Math 285.002
Homework 12
Solutions

17.3 #34.

(a) Notice that

$$\mathbf{F}(x,y,z) = \left\langle \frac{cx}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{cy}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{cz}{\left(x^2+y^2+z^2\right)^{3/2}} \right\rangle.$$

Although we could use the methods of the text to find a function f such that $\mathbf{F} = \nabla f$, it is clear from a few moments' inspection that if $f(x,y,z) = -c\big/\sqrt{x^2+y^2+z^2}$ (that is, $f(\mathbf{r}) = -c/|\mathbf{r}|$), then $\mathbf{F} = \nabla f$. Thus, the work done in moving an object from a point P_1 along a path C to a point P_2 in terms of the distances d_1 and d_2 from these points to the origin is

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = -\frac{c}{d_2} + \frac{c}{d_1} = c\left(\frac{1}{d_1} - \frac{1}{d_2}\right).$$

(b) The work done is

$$-mMG\left(\frac{1}{d_1}-\frac{1}{d_2}\right) = -\left(5.97\times10^{24}\right)\left(1.99\times10^{30}\right)\left(6.67\times10^{-11}\right)\left(\frac{1}{1.52\times10^{11}} - \frac{1}{1.47\times10^{11}}\right)$$
$$= 1.77\times10^{32}\text{ J}.$$

(c) The work done is

$$W = \varepsilon qQ\left(\frac{1}{d_1}-\frac{1}{d_2}\right) = \left(8.985\times10^{10}\right)\left(-1.6\times10^{-19}\right)(1)\left(\frac{1}{10^{-12}} - \frac{1}{0.5\times10^{-12}}\right) = 14{,}376\text{ J}.$$

17.4 #29. From the hypotheses of 16.9.9, the transformation $x = g(u,v)$, $y = h(u,v)$ has continuous first-order partial derivatives on S. We will actually assume that the second-order partial derivatives of h are continuous on an open region containing S so that Clairaut's and Green's Theorems can be applied at appropriate points. Notice that the Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$ is also continuous, and therefore always positive or always negative (by the Intermediate Value Theorem) since it is never zero. We also need to assume that ∂S is a piecewise-smooth simple closed curve. Following the hint,

$$\iint_R dx\,dy = A(R) = \oint_{\partial R} x\,dy. \tag{1}$$

We now let $u = u(t)$ and $v = v(t)$, where $a \le t \le b$, be a parametrization of ∂S so that ∂S is traced with the positive orientation as t increases from a to b. It follows that letting $x(t) = g\left(u(t),v(t)\right)$ and $y(t) = h\left(u(t),v(t)\right)$, where $a \le t \le b$, traces ∂R as t

increases from a to b, but we cannot be certain of the orientation this gives ∂R. Thus, with the plus or minus sign selected to make the area integral $\oint_{\partial R} x\,dy$ nonnegative,

$$\begin{aligned}
\oint_{\partial R} x\,dy &= \pm\int_a^b x(t)y'(t)\,dt \\
&= \pm\int_a^b g\left(u(t),v(t)\right)\frac{d}{dt}h\left(u(t),v(t)\right)dt \\
&= \pm\int_a^b g\left(u(t),v(t)\right)\left(\frac{\partial h}{\partial u}\frac{du}{dt}+\frac{\partial h}{\partial v}\frac{dv}{dt}\right)dt \\
&= \pm\oint_{\partial S}\left(g\left(u,v\right)\frac{\partial h}{\partial u}du+g\left(u,v\right)\frac{\partial h}{\partial v}dv\right).
\end{aligned} \tag{2}$$

Let $P(u,v)=g\left(u,v\right)\frac{\partial h}{\partial u}$ and $Q(u,v)=g\left(u,v\right)\frac{\partial h}{\partial v}$, and observe that P and Q have continuous first-order partial derivatives on an open region containing S. Applying Green's Theorem to this last integral, using Clairaut's Theorem at the appropriate point, and switching notation from partial derivatives of g and h to those of x and y when convenient shows that

$$\begin{aligned}
\oint_{\partial S}\left(g\left(u,v\right)\frac{\partial h}{\partial u}du+g\left(u,v\right)\frac{\partial h}{\partial v}dv\right) &= \iint_S\left(\frac{\partial}{\partial u}\left(g\left(u,v\right)\frac{\partial h}{\partial v}\right)-\frac{\partial}{\partial v}\left(g\left(u,v\right)\frac{\partial h}{\partial u}\right)\right)du\,dv \\
&= \iint_S\left(\frac{\partial g}{\partial u}\frac{\partial h}{\partial v}+g\left(u,v\right)\frac{\partial^2 h}{\partial u\,\partial v}-\frac{\partial g}{\partial v}\frac{\partial h}{\partial u}-g\left(u,v\right)\frac{\partial^2 h}{\partial v\,\partial u}\right)du\,dv \\
&= \iint_S\left(\frac{\partial g}{\partial u}\frac{\partial h}{\partial v}-\frac{\partial g}{\partial v}\frac{\partial h}{\partial u}\right)du\,dv \\
&= \iint_S\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)du\,dv \\
&= \iint_S\frac{\partial(x,y)}{\partial(u,v)}du\,dv,
\end{aligned}$$

which when combined with (1) and (2) gives

$$\iint_R dx\,dy = \pm\iint_S\frac{\partial(x,y)}{\partial(u,v)}du\,dv,$$

where the plus or minus sign is chosen to make the integral positive. Now $\frac{\partial(x,y)}{\partial(u,v)}$ does not change sign on S, so the last integral, with the appropriate choice of sign, just equals $\iint_S\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$. Thus,

$$\iint_R dx\,dy = \iint_S\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv,$$

as claimed.