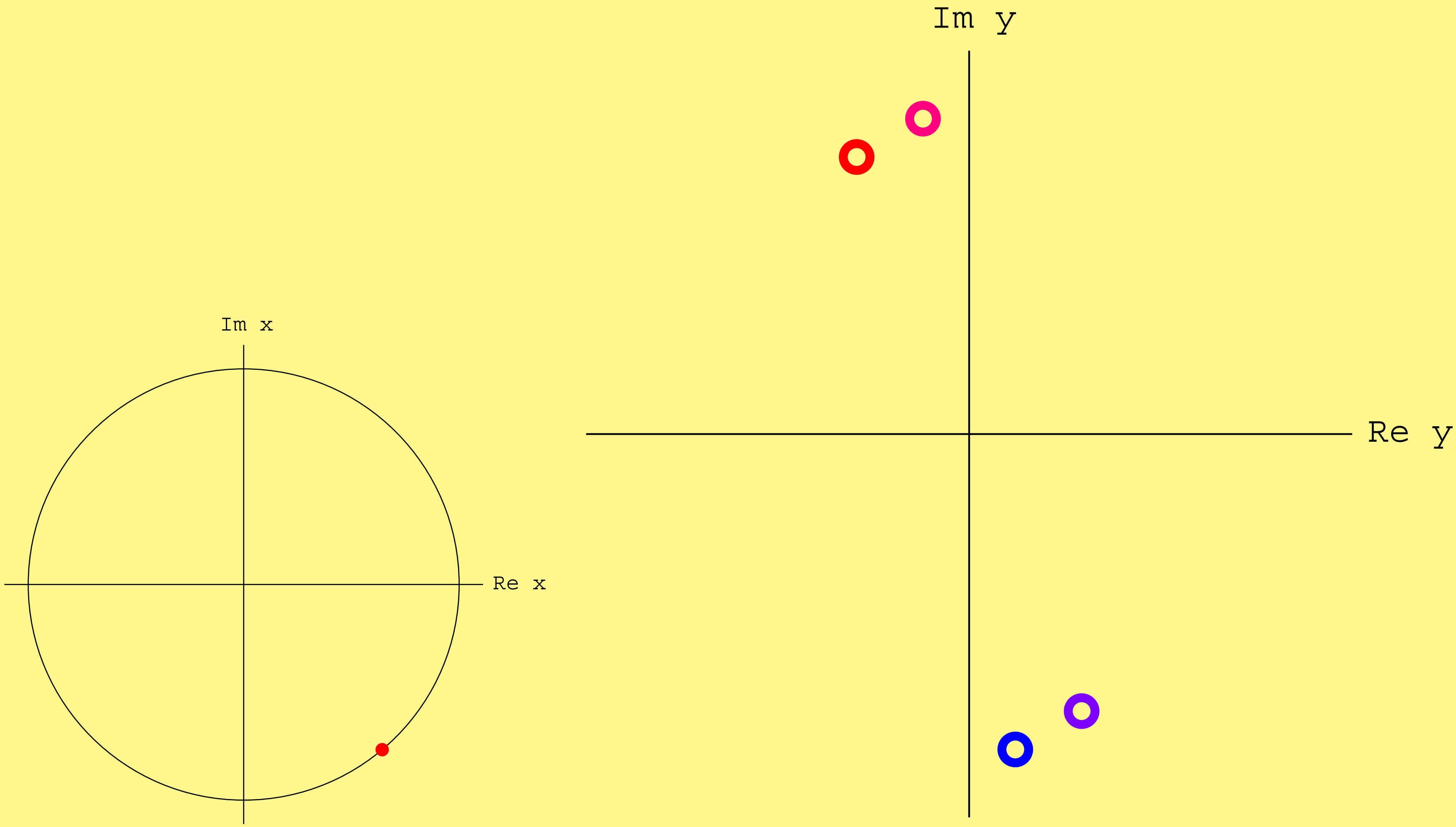
Visualization of toroidal valuations



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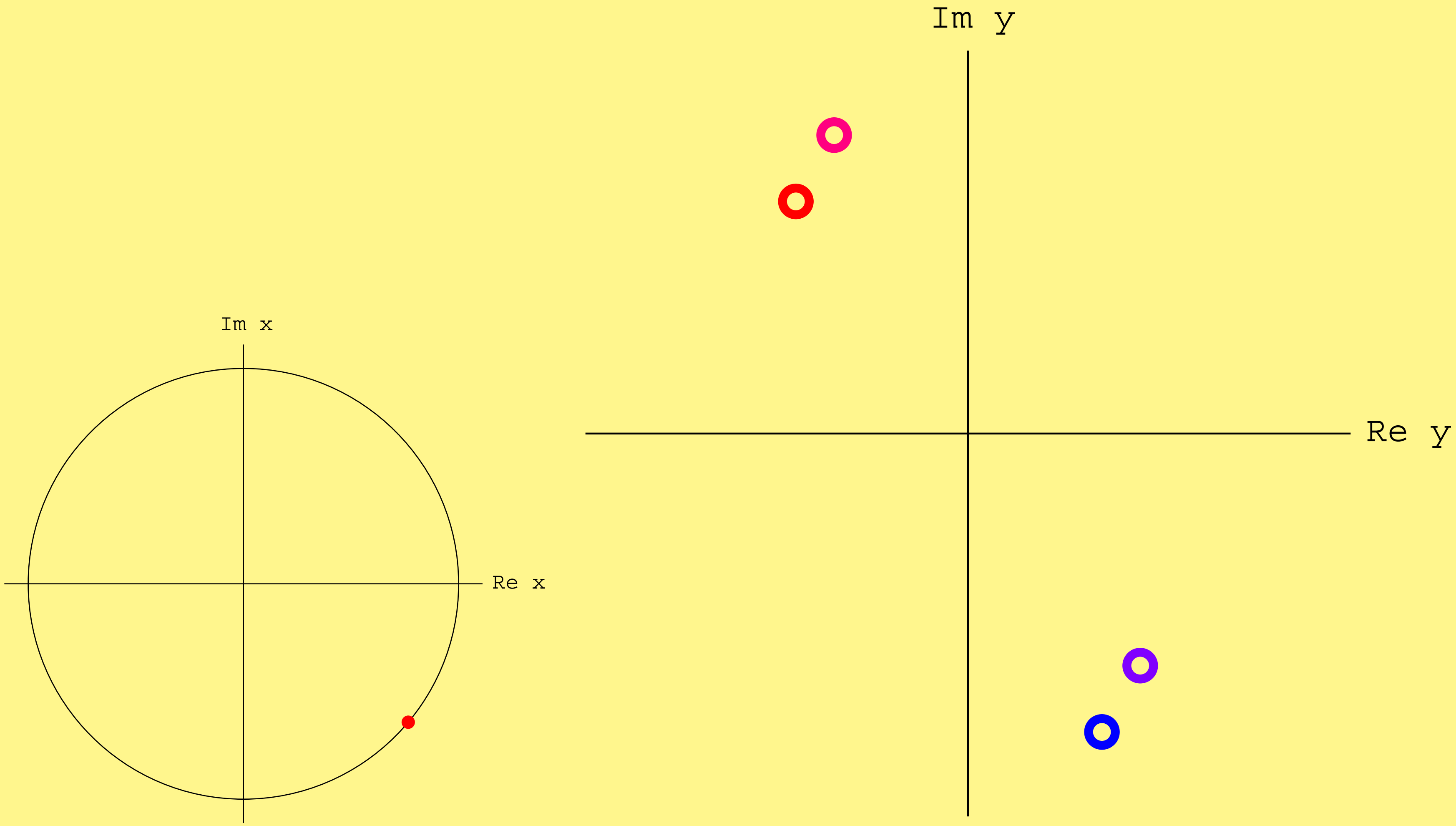
Visualization of toroidal valuations



