**Math 285.002**
**Homework 8**
**Solutions**

*15.6 #56.* **Fact:** The product of the $x$-, $y$-, and $z$-intercepts of any tangent plane to the surface $xyz = c^3$ is a constant.

**Proof:** The authors intended that $c \neq 0$, although it is easy to check that this fact is also true if $c = 0$. In any case, we will assume that $c \neq 0$. Let $(x_0, y_0, z_0)$ be a point on this surface. Notice that $x_0 y_0 z_0 = c^3 \neq 0$, so $x_0$, $y_0$, and $z_0$ are all nonzero. The gradient of $F(x, y, z) = xyz$ at $(x_0, y_0, z_0)$ is $\nabla F(x_0, y_0, z_0) = \langle y_0 z_0, x_0 z_0, x_0 y_0 \rangle$, so the equation of the tangent plane to the surface at that point is

$$y_0 z_0 (x - x_0) + x_0 z_0 (y - y_0) + x_0 y_0 (z - z_0) = 0.$$

The $x$-intercept of this is found by setting $y = z = 0$ and solving for $x$, giving

$$\begin{aligned} y_0 z_0 (x - x_0) - 2 x_0 y_0 z_0 &= 0 \\ x - x_0 &= 2x_0 \\ x &= 3x_0. \end{aligned}$$

Similarly, the $y$- and $z$-intercepts are $3y_0$ and $3z_0$, respectively, so the product of the three intercepts is $3x_0 \cdot 3y_0 \cdot 3z_0 = 27 x_0 y_0 z_0 = 27c^3$, a constant.

*15.7 #50.* We are to maximize $P = 2pq + 2pr + 2rq$ subject to $p + q + r = 1$. We solve for $r$ and substitute back into $P$:

$$r = 1 - p - q \quad \Rightarrow \quad P = 2pq + 2p(1 - p - q) + 2q(1 - p - q).$$

This is a function of $p$ and $q$ that can be maximized subject to $0 \le p \le 1$, $0 \le q \le 1$:

$$\begin{aligned} P_p &= 2q + 2(1 - p - q) + 2p(-1) + 2q(-1) \\ &= 2q + 2 - 2p - 2q - 2p - 2q \\ &= 2 - 4p - 2q. \\ P_q &= 2p + 2p(-1) + 2(1 - p - q) + 2q(-1) \\ &= 2p - 2p + 2 - 2p - 2q - 2q \\ &= 2 - 2p - 4q. \end{aligned}$$

To find the critical points, set $2 - 4p - 2q$ and $2 - 2p - 4q$ both equal to 0 and solve simultaneously to obtain $p = q$. Thus,

$$2 - 4p - 2p = 0 \quad \Rightarrow \quad 2 = 6p \quad \Rightarrow \quad p = \tfrac{1}{3}.$$

Similarly, $q = \frac{1}{3}$, so $(p, q) = \left(\frac{1}{3}, \frac{1}{3}\right)$ is a critical point.

We are finding a maximum on $D=\{(p,q):0\le p\le 1, 0\le q\le 1\}$, and $\left(\frac{1}{3},\frac{1}{3}\right)$ does lie in this region. At that point,

$$P\left(\tfrac{1}{3},\tfrac{1}{3}\right)=2\left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right)+2\left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right)+2\left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right)=\tfrac{6}{9}=\tfrac{2}{3}.$$

On the boundary of the closed bounded region $D$, either $p=0$ (in which case $P=2q(1-q)=2q-2q^2$, which by the usual calculus techniques has maximum $\frac{1}{2}$), or $q=0$ (in which case $P$ again has maximum $\frac{1}{2}$), or $p=1$ (in which case $q=r=0$ and $P=0$), or $q=1$ (in which case again $P=0$), so the maximum value of $P$ on $D$ occurs at the critical point $\left(\frac{1}{3},\frac{1}{3}\right)$ and is $\frac{2}{3}$.