

TOPIC 5: THE INVERSE-SCATTERING TRANSFORM FOR THE DEFOCUSING NONLINEAR SCHRÖDINGER EQUATION

Lax pair and zero-curvature representation. The defocusing nonlinear Schrödinger (NLS) equation

$$(1) \quad i\psi_t + \frac{1}{2}\psi_{xx} - |\psi|^2\psi = 0$$

for a complex-valued field $\psi(x, t)$ can be viewed as a compatibility condition for the simultaneous linear equations of a Lax pair:

$$(2) \quad \frac{\partial \mathbf{w}}{\partial x} = \mathbf{U} \mathbf{w}, \quad \mathbf{U} = \mathbf{U}(x, t, \lambda) := \begin{bmatrix} -i\lambda & \psi \\ \psi^* & i\lambda \end{bmatrix}$$

and

$$(3) \quad \frac{\partial \mathbf{w}}{\partial t} = \mathbf{V} \mathbf{w}, \quad \mathbf{V} = \mathbf{V}(x, t, \lambda) := \begin{bmatrix} -i\lambda^2 - i\frac{1}{2}|\psi|^2 & \lambda\psi + i\frac{1}{2}\psi_x \\ \lambda\psi^* - i\frac{1}{2}\psi_x^* & i\lambda^2 + i\frac{1}{2}|\psi|^2 \end{bmatrix}.$$

In other words, the defocusing NLS equation is equivalent to the *zero-curvature condition*

$$\frac{\partial \mathbf{U}}{\partial t} - \frac{\partial \mathbf{V}}{\partial x} + [\mathbf{U}, \mathbf{V}] = \mathbf{0},$$

i.e., the left-hand side is independent of $\lambda \in \mathbb{C}$ and vanishes when the defocusing NLS equation holds for ψ . Original references for these facts include Zakharov and Shabat [4] and Ablowitz, Kaup, Newell, and Segur [1]. We may consider at the same time several different vector solutions \mathbf{w} of (2) and/or (3) simply by building a matrix \mathbf{W} whose columns are those vectors and then writing the matrix equations $\mathbf{W}_x = \mathbf{U}\mathbf{W}$ and/or $\mathbf{W}_t = \mathbf{V}\mathbf{W}$. Recall that such a matrix solution \mathbf{W} is called *fundamental* if its columns are linearly independent vector solutions.

We are now interested in general initial-value problems in which we seek $\psi(x, t)$ solving (1) for $x \in \mathbb{R}$ and $t \geq 0$, such that $\psi(x, t)$ is smooth and decays to zero for large $|x|$ and satisfies a specified initial condition

$$\psi(x, 0) = \psi_0(x),$$

for some given function $\psi_0(\cdot)$ in a suitable function space.

Quick review of the linearized theory. Consider first the initial-value problem for the linear Schrödinger equation:

$$(4) \quad i\psi_t + \frac{1}{2}\psi_{xx} = 0, \quad \psi(x, 0) = \psi_0(x).$$

This problem may be viewed formally as a limiting case of the nonlinear problem in which the initial data ψ_0 is a “small” function (because then $|\psi|^2\psi$ should be even smaller, and negligible by comparison with the linear terms in (1)). Taking the (direct) Fourier transform via

$$\hat{\psi}(\lambda, t) := \int_{-\infty}^{\infty} \psi(x, t) e^{2i\lambda x} dx,$$

the initial value problem (4) becomes

$$i\hat{\psi}_t - 2\lambda^2 \hat{\psi} = 0, \quad \hat{\psi}(\lambda, 0) = \hat{\psi}_0(\lambda).$$

This ordinary differential equation is solved in closed form as follows:

$$\hat{\psi}(\lambda, t) = e^{-2i\lambda^2 t} \hat{\psi}_0(\lambda) = e^{-2i\lambda^2 t} \hat{\psi}_0(\lambda).$$

Thus, the time evolution of the Fourier transform is given by multiplication by a simple and explicit oscillatory factor. To complete the solution of the initial-value problem (4) we have to invert the Fourier transform by evaluating another integral:

$$\psi(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \hat{\psi}(\lambda, t) e^{-2i\lambda x} d\lambda.$$

Even though the formulae look simple one should keep in mind that only rarely can the direct and inverse transform integrals be evaluated in closed form.