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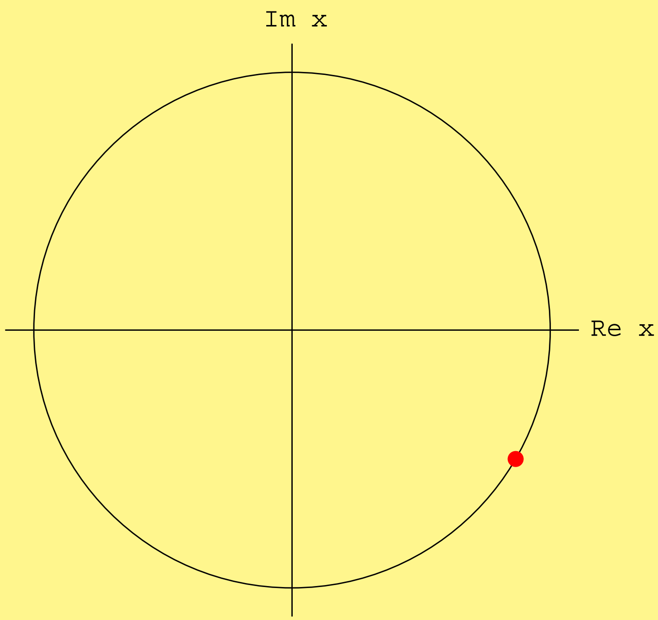
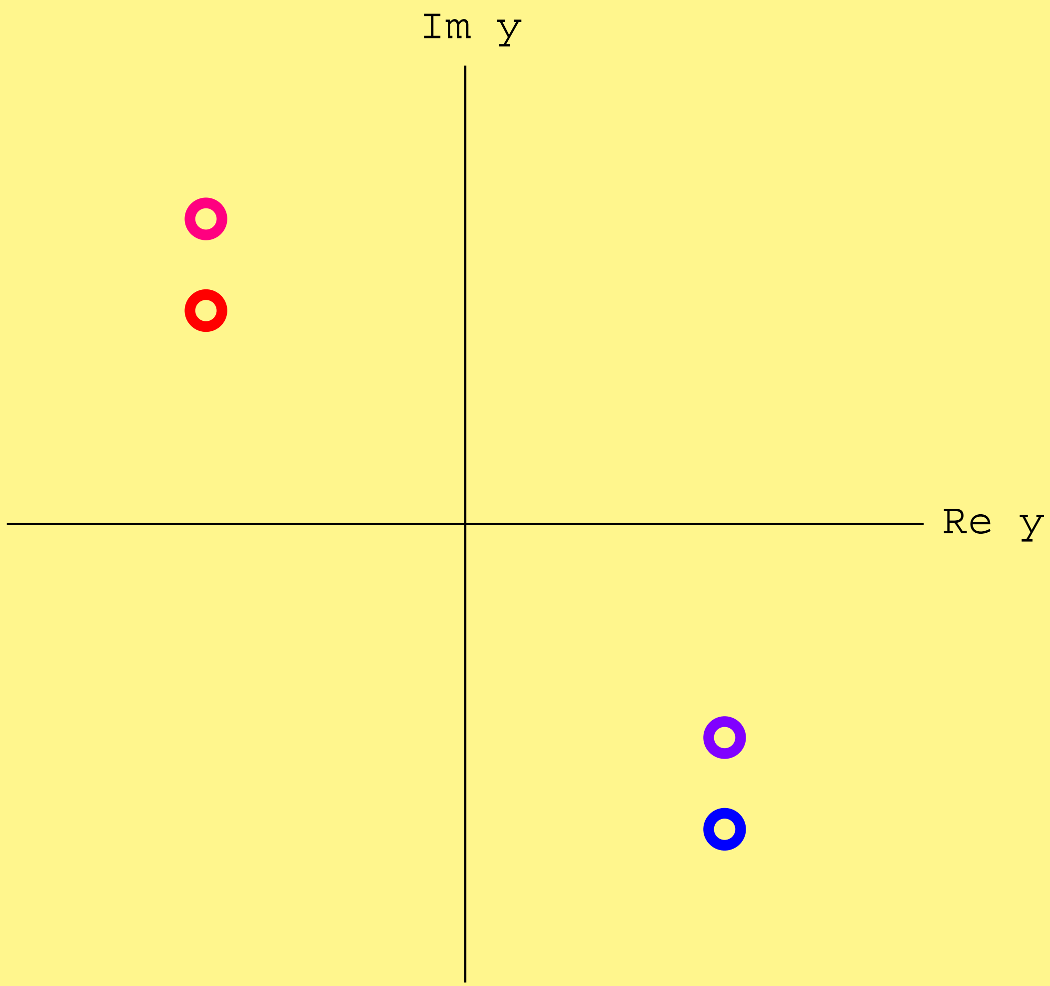
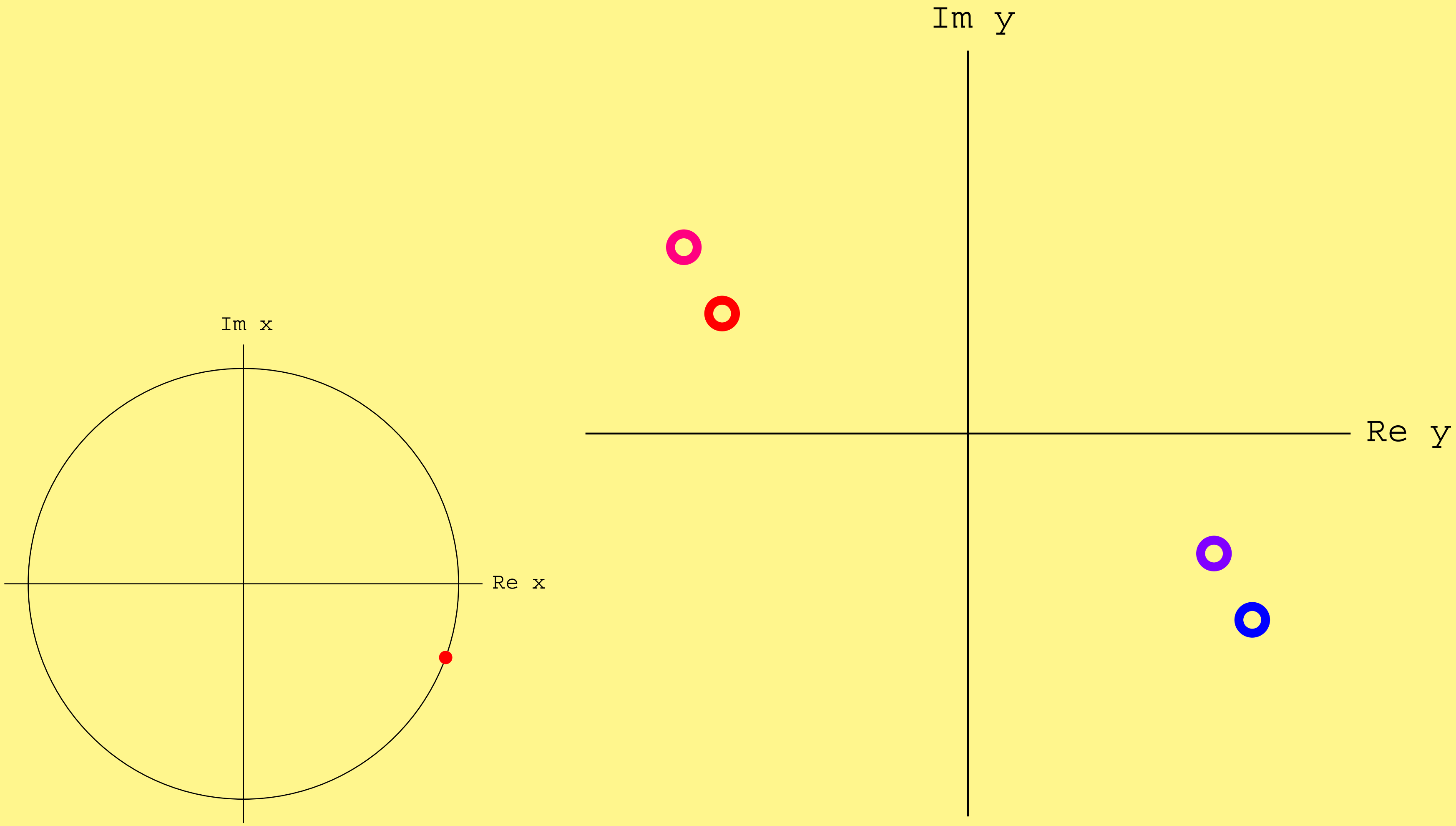




