SOLUTIONS

Math 115 Calculus Exam I
October 6, 1999

Department of Mathematics
University of Michigan

Name: _____ Instructor: _____

Signature: _____ Section: _____

General Instructions: *Do not open this exam until you are told to begin.* This test consists of 8 questions on 11 pages (including this cover sheet), totaling 75 points. When the exam begins, please count all of the pages of the exam and make sure that none are missing. If any pages become detached, write your name on them and point them out to your instructor when you turn in the exam.

Please read the instructions for each individual exercise carefully. Show an appropriate amount of work for each exercise so that the grader can clearly see not just the answer, but also how you obtained it. If you use graphs or tables to obtain an answer, be certain to provide an explanation, with sketches of the graphs or excerpts from the tables to make it clear how you arrived at your solution. Other than this, explanations are only needed where they are requested.

Exercise	Points	Score
1	7	
2	9	
3	11	
4	9	
5	10	
6	8	
7	9	
8	12	
Total	75	

1. (7 points) For most standard computer monitors, the weight of the monitor is proportional to the cube of the diagonal screen measurement. The writer of this test recently purchased a ViewSonic PS790 computer monitor that has a diagonal screen measurement of 19 inches and that weighs 47.4 pounds. Suppose he had instead purchased a monitor with a diagonal screen measurement of 21 inches. What could he have expected it to weigh? Round your answer to one place after the decimal point. (Show your work, but no explanation is necessary.)

Let w be the weight and d the diagonal screen measurement.

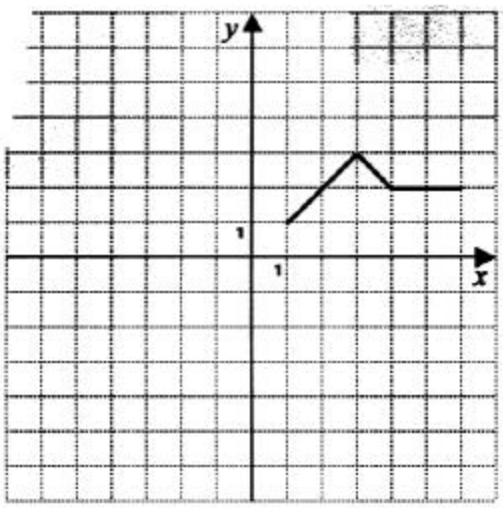
$w = kd^3$, where k is a constant

$47.4 = k \cdot 19^3$, so $k = \frac{47.4}{19^3}$

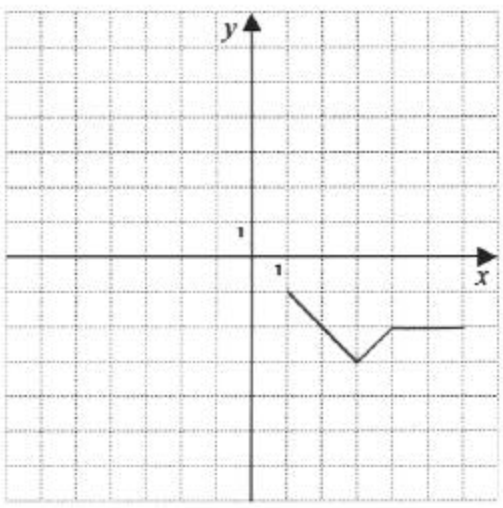
When $d = 21$,

$w = \frac{47.4}{19^3} \cdot 21^3 \approx \boxed{64.0 \text{ lbs.}}$

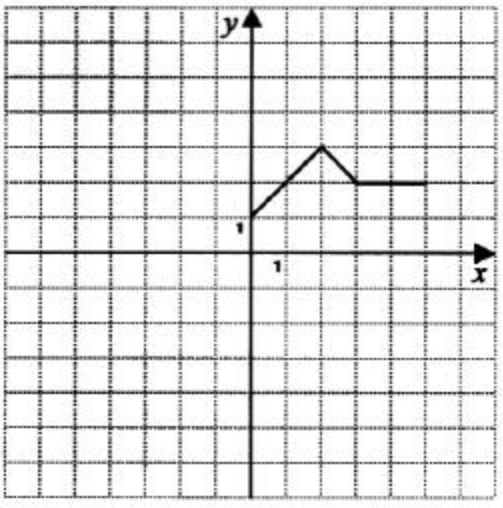
2. (3 points each part) The first grid shown below contains the graph of a function f. In each of the other grids, carefully draw the graph whose formula is given below the grid.



