Introduction to Financial Mathematics

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My Information

- E-mail address: `marymorj (at) umich.edu`
- Financial work experience includes 2 years in public finance investment banking and one summer in securitized products and interest rate strategy
- Second year in AIM masters program with finance as partner discipline

What is an Investment Bank

An investment bank performs the following services

- Underwriting of securities (e.g. stocks and bonds)
- Market maker - principal vs. agency transactions
- Derivatives counterparty
- Advisory services - M&A, change in capital structure, etc.
- Prime brokerage

Bonds

A **bond** is a financial security where the **issuer** borrows from the **bond-holder** with obligation to pay back both principal and interest.

The price of a **zero-coupon bond**, one which pays all interest plus principal in one final payment, paying \$1 at time N years from now can be calculated as

$$P_{0,N} = \frac{1}{(1 + r_{ann})^N}$$

where r_N is the annualized rate of interest corresponding to time N in years.

Bonds Continued

The price of a **coupon-bearing bond** with interest rate r can thus be priced as the sum of zero-coupon bonds as follows:

$$P_{coup} = rP_{0,1} + rP_{0,2} + ... + rP_{0,N-1} + (1 + r)P_{0,N}$$

where the subscripts correspond to the timing of the coupon cash-flows.

Swaps

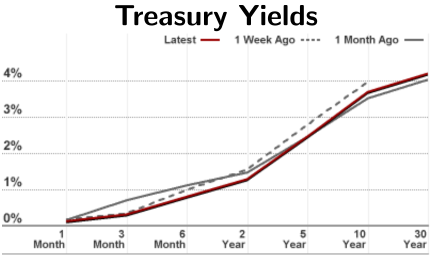
A **swap** is an agreement where one counterparty agrees to receive a fixed-rate payment in exchange for a floating rate payment (usually an index or percent of an index)

The fixed rate of the swap represents the market's assessment of the rate which gives the swap agreement zero value. (Note that swaps can be quoted **off-market**, but we will not consider these transactions)

$$\text{Present Value}(Swap) = \sum_{i=1}^{N} \left(r_{float_{i-1}} - r_{fix} \right) P_{0,i} = 0$$

Bootstrapping the Yield Curve

The **yield curve** is the plot of bond rates vs. maturity. The current Treasury yield curve is plotted below



We may observe that we can extract the **forward rates** from the yield curve by applying **arbitrage-free pricing**.

Options in the Fixed Income Markets

We primarily add optionality by working with the following

- Swaptions
- Caps/floors
- Options on futures

Mortgages are complicated instruments that also have optionality, however this is reserved for a seperate discussion

Caps/Floors and Swaptions

A cap pays the difference if some variable rate goes above the cap strike rate times some notional amount during a series of set time intervals

$$\text{PV(Cap)} = \text{Notional} \times \sum_{i=\alpha}^{N} P_{0,i+1} \left[r_{\text{var}_i} - r_{\text{fix}_i} \right]_+$$

A cap is a series of options called **caplets**

A swaption is an option on a swap rate

$$\text{PV(Swaption)} = \text{Notional} \times P_{0,\alpha} \left[r - K \right]_+ \sum_{i=\beta}^{N} P_{\alpha,i+1}$$

Where α is the option expiry and β is the option start period.

The Black-Scholes (and Merton) Revolution

The Black-Scholes equation calculates call price as a function of

- Underlying current rate/price S_0
- Strike rate/price K
- Time to expiry T
- Variance of the underlying σ^2

$$\text{Call} = BS\left(S_0, K, T, \sigma^2\right)$$

The Nobel Prize in Economics for 1997 was awarded to Robert C. Merton and Myron Scholes. If their partner, Fischer Black, were alive, he would have shared the prize.

Pricing Options

The Black-Scholes SDE is given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (W_t \text{ Brownian Motion})$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S} - rV = 0$$

Options are priced using a variety of techniques

- Equation
- Finite difference schemes
- Monte Carlo simulations

Volatility in the Rates Market

A number of models have evolved through the years, for example a two-factor **Vasicek model** may be given by

$$r_t = x_t + y_t$$
$$dx_t = k_x(\theta_x - x_t)dt + \sigma_x dW_1(t)$$
$$dy_t = k_y(\theta_y - y_t)dt + \sigma_y dW_2(t)$$

- Can usually explain 85% to 90% of variation with just two factors
- How do we interpret this model?

Market Models

- The **lognormal forward-LIBOR model** (LFM) is one of the most popular models and is widely used
- The idea of the LFM is to break the yield curve into segments of forward LIBOR 1-period rates
- For example, if looking at 3-month LIBOR, then break curve into 3-month segments each representing the forward 3-month rate: $F(t; T_1, T_2)$ corresponds to the forward rate from T_1 to T_2
- At each time period (e.g. T_2 above) the evolution of F is a martingale and given by $dF(t; T_1, T_2) = vF(t; T_1, T_2)dW_t$ (W_t Brownian motion)
- Now interpretation is much easier - everything in terms of "instantaneous vol"

Estimating the Instantaneous Volatility Matrix

We can organize the instantaneous volatilities into the following matrix

Instant. Vols	Time: $t \in (0, T_0]$	(T_0, T_1)	(T_1, T_2)	...	$(T_{M-2}, T_{M-1}]$
Fwd Rate: $F_1(t)$	$\sigma_{1,1}$	Dead	Dead	...	Dead
F_2	$\sigma_{2,1}$	$\sigma_{2,2}$	Dead	...	Dead
$\vdots$	...	...	...	...	...
$F_M(t)$	$\sigma_{M,1}$	$\sigma_{M,2}$	$\sigma_{M,3}$	...	$\sigma_{M,M}$

We may reduce the number of parameters by changing the matrix in the following way (ϕ_i constant, and ψ_i parametric form of i, t plus four more variables)

Instant. Vols	Time: $t \in (0, T_0]$	(T_0, T_1)	(T_1, T_2)	...	$(T_{M-2}, T_{M-1}]$
Fwd Rate: $F_1(t)$	$\phi_1\psi_1$	Dead	Dead	...	Dead
F_2	$\phi_2\psi_2$	$\phi_2\psi_1$	Dead	...	Dead
$\vdots$	...	...	...	...	...
$F_M(t)$	$\phi_M\psi_M$	$\phi_M\psi_{M-1}$	$\phi_M\psi_{M-1}$	...	$\phi_M\psi_1$

Time Evolution of Volatility

Using our parametric form, we are able to develop a model for the time evolution of the volatility curve. Further steps include

- Calibration: Need to at least acurately price securities for T_0 volatility curve
- Use historical data to relate all instantaneous rates
- How will model be used - accurate pricing or relative value?
- Model very flexible and many formulations for pricing or comparing securities

Final Remarks

- This is a large (and becoming even more popular) field that has opportunities in higher education, research and industry
- Broad range of problems - we looked at only one small segment of financial mathematics
- Combines elements of PDE's, Probability and Stochastic Processes, Linear Algebra, Computing, Economic Theory, among others
- Allows easy implamentation of ideas - can design products/solutions/research and bring to market rapidly

Thank you, and don't hesitate to contact me at `marymorj (at) umich.edu` with any questions.