Nonetheless we have an algorithmic way to solve general initial-value problems for the linear Schrödinger equation:

- (1) Map the given initial condition $\psi_0(x)$ to the direct transform thereof: $\psi_0 \mapsto \hat{\psi}_0$.
- (2) Evolve the transform explicitly in time: $\hat{\psi}_0(\lambda) \mapsto \hat{\psi}(\lambda, t) := \hat{\psi}_0(\lambda)e^{-2i\lambda^2 t}$.
- (3) Since $\hat{\psi}(\lambda, t)$ is the direct transform of $\psi(x, t)$, find $\psi(x, t)$ at a later time t by applying the inverse transform to $\hat{\psi}(\lambda, t)$.

For the corresponding initial-value problem for the nonlinear equation (1) we will have instead an inverse-scattering transform; it has the same kinds of benefits and shortcomings as the Fourier transform method for the linear problem (4):

- It gives an algorithmic way to solve initial-value problems with arbitrary given initial data.
- Just as the Fourier integrals can rarely be computed in closed form, the individual steps in the solution algorithm can rarely be carried out in closed form.

If the steps of the algorithm cannot be carried out in closed form, then what is the use? Well, in the case of Fourier transforms, we can analyze the solutions of initial-value problems with great precision in asymptotic regimes like long time limits due to the existence of analytical tools like the method of steepest descents and the method of stationary phase for asymptotic expansions of integrals. It turns out that similar methods exist at the integrable nonlinear level, so similar questions can be addressed as in the linear case.

The direct transform for the defocusing NLS equation. Suppose that t is fixed and $\psi(x) = \psi(x, t)$ is a rapidly decreasing function of $x \in \mathbb{R}$. We will impose more specific conditions on $\psi(x)$ as we go. We are interested in the properties of solutions $\mathbf{w} = \mathbf{w}(x; \lambda)$ of the equation (2) when λ is assumed to be a real number. Since $\psi(x)$ decays rapidly for large $|x|$, we have the following asymptotic behavior for the coefficient matrix:

$$\mathbf{U}(x, t, \lambda) = -i\lambda\sigma_3 + o(1), \quad |x| \rightarrow \infty, \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This in turn suggests that when $|x|$ is large, we could approximate solutions $\mathbf{w}(x; \lambda)$ by the solutions $\mathbf{w}_0(x; \lambda)$ of the $\psi \equiv 0$ system:

$$\mathbf{w}_0(x; \lambda) = c_1 \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix},$$

where c_1 and c_2 are arbitrary constants. The reason we are interested especially in real λ is that this makes up the *continuous spectrum* for the equation $\mathbf{w}_x = \mathbf{U}\mathbf{w}$ when ψ is rapidly decreasing for large $|x|$ because $\lambda \in \mathbb{R}$ if and only if $\mathbf{w}_0(x; \lambda)$ is oscillatory rather than exponentially growing or decaying.

Jost solutions. We can isolate particular solutions of (2) for $\lambda \in \mathbb{R}$ by insisting on certain values of the constants c_1 and c_2 in the limit $x \rightarrow -\infty$ or $x \rightarrow +\infty$. For example, we can look for a solution $\mathbf{w} = \mathbf{j}^{-,1}(x; \lambda)$ that satisfies a boundary condition of the form

$$\mathbf{j}^{-,1}(x; \lambda) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \quad \text{as } x \rightarrow -\infty$$

which corresponds to insisting that $\mathbf{j}^{-,1}(x; \lambda)$ is asymptotically of the form $\mathbf{w}_0(x; \lambda)$ with $c_1 = 1$ and $c_2 = 0$ as $x \rightarrow -\infty$. Similarly, we can look for a solution $\mathbf{w} = \mathbf{j}^{-,2}(x; \lambda)$ that satisfies the boundary condition

$$\mathbf{j}^{-,2}(x; \lambda) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \quad \text{as } x \rightarrow -\infty$$

so we're taking $c_1 = 0$ and $c_2 = 1$ as $x \rightarrow -\infty$. These two solutions are called *Jost solutions* normalized as $x \rightarrow -\infty$. The Jost solutions $\mathbf{j}^{-,1}(x; \lambda)$ and $\mathbf{j}^{-,2}(x; \lambda)$ are linearly independent. Indeed, forming a 2×2 solution matrix from the column vectors:

$$\mathbf{J}^-(x; \lambda) := [\mathbf{j}^{-,1}(x; \lambda), \mathbf{j}^{-,2}(x; \lambda)]$$

the Wronskian of the Jost solutions $\mathbf{j}^{-,1}(x; \lambda)$ and $\mathbf{j}^{-,2}(x; \lambda)$ is just the determinant of $\mathbf{J}^-(x; \lambda)$. Since the trace of the coefficient matrix \mathbf{U} is zero, Abel's Theorem says that Wronskians for the system (2) are independent of x . Therefore we may compute the Wronskian of the Jost solutions for any x we like, and a good choice is the limit $x \rightarrow -\infty$ since this is where we have information about the columns of $\mathbf{J}^-(x; \lambda)$. Therefore,

$$\det(\mathbf{J}^-(x; \lambda)) = \lim_{x \rightarrow -\infty} \det(\mathbf{J}^-(x; \lambda)) = \lim_{x \rightarrow -\infty} \det(e^{-i\lambda x\sigma_3} + o(1)) = 1.$$

Thus, for all $\lambda \in \mathbb{R}$ for which the Jost solutions exist they are linearly independent¹ of each other, and the matrix $\mathbf{J}^-(x; \lambda)$ is therefore a fundamental solution matrix for the system. We can express the normalization condition for the Jost solutions in terms of $\mathbf{J}^-(x; \lambda)$ by the equation

$$\lim_{x \rightarrow -\infty} \mathbf{J}^-(x; \lambda) e^{i\lambda x \sigma_3} = \mathbb{I}, \quad \lambda \in \mathbb{R}.$$

In a similar way, we can also construct Jost solutions that are normalized in the limit $x \rightarrow +\infty$, where again the coefficient $\psi(x)$ decays to zero and the coefficient matrix \mathbf{U} can be approximated by $-i\lambda\sigma_3$. Thus, we may seek a solution $\mathbf{w} = \mathbf{j}^{+,1}(x; \lambda)$ that satisfies the boundary condition

$$\mathbf{j}^{+,1}(x; \lambda) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \quad \text{as } x \rightarrow +\infty$$

and a solution $\mathbf{w} = \mathbf{j}^{+,2}(x; \lambda)$ that satisfies the boundary condition

$$\mathbf{j}^{+,2}(x; \lambda) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \quad \text{as } x \rightarrow +\infty$$

and we will find that for real λ the Wronskian determinant of the solution matrix

$$\mathbf{J}^+(x; \lambda) := [\mathbf{j}^{+,1}(x; \lambda), \mathbf{j}^{+,2}(x; \lambda)]$$

is exactly equal to 1, so $\mathbf{J}^+(x; \lambda)$ is also a fundamental solution matrix for our system of equations. Its normalization condition can be written in the form

$$\lim_{x \rightarrow +\infty} \mathbf{J}^+(x; \lambda) e^{i\lambda x \sigma_3} = \mathbb{I}.$$

Scattering matrix. The Jost solution matrices $\mathbf{J}^\pm(x; \lambda)$ are necessarily both nonsingular matrices satisfying the differential equation

$$\frac{\partial \mathbf{J}^\pm}{\partial x} = \mathbf{U} \mathbf{J}^\pm.$$

This is a 2×2 linear system of equations, and so can only have two linearly independent column vector solutions. We have, however, discussed four solutions: the two columns of $\mathbf{J}^-(x; \lambda)$ and the two columns of $\mathbf{J}^+(x; \lambda)$. These cannot all four be independent of each other. But, the columns of $\mathbf{J}^+(x; \lambda)$ form a basis of the space of solutions, as do those of $\mathbf{J}^-(x; \lambda)$. Therefore, we may write any column of $\mathbf{J}^+(x; \lambda)$ as a linear combination of the columns of $\mathbf{J}^-(x; \lambda)$. The constants involved in the linear combinations may depend on $\lambda \in \mathbb{R}$. We may write the linear combinations compactly in the form

$$(5) \quad \mathbf{J}^+(x; \lambda) = \mathbf{J}^-(x; \lambda) \mathbf{S}(\lambda),$$

for some 2×2 matrix $\mathbf{S}(\lambda)$. Indeed, the first column of this formula simply says that

