Conformal Field Theory and 2-D Critical Systems

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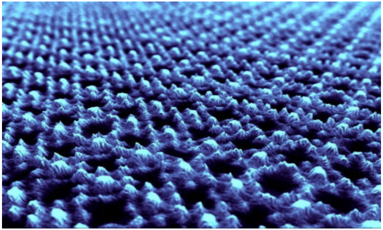
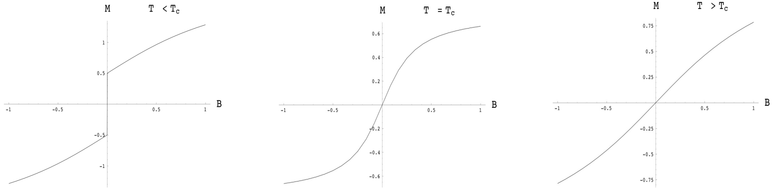
Outline

The 2-D Ising Ferromagnet



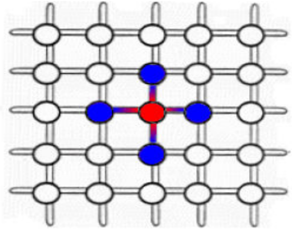
- 2-D lattice of atoms in an external magnetic field, B, at some temperature T, spaced a units apart.
- Atom spins generally align in direction of B. Magnetization M and B have the same sign.
- Lattice spacing a is much less than the magnet dimensions. Away from the magnet boundary, we have translation/rotation invariance.

Clusters, Correlation Length, Critical Temperature



- There exists a critical temperature T_c such that if $B = 0$
    - $T > T_c \quad \Rightarrow \quad M = 0.$
    - $T < T_c \quad \Rightarrow \quad M > 0$ or $M < 0$.
- So if $T < T_c$, $B = 0$, like spins arrange into clusters of average size ξ, "correlation length."
- As $T \nearrow T_c$, $B = 0$, $\xi \to \infty$. Clusters appear on all length scales much greater than a. System is now also "scale invariant." When $T = T_c$, system is called "critical."

Q-State Potts Model



- Lattice site i randomly assigned number $\sigma_1 \in \{1, 2, \ldots, q\}$.
- Total energy of configuration $\{\sigma_i\}$: $H[\{\sigma_i\}] = J \sum_{\langle ij \rangle} \delta_{\sigma_i, \sigma_j}$.
- Probability of configuration $\{\sigma_i\}$: $P(\{\sigma_i\}) = e^{-\beta H[\{\sigma_i\}]}/Z$.
- $q = 2$ corresponds to two "spin"-states: the Ising model.

Spin Correlators in the Potts Model

- Critical Temperature $\beta_c = \ln(\sqrt{q}+1)$.
- 2-point correlator $(q = 2, \sigma_i = \pm 1)$:

$$\begin{aligned} \langle \sigma_i \sigma_j \rangle &= \frac{1}{Z} \sum_{\{\sigma_k\}} \sigma_i \sigma_j e^{-\beta H[\{\sigma_k\}]} \\ &\sim dist(i,j)^{-2\Delta} e^{-dist(i,j)/\xi}. \end{aligned}$$

- As $\beta \to \beta_c$, $\xi \to \infty$, $\langle \sigma_i \sigma_j \rangle \to dist(i,j)^{-2\Delta}$.
- Covariance: If we dilate, that is, $a \mapsto \lambda a$,

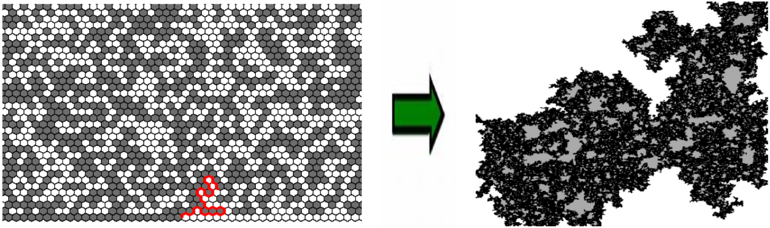
$$\langle \sigma_i \sigma_j \rangle \mapsto \langle \sigma_i' \sigma_j' \rangle = \lambda^{-2\Delta} \langle \sigma_i \sigma_j \rangle.$$

Hence, Δ is the "scaling dimension" of σ_i. We infer that

$$\sigma_i \mapsto \sigma_i' = \lambda^{-\Delta} \sigma_i.$$

The Continuum Limit

Take the continuum limit $a \to 0$:



$$\begin{aligned} \sigma_i &\to \phi(z_i), \quad z_i \in \mathbb{C} \\ \beta H[\{\sigma_i\}] &\to S[\phi] \equiv \int d^2x \, \nabla^2 \phi \\ \langle \sigma_i \sigma_j \rangle &\to \langle \phi(z_1)\phi(z_2) \rangle = \frac{1}{Z} \int \mathcal{D}\phi \, \phi(z_i)\phi(z_j) \, e^{-S[\phi]} \end{aligned}$$

