Math 285.002
Homework 10
Solutions

16.5 #28. The key to this exercise lies in the observation that the desired event, Yolanda's meeting Xavier, occurs if and only if Xavier arrives after Yolanda ($Y \le X$) but no more than thirty minutes after Yolanda ($X \le Y+30$). Thus, we are looking for $P(Y \le X \le Y+30)$. By independence, the joint p.d.f. of X and Y is $f_1(x)f_2(y)$, so the desired probability is obtained in the following computation. Notice that integration by parts can be used at the appropriate place to find that an antiderivative for ye^{-y} is $-(y+1)e^{-y}$.

$$\begin{aligned}
P(Y \le X \le Y+30) &= \int_0^{10}\int_y^{y+30} f_1(x)f_2(y)\,dx\,dy \\
&= \int_0^{10} \tfrac{1}{50}\, y \int_y^{y+30} e^{-x}\,dx\,dy \\
&= \tfrac{1}{50}\int_0^{10} y\left(-e^{-x}\right)\Big|_{x=y}^{y+30} dy \\
&= \tfrac{1}{50}\int_0^{10} y\left(e^{-y}-e^{-y-30}\right)dy \\
&= \frac{1-e^{-30}}{50}\int_0^{10} ye^{-y}\,dy \\
&= \frac{1-e^{-30}}{50}\left(-(y+1)e^{-y}\right)\Big|_0^{10} \\
&= \frac{1-e^{-30}}{50}\left(1-11e^{-10}\right) \\
&\approx 0.01999.
\end{aligned}$$

Thus, the probability is only about 2% that Xavier and Yolanda will meet. Bummer!

16.8 #40 (a). Let E be the cone, let B be a rectangular box containing the cone, and suppose that B is partitioned into sub-boxes as on pages 1042–1043. Let g be extended from E to all of B by letting $g(P)=0$ for points P of B that are not in E. Let $\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right)$ represent a point in sub-box B_{ijk}, and let ΔV be the volume of each of these sub-boxes. Then the weight of the matter in sub-box B_{ijk} is approximately $g\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right)\Delta V$, so the work needed to raise that matter to the appropriate height is approximately $g\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right)h\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right)\Delta V$. The total work W thus needed to form the mountain is given by this approximation:

$$W = \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} g\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right)h\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right)\Delta V.$$

The exact value for the work can be found by taking the limit as $l, m, n \to \infty$, which produces the following integral:

$$W = \iiint_B g(P)h(P)\,dv = \iiint_E g(P)h(P)\,dv,$$

where the last equality comes about because $g(P) = 0$ outside E.

(b). In this case, $g(P) = 200$, $h(P) = z$, and the height of the mountain as a function of the distance r from the center of its base is easily seen to be $12400 - \frac{1}{5}r$ since that height is 12,400 ft when $r = 0$ and falls off linearly to 0 when $r = 62000$. Letting the origin be at the center of the base of the mountain and converting the triple integral from (a) to cylindrical coordinates produces

$$\begin{aligned}
W &= \iiint_E 200z\,dV \\
&= \int_0^{2\pi}\int_0^{62000}\int_0^{12400-\frac{1}{5}r} 200z\,r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{62000} \left(100z^2\right)\Big|_{z=0}^{12400-\frac{1}{5}r} r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{62000} 100\left(12400-\tfrac{1}{5}r\right)^2 r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{62000} \left(15376000000r - 496000r^2 + 4r^3\right)dr\,d\theta \\
&= \int_0^{2\pi} d\theta \int_0^{62000} \left(15376000000r - 496000r^2 + 4r^3\right)dr \\
&= 2\pi\left(7688000000r^2 - \tfrac{496000}{3}r^3 + r^4\right)\Big|_0^{62000} \\
&= 2\pi\left(7688000000 \cdot 62000^2 - \tfrac{496000}{3} \cdot 62000^3 + 62000^4\right) \\
&\approx 3.09475 \times 10^{19} \text{ ft-lb},
\end{aligned}$$

so about 3.1×10^{19} ft-lb of work are needed to build this mountain.