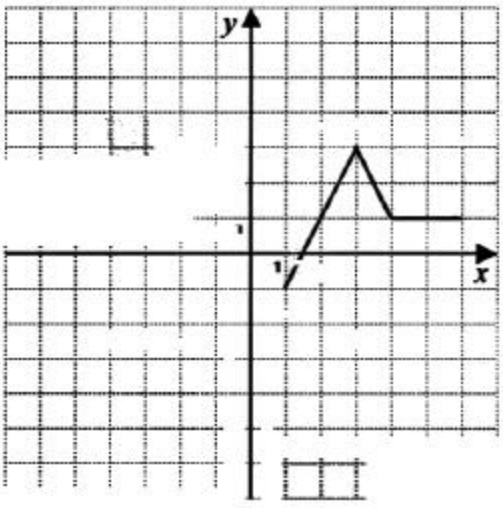
$y = f(x)$



$y = -f(x)$



$y = f(x+1)$



$y = 2f(x) - 3$

3. (This problem is based on the way commodities were once distributed to Native Americans by federal agencies on some reservations.) Suppose that you are to obtain 60 units of food consisting of flour and butter, where each pound of flour counts as 2 units and each pound of butter counts as 3 units. You may divide the 60 units between the flour and butter any way you wish; for example, you could get 18 pounds of flour (which counts as 36 units of food) and 8 pounds of butter (which counts as 24 units) to make up your 60 units. You do not have to get the flour and butter in whole units; for example, you could get $\sqrt{2}$ units of flour if you wish.

   Let x be the number of pounds of flour you obtain, and let y be the number of pounds of butter.

   (a) (3 points) Find a formula for y in terms of x. (Note carefully what is requested. This should *not* just be an equation that involves both x and y, but specifically a formula for y in terms of x.)

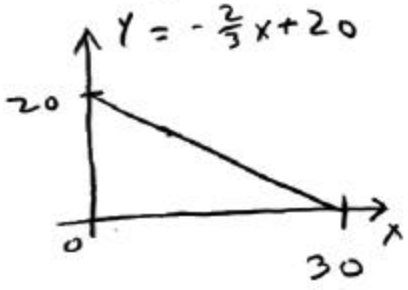
$2x + 3y = 60$

$3y = -2x + 60$

$\boxed{y = -\frac{2}{3}x + 20}$

   (b) (2 points) Your formula from (a) defines y as a function f of x; $y = f(x)$. Given the real-world nature of this problem, find the domain and range of this function, carefully labeling which is which and being sure to use correct mathematical notation.

$y = -\frac{2}{3}x + 20$ (Note: $y = 0 \Rightarrow x = 30$)



Domain: $0 \le x \le 30$

Range: $0 \le y \le 20$

(c) (3 points) Find the values of the x-intercept, y-intercept, and slope of the graph of f, carefully stating which is which.

x-intercept: 30
y-intercept: 20
slope: $-\frac{2}{3}$

(d) (2 points) In real-world terms involving butter and flour, what do the x- and y-intercepts of the graph of f represent? Answer in one or more complete sentences.

The x-intercept represents the amount of flour you can get if you get no butter.

The y-intercept represents the amount butter you can get if you get no flour

(e) (1 point) In real-world terms involving butter and flour, what does the slope of the graph of f represent? Answer in one or more complete sentences.

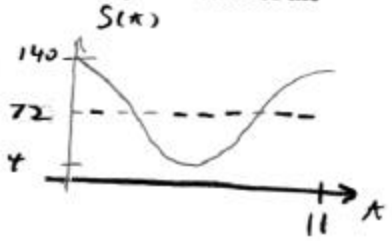
The slope represents the change in the amount of butter you get ($-\frac{2}{3}$ pound) for each additional pound of flour you take

4. Sunspots are dark spots on the surface of the sun that are often larger than the earth. The number of sunspots oscillates regularly between a maximum of about 140 and a minimum of about 4, with about 11 years between successive peaks. The next maximum is predicted to occur just about at the beginning of the year 2001.

(a) (1 point) What types of functions have been used so far in this course to model oscillatory behavior of this type?

This type of behavior has been modeled by trigonometric functions, specifically the sine and cosine.

(b) (6 points) Let $S(t)$ be the number of sunspots t years after the next peak at the beginning of the year 2001. Assuming that $S(t)$ can be modeled by a function of the type you mentioned in (a), find a formula for $S(t)$.



Amplitude $= \frac{140-4}{2} = 68$

$\frac{2\pi}{\text{period}} = \frac{2\pi}{11}$

Upward shift $= 4+68 = 72$

The basic curve that models this is the cosine curve, so

$$S(t) = 68\cos\left(\frac{2\pi}{11}t\right) + 72$$

(c) (1 point) It is about fifteen months until the next maximum point in the sunspot cycle. What would the value of t be today?

-1.25

(d) (1 point) What is the number of sunspots that your formula would predict for today?

$68\cos\left(\frac{2\pi}{11}(-1.25)\right) + 72 \approx$ 123 sunspots

5. The following table gives some values of the functions f, g, and h. Of the three functions, one is linear, another is exponential, and the remaining one is a quadratic power function. The numbers a, b, and c are positive constants whose values you do not know, and the other numbers appearing in the right three columns of the table have been rounded to three places after the decimal point.

x	$f(x)$	$g(x)$	$h(x)$
4	7.236	$7.236b$	$7.236c$
5	$10.997a^2$	$11.974b$	$11.306c$
6	$16.712a^4$	$16.712b$	$16.281c$

(a) (5 points) Using only methods covered in the text for this course, determine which of these three functions is linear, and justify your answer clearly.