$$\mathbf{j}^{+,1}(x; \lambda) = S_{11}(\lambda) \mathbf{j}^{-,1}(x; \lambda) + S_{21}(\lambda) \mathbf{j}^{-,2}(x; \lambda),$$

and the second column says

$$\mathbf{j}^{+,2}(x; \lambda) = S_{12}(\lambda) \mathbf{j}^{-,1}(x; \lambda) + S_{22}(\lambda) \mathbf{j}^{-,2}(x; \lambda),$$

so the matrix $\mathbf{S}(\lambda)$ contains the four constants involved in the linear combinations. The matrix $\mathbf{S}(\lambda)$ is called the *scattering matrix* associated with the coefficient $\psi(x)$ in the matrix \mathbf{U} .

Let us now draw a premature analogy with Fourier transform theory for the linear initial-value problem (4). The first step of the solution algorithm in the linear case is the association of a function $\hat{\psi}_0(\lambda)$ with the given initial condition $\psi_0(x)$, via the Fourier transform. Here we see another way, via the Lax pair for defocusing NLS, to associate some functions of an auxiliary variable λ with a given initial condition $\psi_0(x)$: make $\psi_0(x)$ the known function in the coefficient matrix \mathbf{U} and solve the linear ODE system subject to the appropriate boundary conditions to find the Jost matrices $\mathbf{J}^\pm(x; \lambda)$. Then the matrix $\mathbf{S}(\lambda) := \mathbf{J}^-(x; \lambda)^{-1} \mathbf{J}^+(x; \lambda)$ is guaranteed to be independent of x and its elements are four functions of the “dual variable” λ that can be viewed as “transforms” of $\psi_0(x)$.

¹Observe how the conclusion that $\det(\mathbf{J}^-(x; \lambda)) = 1$ relies on the fact that $\lambda \in \mathbb{R}$. If $\lambda \notin \mathbb{R}$, either $e^{i\lambda x}$ or $e^{-i\lambda x}$ is exponentially growing as $x \rightarrow -\infty$ depending on whether λ is in the upper or lower half-plane, and one cannot conclude that the product of the growing exponential with the unspecified decaying error term represented by the symbol $o(1)$, as will arise in computing the determinant, decays to zero.

Elementary properties of the scattering matrix. First consider taking determinants in the defining relation (5) and using the fact that the Jost matrices have determinant 1. Thus,

$$\det(\mathbf{S}(\lambda)) = 1, \quad \lambda \in \mathbb{R}.$$

Next, note that the coefficient matrix $\mathbf{U}(x, t, \lambda)$ of the linear system (2) has the following symmetry²

$$\mathbf{U}(x, t, \lambda)^* = \sigma_1 \mathbf{U}(x, t, \lambda) \sigma_1, \quad \lambda \in \mathbb{R}, \quad \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It follows that if $\mathbf{w}(x; \lambda)$ is a solution of (2) for $\lambda \in \mathbb{R}$, then so also is $\sigma_1 \mathbf{w}(x; \lambda)^*$. Considering the asymptotic behavior of the Jost solutions, this means that (assuming, as will be shown, that the Jost solutions are uniquely determined by their asymptotic behavior and the fact that they solve (2))

$$\mathbf{j}^{\pm, 2}(x; \lambda) = \sigma_1 \mathbf{j}^{\pm, 1}(x; \lambda)^*, \quad \lambda \in \mathbb{R}.$$

From this it follows that the Jost matrices $\mathbf{J}^\pm(x; \lambda)$ satisfy

$$\mathbf{J}^\pm(x; \lambda)^* = \sigma_1 \mathbf{J}^\pm(x; \lambda) \sigma_1.$$

Taking conjugates in the relation $\mathbf{S}(\lambda) = \mathbf{J}^-(x; \lambda)^{-1} \mathbf{J}^+(x; \lambda)$ we therefore get

$$\mathbf{S}(\lambda)^* = \sigma_1 \mathbf{S}(\lambda) \sigma_1.$$

This latter relation implies that $\mathbf{S}(\lambda)$ can be written in the form

$$\mathbf{S}(\lambda) = \begin{bmatrix} a(\lambda)^* & -b(\lambda)^* \\ -b(\lambda) & a(\lambda) \end{bmatrix}, \quad \lambda \in \mathbb{R}$$