Conformal Transformations

Conformal Transformations

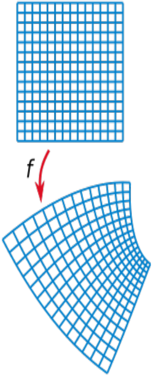
- are transformations mapping of $\Omega \subset \mathbb{C}^e$ into $\mathbb{C}^e$ in such a way that angles between curves are preserved.
- can be thought of as any smooth transformation that dilates our "meter-stick" (metric tensor) by a local scale factor:

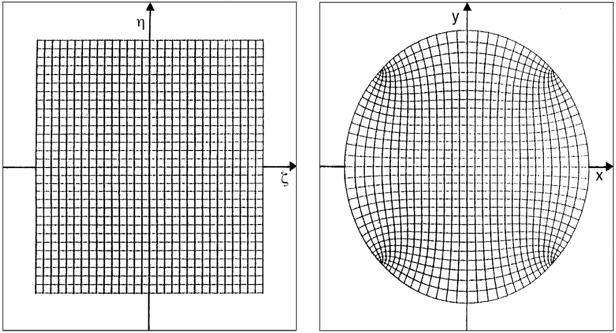
$$g_{\mu\nu} \mapsto g'_{\mu\nu} = \Lambda(z) g_{\mu\nu}.$$

- Local Conformal Transformations: $\Omega \neq \mathbb{C}$
- Global Conformal Transformations: Conformal transformations that are bijections of $\mathbb{C}^e$ onto itself. These have the form

$$z \mapsto z' = \frac{a + bz}{c + dz}.$$

Examples of Conformal Transformations





ω - plane   z - plane

$z = \mathrm{sn}(\omega)\,\mathrm{dn}(\omega)/\mathrm{cn}(\omega)$

Infinitesimal Conformal Transformations (ICTs)

- Conformal transformation with the form $z \mapsto z' = z + \epsilon(z)$, where $\epsilon(z)$ is "small" in a neighborhood of z are "infinitesimal conformal transformations (ICTs)."
- Two kinds
  - Local: $\epsilon(z) = \sum_{-\infty}^{\infty} c_n z^n$.
  - Global: $\epsilon(z) = \alpha + \beta z + \gamma z^2$
    - α: translation by a units.
    - β: dilation by $|1 + \beta|$, rotation by $\arg(1 + \beta)$.
    - γ: "special conformal" (invert, translate, invert again).

Covariance under ICTs

- Covariance: Under infinitesimal conformal transformations

$$\phi(z) \mapsto \phi'(z') = \left(\frac{dz'}{dz}\right)^{-h} \left(\frac{d\bar{z}'}{d\bar{z}}\right)^{-\bar{h}} \phi(z),$$

  where h, $\bar{h}$ are "conformal dimensions," (same as $\Delta/2$ if ϕ doesn't change under rotations).

- Correlators transform as

$$\delta_\epsilon \langle \phi(z_1) \dots \phi(z_n) \rangle = -\sum_i (\epsilon(z_i)\partial_{z_i} + \partial_{z_i}\epsilon(z_i)h_i)\langle \phi_1(z_1) \dots \phi_n(z_n) \rangle$$

Invariance of Correlators under Global ICTs

- It is theorized that at their critical point, 2-D lattice systems are "invariant under global ICTs." That is

$$\delta_\epsilon \langle \phi_1(z_1) \dots \phi_n(z_n) \rangle = 0.$$

- Global ICTs are $\epsilon = \alpha, \beta z, \gamma z^2$. Plugging into the above ...

$$\begin{aligned} \langle \phi_1(z_1) \dots \phi_n(z_n) \rangle &= G(\text{cross-ratios}) \prod_{i<j} (z_i - z_j)^{\mu_{ij}} \\ \sum_{i, i \neq j} \mu_{ij} &= -2h_j, \quad \mu_{ij} = \mu_{ji}, \quad \mu_{ii} = 0. \end{aligned}$$

Correlators

The forms of corrlators are thus restricted to the following: $(z_{ij} = z_i - z_j)$

- $n = 2$: $\langle \phi_1(z_1)\phi_2(z_2)\rangle = \frac{C_{12}\delta_{h_1,h_2}}{z_{12}^{2h_1}} \times c.c.$
- $n = 3$:
  $\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{12}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_1+h_3-h_2}} \times c.c.$
- $n = 4$:
  $\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle = G\left(\frac{z_{12}z_{34}}{z_{13}z_{24}}\right) \prod_{i<j} {z_{ij}}^{\mu_{ij}} \times c.c.$

Some Questions

This is all good, but ...

- how many different species of fields ϕ are in a generic CFT?
- are there theories with only a finite number of fields?
- what physical phenomenon can those theories describe?
- what additional restrictions do such theories contain that allow us to further these calculations by, say, computing G?