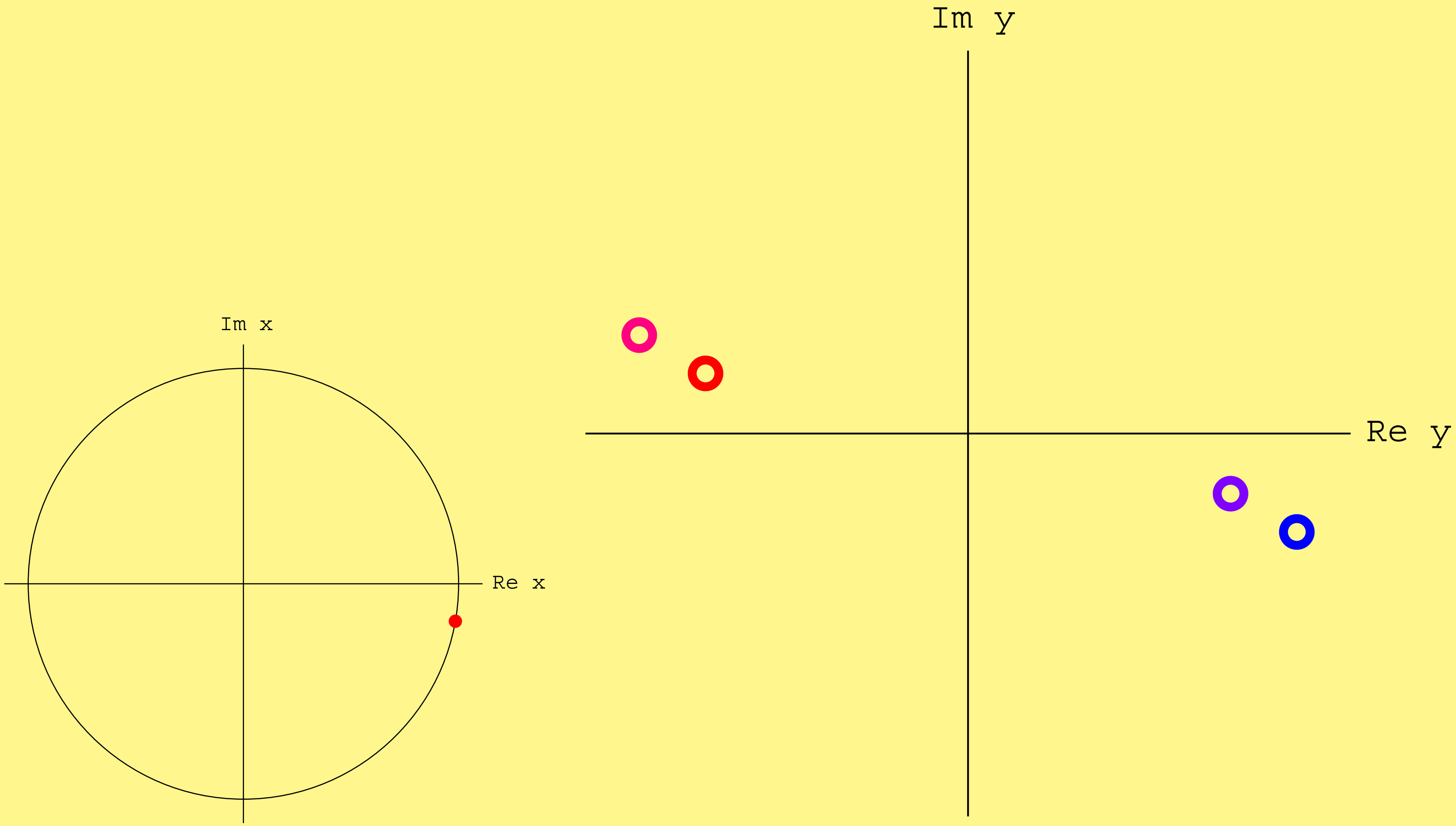
$$|x| \sim r, \qquad |(y^2 - x^3)^2 - x^9| \sim r^{10}$$

Visualization of toroidal valuations



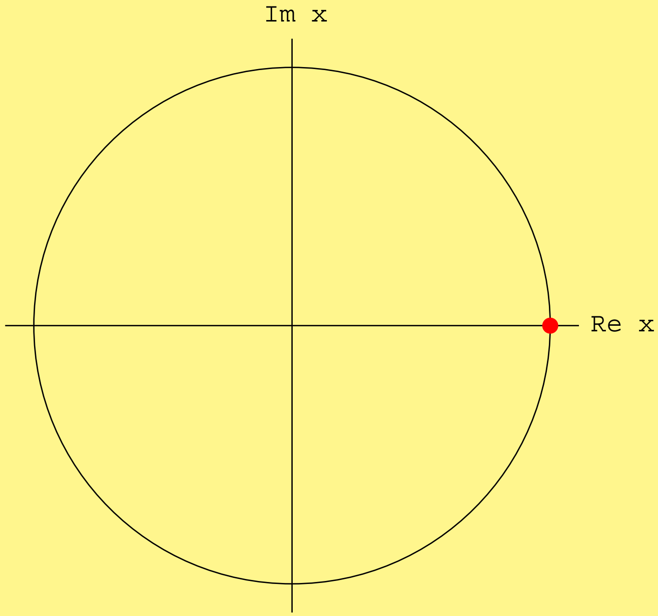
$$|x| \sim r, \qquad |(y^2 - x^3)^2 - x^9| \sim r^{10}$$

