Math 285.002
Homework 11
Solutions

16.9 #24. Suppose that f is continuous on $[0,1]$ and that R is the triangular region with vertices $(0,0)$, $(1,0)$, and $(0,1)$. We are to show that

$$\iint_R f(x+y)\,dA = \int_0^1 u\,f(u)\,du.$$

This suggests that we let $u = x + y$, which gives us half of a transformation to which we might be able to apply the methods of this section. As hinted in class, we can try letting $v = x - y$ to get the other half of the transformation, which can then be solved for x and y in terms of u and v:

$$x = \tfrac{1}{2}(u+v), \qquad y = \tfrac{1}{2}(u-v).$$

This transformation is one-to-one, by the very fact that we can solve for x and y. The sides of the original region are the lines $y = 0$, $x = 0$, and $x + y = 1$, which after the transformation is performed become the lines $v = u$, $v = -u$, and $u = 1$, which enclose the triangular region S in the uv-plane with vertices $(0,0)$, $(1,1)$, and $(1,-1)$. The Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

Therefore

$$\begin{aligned} \iint_R f(x+y)\,dA &= \iint_S f(u)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \\ &= \tfrac{1}{2}\int_0^1 \int_{-u}^{u} f(u)\,dv\,du \\ &= \tfrac{1}{2}\int_0^1 2u\,f(u)\,du \\ &= \int_0^1 u\,f(u)\,du, \end{aligned}$$

as claimed.

17.2 #44. As indicated in class, you may assume that $B = |\mathbf{B}|$ depends only on the distance from the wire. Let C be the indicated circle of radius r, parametrized so that the circle is traversed counterclockwise as shown in the illustration, and let $\mathbf{T}(x, y, z)$ be the unit tangent vector to this curve at the point (x, y, z) on it. Since $\mathbf{B}$ and $\mathbf{T}$ point the same direction, $\mathbf{B} \cdot \mathbf{T} = |\mathbf{B}||\mathbf{T}|\cos 0 = B$, so

$$\mu_0 I = \int_C \mathbf{B} \cdot d\mathbf{r} = \int_C \mathbf{B} \cdot \mathbf{T}\, ds = \int_C B\, ds = 2\pi r B.$$

Solving this for B produces $B = \dfrac{\mu_0 I}{2\pi r}$.