for some complex-valued functions $a(\lambda)$ and $b(\lambda)$. The condition that $\det(\mathbf{S}(\lambda)) = 1$ then implies that

$$|a(\lambda)|^2 - |b(\lambda)|^2 = 1 \quad \text{or, equivalently} \quad |R(\lambda)|^2 + |T(\lambda)|^2 = 1$$

where the quantity $T(\lambda) := 1/a(\lambda)$ is sometimes called the *transmission coefficient* and the ratio $R(\lambda) := b(\lambda)/a(\lambda)$ is sometimes called the *reflection coefficient*. Indeed, from

$$\mathbf{J}^-(x; \lambda) = \mathbf{J}^+(x; \lambda) \mathbf{S}(\lambda)^{-1} = \mathbf{J}^+(x; \lambda) \begin{bmatrix} a(\lambda) & b(\lambda)^* \\ b(\lambda) & a(\lambda)^* \end{bmatrix},$$

we have in particular that

$$\mathbf{j}^{-, 1}(x; \lambda) = a(\lambda) \mathbf{j}^{+, 1}(x; \lambda) + b(\lambda) \mathbf{j}^{+, 2}(x; \lambda),$$

which means, upon dividing through by $a(\lambda)$, that there is for each $\lambda \in \mathbb{R}$ a solution $\mathbf{w}(x; \lambda) = a(\lambda)^{-1} \mathbf{j}^{-, 1}(x; \lambda)$ satisfying

$$\mathbf{w}(x; \lambda) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + R(\lambda) \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \quad x \rightarrow +\infty$$

and also

$$\mathbf{w}(x; \lambda) = T(\lambda) \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \quad x \rightarrow -\infty.$$

Therefore, interpreting $e^{i\lambda x}$ ($e^{-i\lambda x}$) as a “wave” propagating to the right (left), we see that a unit wave traveling to the left incident on the disturbance modeled by the potential $\psi(x)$ causes a reflected wave of complex amplitude $R(\lambda)$ and admits a transmitted wave of complex amplitude $T(\lambda)$. In the inverse-scattering transform theory, the reflection coefficient $R(\lambda)$ is a nonlinear analogue of the Fourier transform of $\psi(x)$.

Definition 1 (Direct transform for defocusing NLS). *For a suitable function $\psi : \mathbb{R} \rightarrow \mathbb{C}$ decaying as $x \rightarrow \pm\infty$, the direct transform for the defocusing NLS equation is the mapping $\psi \mapsto R$ associating to ψ its reflection coefficient $R = R(\lambda)$, $\lambda \in \mathbb{R}$.*

²We use an asterisk to indicate complex conjugation, and when the asterisk appears on a matrix or a vector we mean complex conjugation of the individual elements (without any kind of transposition of matrix elements). We use the “dagger” superscript \dagger to indicate simultaneous complex conjugation and transposition of vectors and matrices.

Example 1: zero potential. The simplest example is to consider (2) in the special case that $\psi(x) = 0$ for all $x \in \mathbb{R}$. Then it is easy to see that the only matrix solutions of (2) that are consistent with the required asymptotic behavior of the Jost matrices as $x \rightarrow \pm\infty$ are simply $\mathbf{J}^-(x; \lambda) = \mathbf{J}^+(x; \lambda) = e^{-i\lambda x \sigma_3}$. Therefore $\mathbf{S}(\lambda) = \mathbf{J}^-(x; \lambda)^{-1} \mathbf{J}^+(x; \lambda) = \mathbb{I}$. Thus when $\psi(x) \equiv 0$, $a(\lambda) \equiv 1$ and $b(\lambda) \equiv 0$, so the reflection coefficient is $R(\lambda) = b(\lambda)/a(\lambda) \equiv 0$.

Example 2: barrier potential. As a simple nontrivial example, suppose that $\psi(x)$ is the piecewise-constant function

$$\psi(x) = \begin{cases} 0, & |x| > L, \\ B, & |x| \leq L. \end{cases}$$

Thus, ψ represents a barrier of width $2L$ and complex “height” $B \in \mathbb{C}$. Then, for $|x| > L$ we have the simple diagonal system:

$$\frac{\partial \mathbf{w}}{\partial x} = -i\lambda \sigma_3 \mathbf{w}, \quad |x| > L,$$

and for $|x| \leq L$ we have a non-diagonal but still constant-coefficient system:

$$\frac{\partial \mathbf{w}}{\partial x} = \mathbf{U}_B(\lambda) \mathbf{w}, \quad \mathbf{U}_B(\lambda) := \begin{bmatrix} -i\lambda & B \\ B^* & i\lambda \end{bmatrix}.$$

We may construct the Jost solutions piecewise, and join them at $x = \pm L$ by continuity.