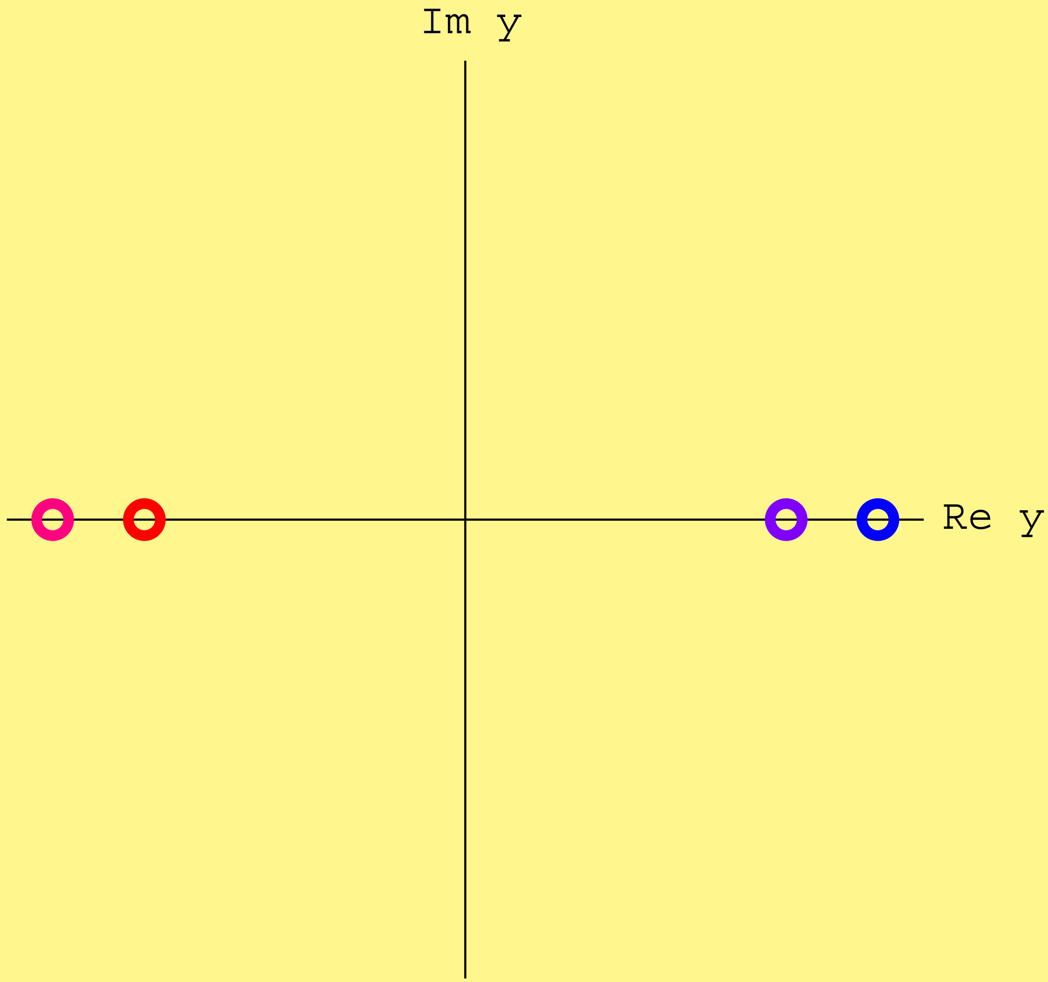
Visualization of toroidal valuations



$$|x| \sim r, \qquad |(y^2 - x^3)^2 - x^9| \sim r^{10}$$

Visualization of toroidal valuations





$$|x| \sim r, \qquad |(y^2 - x^3)^2 - x^9| \sim r^{10}$$

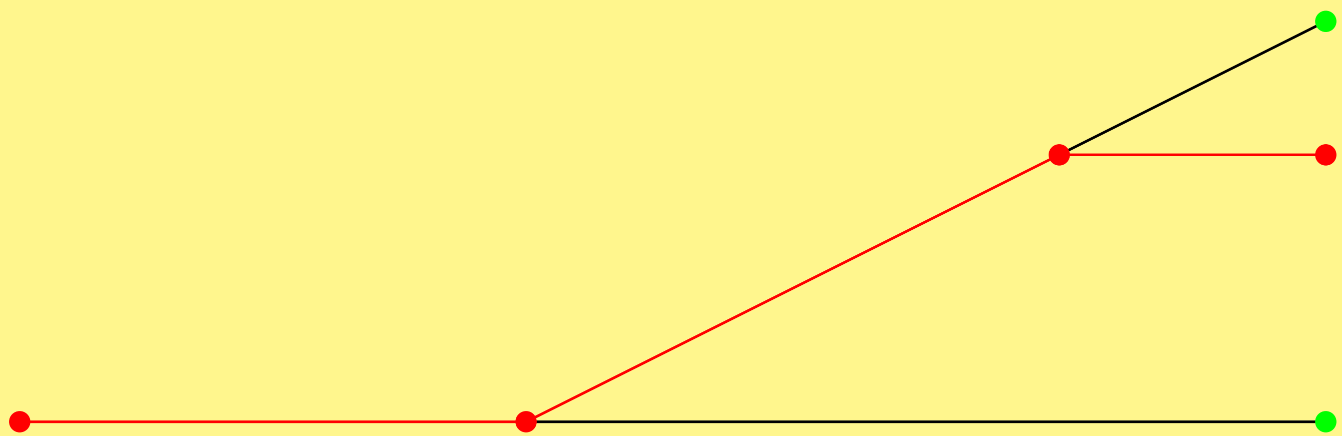
A walk in valuation space

The following slides illustrate the parameterization of the segment $[\nu_{\mathfrak{m}}, \nu_\phi]$ by $\nu_{\phi,\alpha}$, $\alpha \in [1, \infty]$ for

$$\phi = (y^2 - x^3)^2 - x^9$$

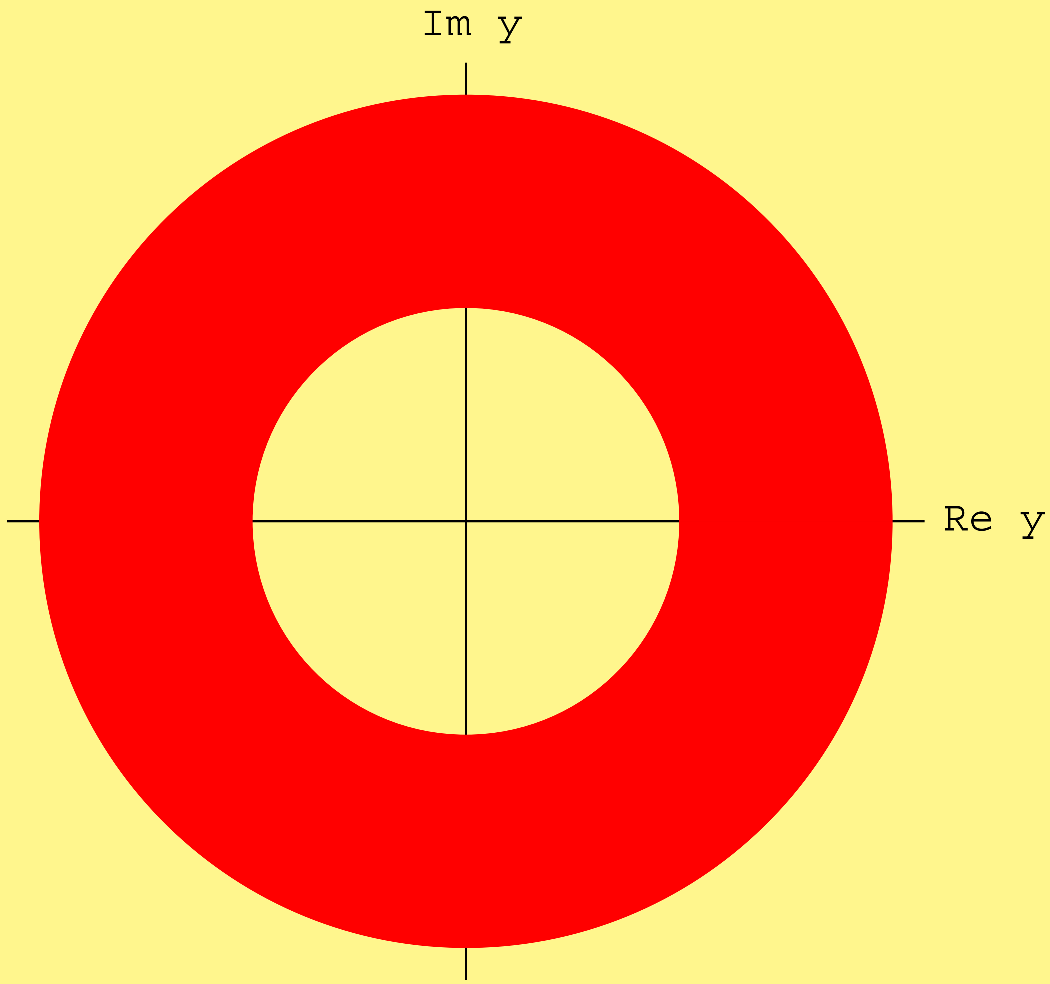
and

- $1 \leq \alpha < \frac{3}{2}$ (multiplicity 1)
- $\frac{3}{2} < \alpha < \frac{9}{4}$ (multiplicity 2)
- $\frac{9}{4} < \alpha < \infty$ (multiplicity 4)
- $\alpha = \infty$ (multiplicity 4)