$g(5) - g(4) = 4.738b$

$g(6) - g(5) = 4.738b$

Since g is giving equal differences over equal intervals, $\boxed{g}$ is the linear function.

(b) (5 points) Using only methods covered in the text for this course, determine which of these three functions is exponential, and justify your answer clearly.

$$\frac{f(6)}{f(5)} = \frac{16.712a^4}{10.997a^2} \approx 1.520a^2$$

$$\frac{f(5)}{f(4)} = \frac{10.997a^2}{7.236} \approx .520a^2$$

Since f is giving equal ratios over equal intervals, $\boxed{f}$ is the exponential function.

6. (8 points) Suppose that you have \$1000 to invest on January 1, 2000, and that you have a choice of two different investments you can make. If you invest in Spartan Securities at that time, your investment will grow linearly as a function of the number of years you invest your money, and your initial investment will double in only two years. If instead you invest in Wolverine Bonds, your investment will grow exponentially as a function of the number of years you make the investment, but it will take ten years for your initial investment to double.

   If the year in which you will withdraw your money from the investment is 2002, then Spartan Securities is clearly the better investment, since that investment doubles your money while the other one does not. For what withdrawal year does Wolverine Bonds first become the better investment?

   Give your final answer as a complete sentence.

Let t be the number of years since January 1, 2000, let s be the value of your investment at time t if the money is put into Spartan Securities, and let w be the value if put into Wolverine Bonds

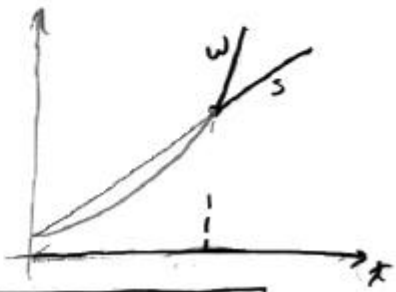
Since s is a linear function of t that grows at the rate of \$500 per year and starts at \$1000,

$$s = 500t + 1000$$

Since w is exponential with a starting value of \$1000 and a doubling time of 10 years,

$$w = 1000 \cdot 2^{\frac{t}{10}}.$$

A calculator computation shows that the graphs cross when $t \approx 45.8$.



Wolverine Bonds first becomes the better investment in the year 2045.

7. (9 points) Suppose that h is a rational function with all of these properties:

   (a) The graph of h has $x=-2$ and $x=5$ as vertical asymptotes.

   (b) The graph of h has $y=3$ as a horizontal asymptote.

   (c) $h(1)=0$.

   Find a possible formula for $h(x)$. (There are many different possible correct answers. You only need to find one.)

One possibility is $h(x) = \dfrac{3(x-1)^2}{(x+2)(x-5)}$ There are infinitely many others

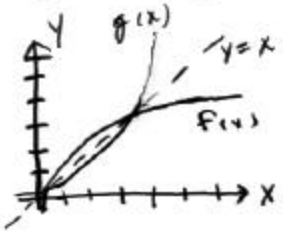
Notice that the numerator has to have the same degree as the denominator for $y=3$ to be a horizontal asymptote.

ESSAY QUESTION

Where explanations are required in this exercise, write clearly using complete sentences.

8. Let f and g be the functions given by the formulas $f(x) = 2 + \ln(x)$ and $g(x) = e^x / e^2$.

(a) (4 points) Graph both functions on the same screen on your calculator using the window $-6 \le x \le 6$, $-2 \le y \le 6$. (If you have a TI-92 or other calculator with a screen much wider than it is high, you may want to replace the range for x by $-10 \le x \le 6$.) How do the graphs of f and g suggest that the two functions are related to each other? Explain, referring to a rough sketch of the graph.



Since the graph of g appears to be the reflection of that of f about the line $y = x$, it appears that g is the inverse of f.

(b) (4 points) In the first of the following tables, fill in the values for $f(x)$, giving your answers rounded to four places after the decimal point. Now fill in the second table with some three values for x and the corresponding values for $g(x)$, so that the two tables together provide further evidence for the relationship between f and g suggested by their graphs. Explain on the next page.

x	$f(x)$
1	2
2	2.6931
3	3.0986

x	$g(x)$
2	1
2.6931	2
3.0986	3

Explanation for (b):

Since $g(f(x)) = x$ for $x = 1, 2, 3$, this reinforces our belief that g is the inverse of f.

(c) (4 points) Now start with the formula for $f(x)$ and show how the formula for $g(x)$ can be derived from it, to show for certain that the relationship between f and g suggested by (a) and (b) really does hold. Explain.

$$2 + \ln x$$

$$x = 2 + \ln y$$

$$\ln y = x - 2$$

$$= e^{x-2} = \frac{e^x}{e^2} = f^{-1}(x)$$

$$\therefore g(x) = f^{-1}(x)$$

Since the standard procedure for finding inverses shows that $g = f^{-1}$, this settles the matter.