Math 285.002
Homework 13
Solutions

17.5 #34. We are to prove **Green's second identity:** If D and C satisfy the hypotheses of Green's theorem and f and g have partial derivatives with the amount of continuity needed to satisfy Green's first identity, then

$$\iint_D \left(f\nabla^2 g - g\nabla^2 f\right)dA = \oint_C \left(f\nabla g - g\nabla f\right)\cdot \mathbf{n}\,ds.$$

It turns out that there is almost nothing to this, since applying Green's first identity twice shows that

$$\begin{aligned}\iint_D \left(f\nabla^2 g - g\nabla^2 f\right)dA &= \iint_D f\nabla^2 g\,dA - \iint_D g\nabla^2 f\,dA \\ &= \left(\oint_C f\left(\nabla g\right)\cdot \mathbf{n}\,ds - \iint_D \nabla f\cdot\nabla g\,dA\right) - \left(\oint_C g\left(\nabla f\right)\cdot \mathbf{n}\,ds - \iint_D \nabla g\cdot\nabla f\,dA\right) \\ &= \oint_C f\left(\nabla g\right)\cdot \mathbf{n}\,ds - \oint_C g\left(\nabla f\right)\cdot \mathbf{n}\,ds \\ &= \oint_C \left(f\nabla g - g\nabla f\right)\cdot \mathbf{n}\,ds,\end{aligned}$$

where the fact that $\mathbf{x}\cdot\mathbf{y} = \mathbf{y}\cdot\mathbf{x}$ for any two vectors $\mathbf{x}$ and $\mathbf{y}$ was used at one point to cancel two equal integrals.

17.6 #48 (a). With θ and α as shown in the figure in the text, we notice that the point $(x, y, 0)$ lies at distance $r = b + a\cos\alpha$ from the origin, so the parametric representation of a point (x, y, z) on the torus is $x = (b + a\cos\alpha)\cos\theta$, $y = (b + a\cos\alpha)\sin\theta$, $z = a\sin\alpha$, where $0 \le \theta \le 2\pi$, $0 \le \alpha \le 2\pi$.

(c). We compute that

$$\mathbf{r}_\theta = \left\langle \frac{\partial x}{\partial\theta}, \frac{\partial y}{\partial\theta}, \frac{\partial z}{\partial\theta} \right\rangle = \left\langle -(b + a\cos\alpha)\sin\theta, (b + a\cos\alpha)\cos\theta, 0 \right\rangle$$

and

$$\mathbf{r}_\alpha = \left\langle \frac{\partial x}{\partial\alpha}, \frac{\partial y}{\partial\alpha}, \frac{\partial z}{\partial\alpha} \right\rangle = \left\langle -a\sin\alpha\cos\theta, -a\sin\alpha\sin\theta, a\cos\alpha \right\rangle.$$

It is then easy to compute that

$$\mathbf{r}_\theta \times \mathbf{r}_\alpha = a(b + a\cos\alpha)\left\langle \cos\alpha\cos\theta, \cos\alpha\sin\theta, \sin\alpha \right\rangle,$$

from which it follows that

$$\left|\mathbf{r}_\theta \times \mathbf{r}_\alpha\right| = a(b + a\cos\alpha)\sqrt{\cos^2\alpha\cos^2\theta + \cos^2\alpha\sin^2\theta + \sin^2\alpha} = a(b + a\cos\alpha).$$

Therefore the surface area of the torus is

$$\iint_{[0,2\pi]\times[0,2\pi]} |\mathbf{r}_\theta \times \mathbf{r}_\alpha| \, dA = \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos\alpha)\, d\theta \, d\alpha$$

$$= 2\pi \int_0^{2\pi} a(b + a\cos\alpha)\, d\alpha$$

$$= 2\pi \cdot 2\pi ab$$

$$= 4\pi^2 ab.$$