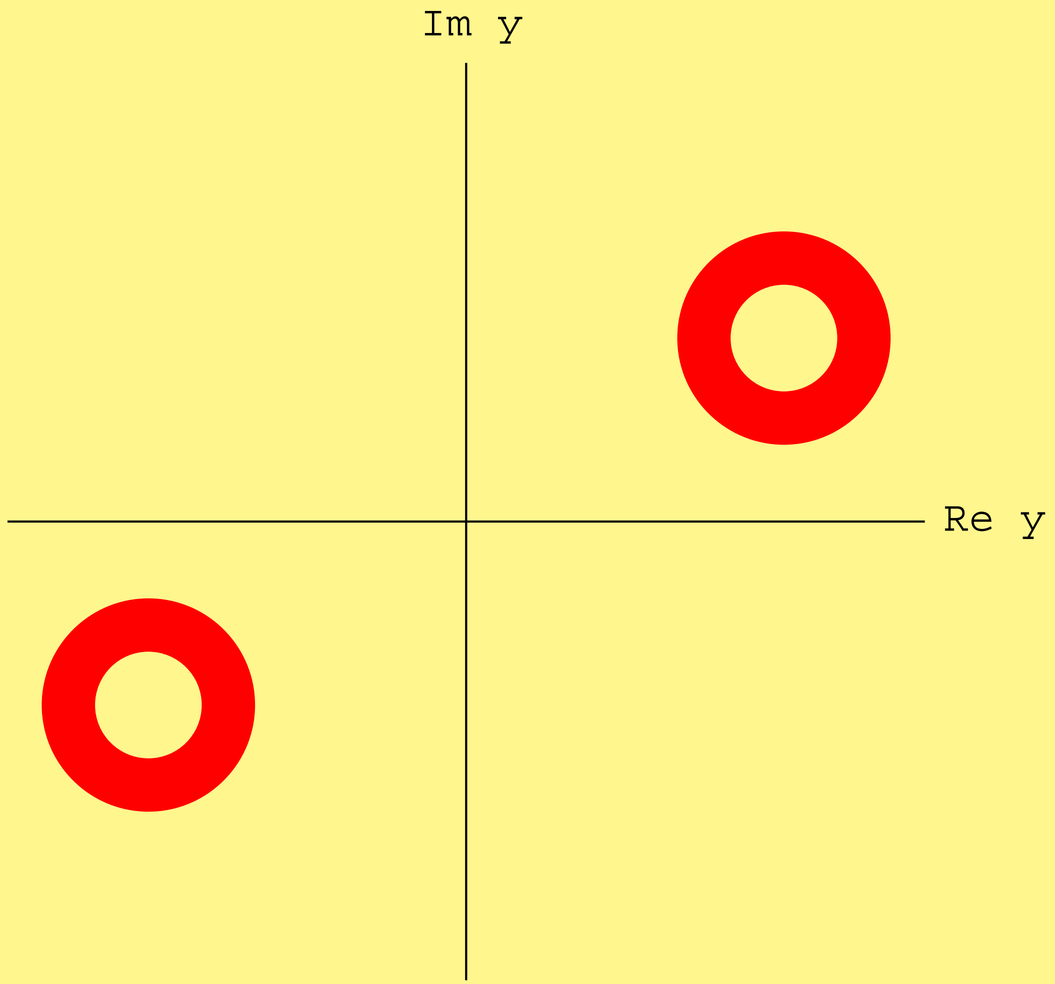
A walk in valuation space

$$1 \leq \alpha < \frac{3}{2}$$



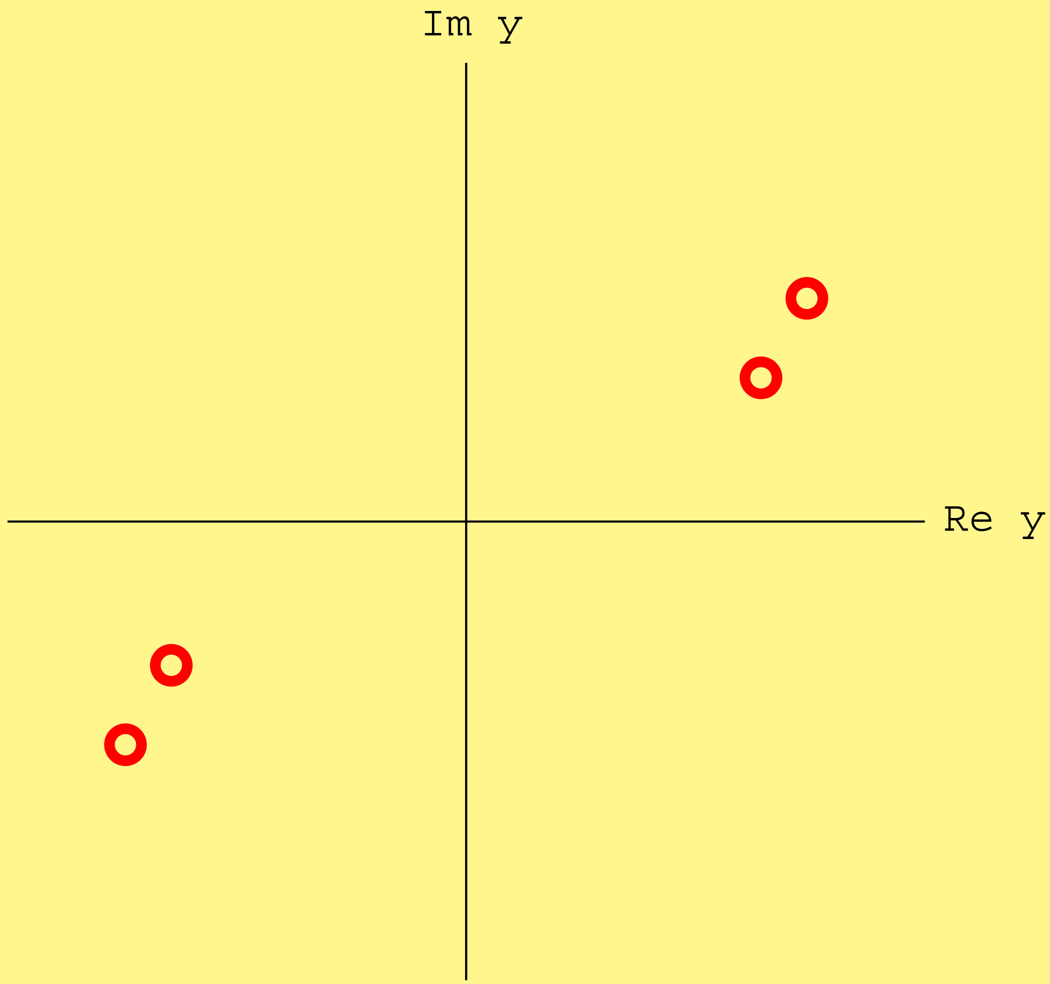
A walk in valuation space

$$\frac{3}{2} < \alpha < \frac{9}{4}$$