Kac's Dimensions, Minimal Models

- $h_{r,s}$ is called a Kac Dimension; a "special dimension" indexed by integers r and s.
- There exist CFTs called "minimal models" with finitely many species of fields ϕ of different Kac's Dimensions $h_{r,s}$.
- These fields are called Kac operators, and are denoted $\phi_{r,s}$.
- Correlators of Kac operators obey special PDEs. Often, these extra constraints lets us compute G in the four point correlator.
- Minimal Models are indexed by integers $m = 2, 3, 4, \ldots$. Such a model has Kac operators with $r = 1, \ldots m-1$, $s = 1, \ldots, m$. Several of these fields are identical.

CFT and the Critical Ising Model

- It can be shown (without CFT) that in the Ising model,
  - the spin field σ has $h_\sigma = 1/16$.
  - the energy field ϵ has $h_\epsilon = 1/2$.
- Take a look at the $m = 3$ minimal model. This has the fields

$$\{\phi_{1,1} = \phi_{2,3}, \phi_{1,2} = \phi_{2,2}, \phi_{1,3} = \phi_{2,1}\}.$$

- It's known that $h_{1,2} = 1/16$, $h_{1,3} = 1/2$. The Ising Model!?
- We correspond these fields to the continuum limit of σ, ϵ via
  - $h_{1,2} = 1/16$: $\sigma \longrightarrow \phi_{1,2}$
  - $h_{1,3} = 1/2$: $\epsilon \longrightarrow \phi_{1,3}$
  - $h_{1,1} = 0$ corresponds to an "identity field" $\mathbb{I}$.

Computation of Correlators in the Continuum

- CFT says that in the continuum limit of the Ising model,

$$\langle\sigma(z_1)\dots\sigma(z_4)\rangle = \frac{1}{|z_{14}|^{1/4}}\frac{1}{|z_{32}|^{1/4}} G\left(\frac{z_{12}z_{34}}{z_{32}z_{14}}, \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{32}\bar{z}_{14}}\right)$$

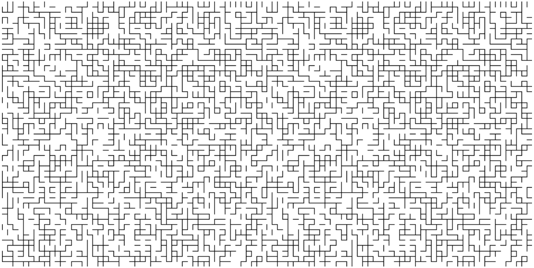
- Correspond this to the $m = 3$ minimal model, $\sigma = \phi_{1,2}$, so the correlator has $\phi_{1,2}$ operators. This forces G to satisfy a 2nd order ODE with solutions

$$G^{\pm}(\eta,\bar{\eta}) \propto (\eta(1-\eta))^{-1/8}\sqrt{1 \pm \sqrt{1-\eta}} \times c.c.$$

- Modronomy issues and normalization force the solution to thus be

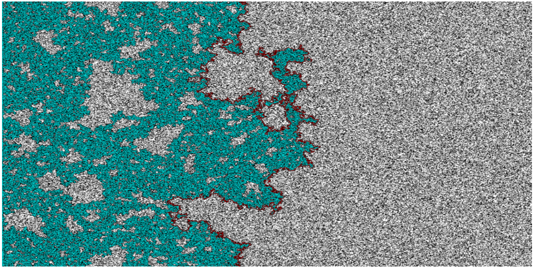
$$\langle\sigma(z_1)\dots\sigma(z_4)\rangle = \frac{1}{|z_{14}|^{1/4}}\frac{1}{|z_{32}|^{1/4}}(G^{+} + G^{-})$$

Percolation



- Bonds on a square lattice inside a rectangle of aspect ratio r are "activated" with probability p.
- Corresponds to $q \to 1$ limit of Potts model if you count clusters of "size 0."
- In the continuum limit, this is an "$m = 2$" minimal model.

Crossings in Percolation



- In the continuum limit, what is the probability that a cluster connects opposite sides of the rectangle?
  - If $p < 1/2$, probability is 0
  - If $p > 1/2$, probability is 1
  - If $p = 1/2$ (critical point), probability is what?
  - We can use CFT to compute the crossing probability at the critical point.

Cardy's Formula

- $q \to 1$ says an "$m = 2$" minimal model describes percolation.
- It can be argued that the crossing probability is the correlator

$$\langle \phi_{1,2}(0)\phi_{1,2}(1)\phi_{1,2}(\eta)\phi_{1,2}(\infty)\rangle$$

where η is related to the aspect ratio r via

$$r = \frac{K(1-k^2)}{2K(k^2)}, \quad \eta = \frac{(1-k)^2}{(1+k)^2}.$$

- The correlator contains $\phi_{1,2}$, so it satisfies a particular 2nd order ODE. One solution is "Cardy's formula." It is the crossing probability we seek:

$$\Pi(r) = \frac{3\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)^2}\eta^{1/3}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{4}{3};\eta\right)$$

For Further Reading

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- John Cardy, *SLE for theoretical physicists*, Annals Phys. 318 (2005) 81-118, [arXiv:cond-mat/0503313].
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- A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. B 241, 333 (1984).
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