Math 285.002
Homework 9
Solutions

16.4 #34.

(a) We are to show that $\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \pi$. Letting D_a be the disk with radius a and center at the origin, we have

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)}\, dA = \lim_{a\to\infty} \int_0^{2\pi}\int_0^a e^{-r^2} r\, dr\, d\theta$$

$$= \lim_{a\to\infty} \int_0^a \int_0^{2\pi} e^{-r^2} r\, d\theta\, dr = 2\pi \lim_{a\to\infty} \int_0^a re^{-r^2}\, dr = 2\pi \lim_{a\to\infty} \left.\frac{e^{-r^2}}{-2}\right|_0^a$$

$$= 2\pi \lim_{a\to\infty}\left(\frac{e^{-a^2}}{-2} + \frac{1}{2}\right) = 2\pi\left(\frac{1}{2}\right) = \pi.$$

(b) We are to show that $\int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \pi$. Applying what was shown in (a), we get

$$\pi = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \lim_{a\to\infty} \iint_{S_a} e^{-(x^2+y^2)}\, dA = \lim_{a\to\infty} \int_{-a}^{a}\int_{-a}^{a} e^{-x^2} e^{-y^2}\, dy\, dx$$

$$= \lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\, dx \int_{-a}^{a} e^{-y^2}\, dy\right) = \left(\lim_{a\to\infty}\int_{-a}^{a} e^{-x^2}\, dx\right)\left(\lim_{a\to\infty}\int_{-a}^{a} e^{-y^2}\, dy\right) = \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy.$$

(c) Notice that $\int_{-\infty}^{\infty} e^{-x^2}\, dx \ge 0$, since the integrand is positive. Thus,

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2} = \sqrt{\int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy} = \sqrt{\pi}.$$

(d) Finally, using the substitution $t = \sqrt{2}x$ at the appropriate point produces

$$\int_{-\infty}^{\infty} e^{-t^2/2} dt = \lim_{a\to\infty}\int_{-a}^{a} e^{-\left(t/\sqrt{2}\right)^2}\, dt = \lim_{a\to\infty}\int_{-a/\sqrt{2}}^{a/\sqrt{2}} e^{-x^2}\sqrt{2}\, dx = \sqrt{2}\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{2}\sqrt{\pi} = \sqrt{2\pi}.$$

Since the letter representing the variable of integration is unimportant, we have $\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$, as claimed.