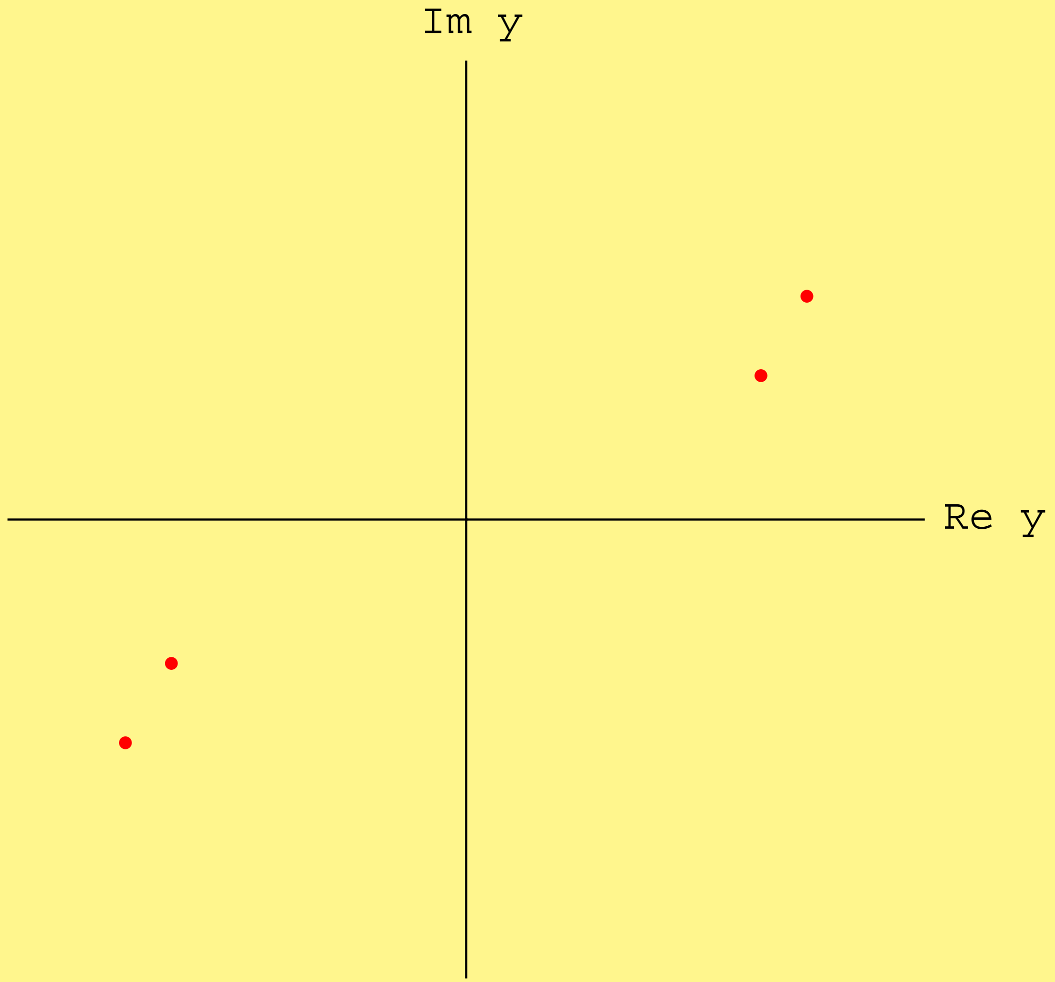
A walk in valuation space

$$\frac{9}{4} < \alpha < \infty$$



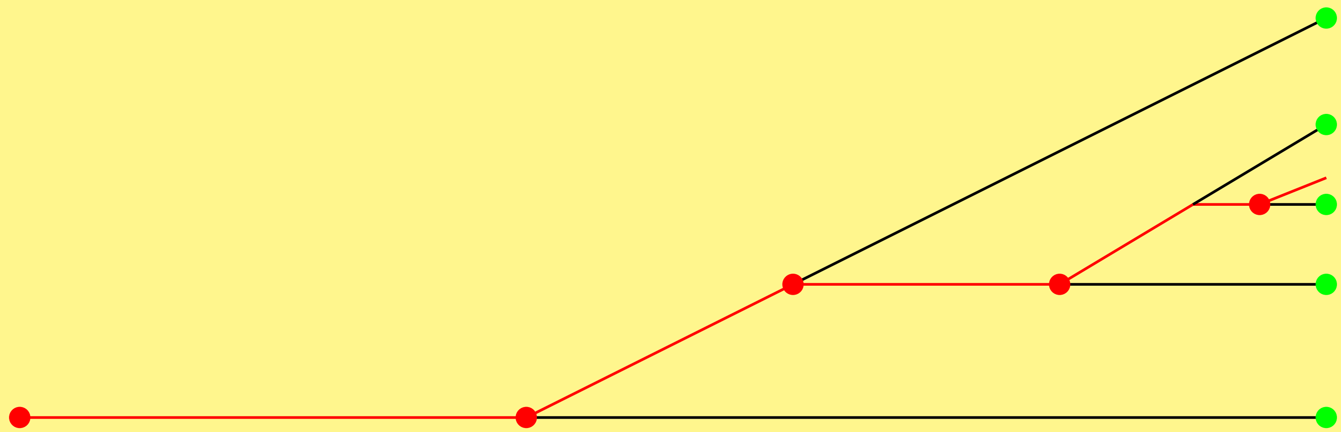
A walk in valuation space

$$\alpha = \infty$$

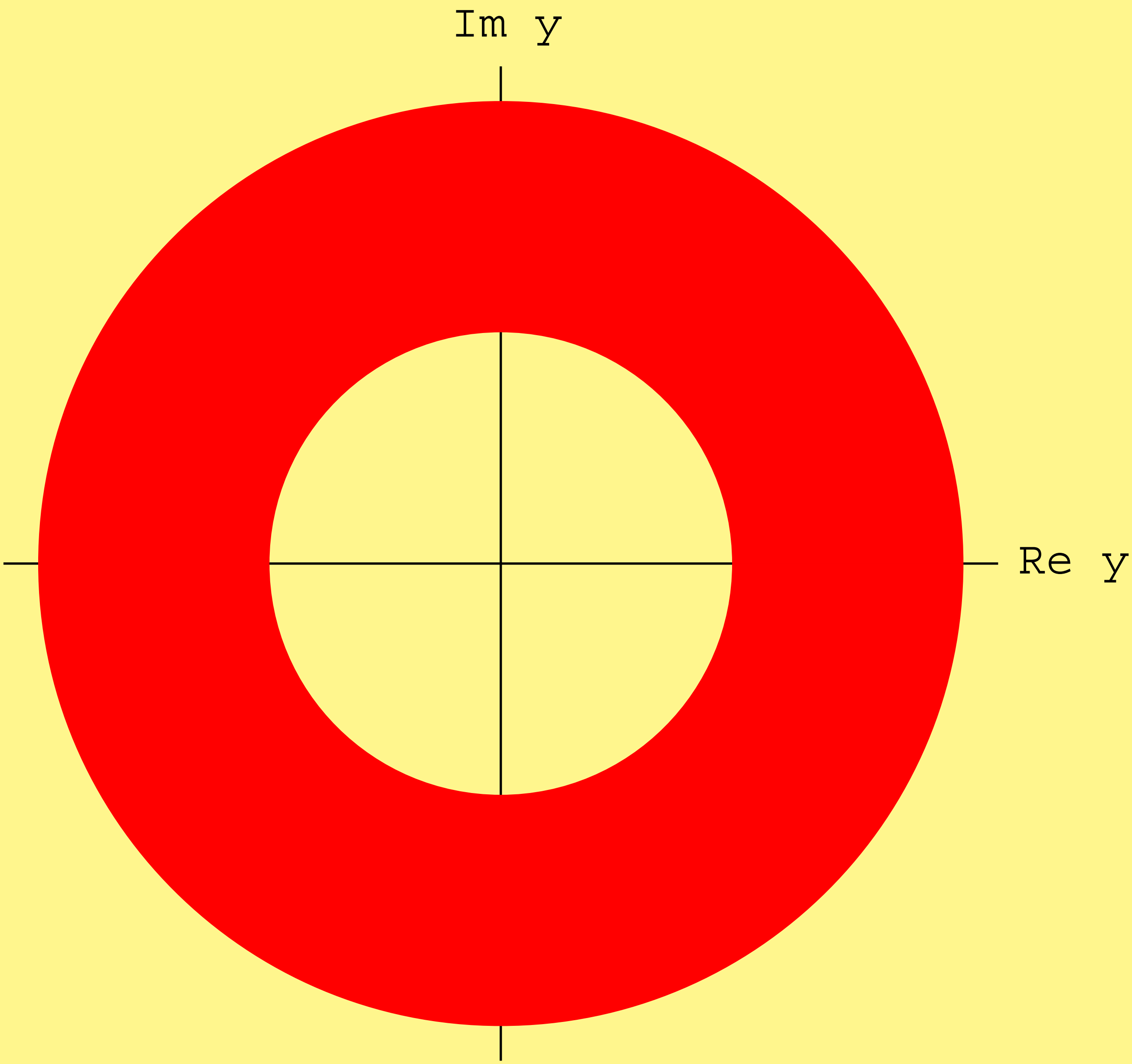


A walk on the wild side

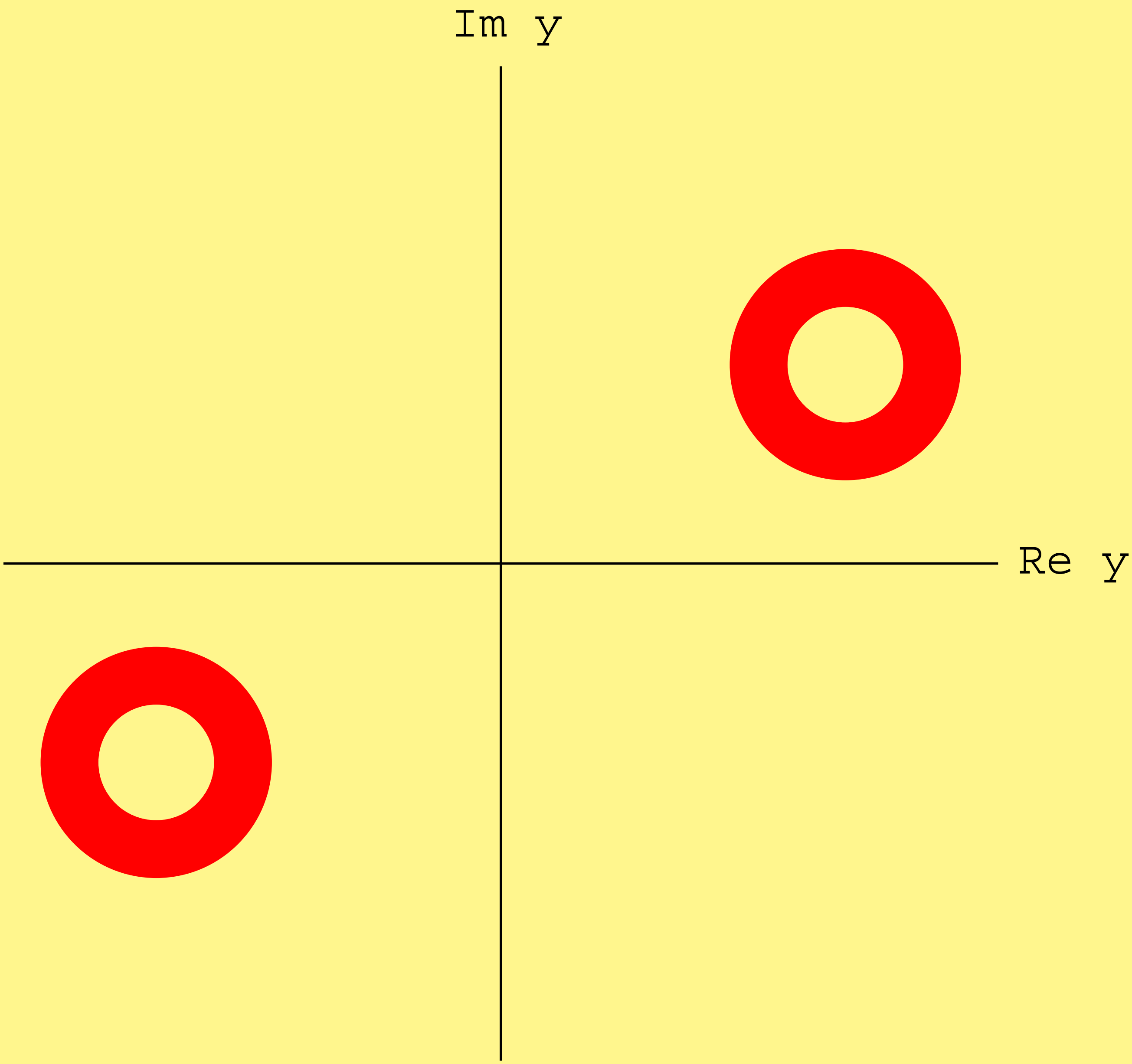
The following slides illustrate a segment $[\nu_{\mathfrak{m}}, \nu]$ where ν is a solenoidal valuation.



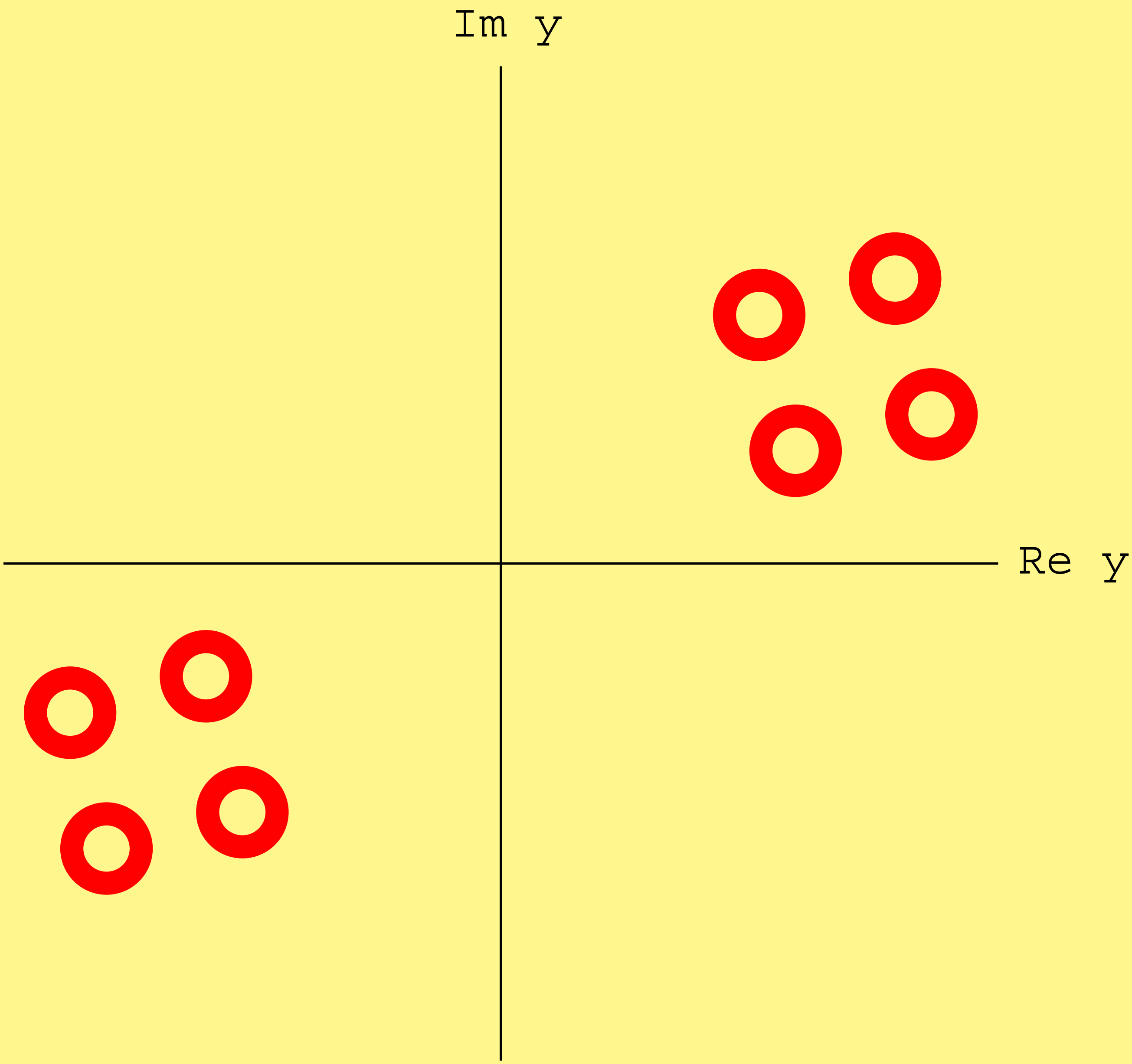
A walk on the wild side



A walk on the wild side



A walk on the wild side



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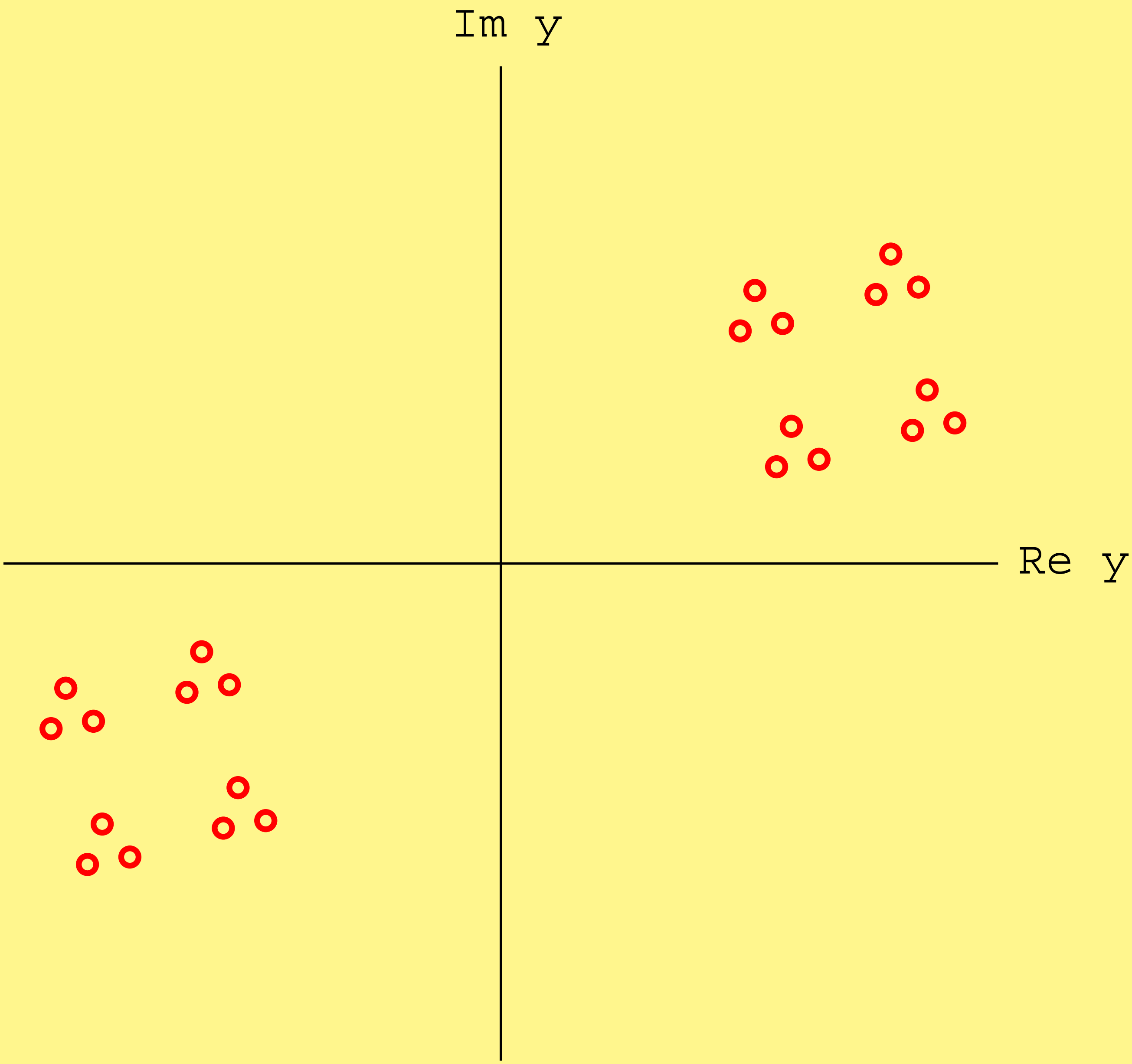

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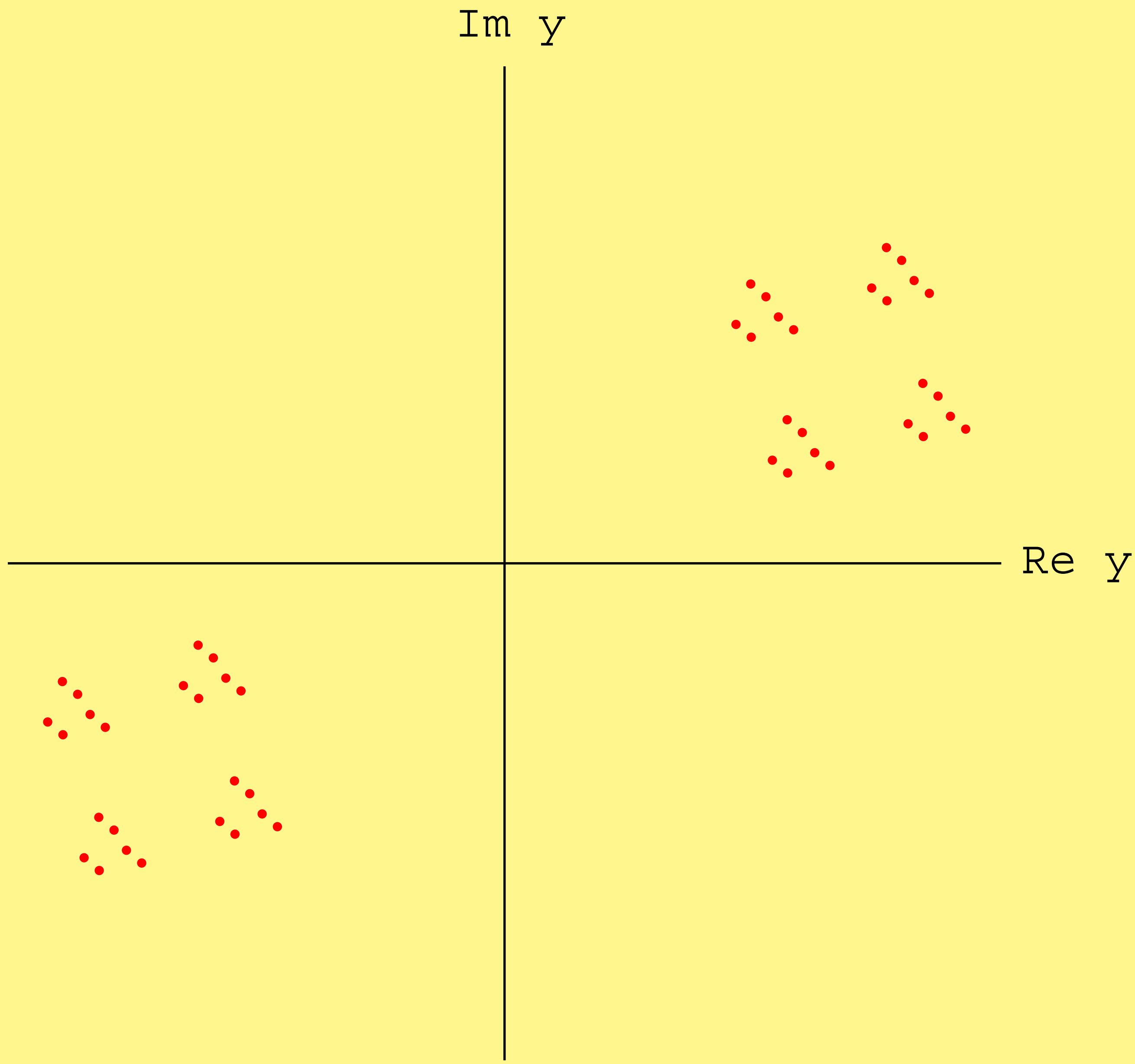

A walk on the wild side

. . . and so on.

This gives rise to a (formal) solenoid, i.e. a nontrivial fibration with Cantor sets as fibers.

Future research

- Mapping properties of germs $f : R \to R$ (work in progress).
- Dynamics (work in progress).
- Surface singularities.
- Valuations on $\mathbf{C}[[x_1, \ldots, x_n]]$. Buildings?