The general solution for $|x| > L$ is as we found earlier, $\mathbf{w} = \mathbf{w}_0(x; \lambda)$. Thus, the Jost matrix $\mathbf{J}^-(x; \lambda)$ satisfies exactly

$$\mathbf{J}^-(x; \lambda) = e^{-i\lambda x \sigma_3}, \quad x < -L.$$

Similarly, the Jost matrix $\mathbf{J}^+(x; \lambda)$ satisfies exactly

$$\mathbf{J}^+(x; \lambda) = e^{-i\lambda x \sigma_3}, \quad x > L.$$

The general solution of the differential equation for $|x| < L$ can be written in terms of the matrix exponential $\exp(x \mathbf{U}_B(\lambda))$. Thus, for some constant matrix $\mathbf{C} = \mathbf{C}(\lambda)$, we may write the Jost matrix $\mathbf{J}^-(x; \lambda)$ for $|x| < L$ in the form

$$\mathbf{J}^-(x; \lambda) = e^{x \mathbf{U}_B(\lambda)} \mathbf{C}(\lambda), \quad |x| < L,$$

and we may find $\mathbf{C}(\lambda)$ by demanding continuity of $\mathbf{J}^-(x; \lambda)$ at $x = -L$:

$$\lim_{x \uparrow -L} \mathbf{J}^-(x; \lambda) = e^{i\lambda L \sigma_3} \quad \text{and} \quad \lim_{x \downarrow -L} \mathbf{J}^-(x; \lambda) = e^{-L \mathbf{U}_B(\lambda)} \mathbf{C}(\lambda)$$

so matching gives

$$\mathbf{C}(\lambda) = e^{L \mathbf{U}_B(\lambda)} e^{i\lambda L \sigma_3}.$$

Now, $\mathbf{J}^-(x; \lambda)$ and $\mathbf{J}^+(x; \lambda)$ are both known at $x = L$, so the scattering matrix may be calculated explicitly:

$$\mathbf{S}(\lambda) = \mathbf{J}^-(L; \lambda)^{-1} \mathbf{J}^+(L; \lambda) = \mathbf{C}(\lambda)^{-1} e^{-L \mathbf{U}_B(\lambda)} \cdot e^{-i\lambda L \sigma_3} = e^{-i\lambda L \sigma_3} e^{-2L \mathbf{U}_B(\lambda)} e^{-i\lambda L \sigma_3}.$$

An interesting calculation is to consider the expansion of the scattering matrix elements in the small-amplitude limit, $B \rightarrow 0$. Expanding the matrix exponential in Taylor series gives:

$$\begin{aligned} \mathbf{S}(\lambda) &= e^{-i\lambda L \sigma_3} \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \left(2i\lambda L \sigma_3 - \begin{bmatrix} 0 & 2LB \\ 2LB^* & 0 \end{bmatrix} \right)^n \right\} e^{-i\lambda L \sigma_3} \\ &= e^{-i\lambda L \sigma_3} \left\{ e^{2i\lambda L \sigma_3} - \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=1}^n (2i\lambda L \sigma_3)^{k-1} \begin{bmatrix} 0 & 2LB \\ 2LB^* & 0 \end{bmatrix} (2i\lambda L \sigma_3)^{n-k} \right\} e^{-i\lambda L \sigma_3} + O(B^2) \\ &= e^{-i\lambda L \sigma_3} \left\{ e^{2i\lambda L \sigma_3} - \begin{bmatrix} 0 & 2LB \\ 2LB^* & 0 \end{bmatrix} \sum_{n=1}^{\infty} \frac{1}{n!} (2i\lambda L \sigma_3)^{n-1} \sum_{k=1}^n (-1)^{k-1} \right\} e^{-i\lambda L \sigma_3} + O(B^2) \\ &= \mathbb{I} - \begin{bmatrix} 0 & 2LB \\ 2LB^* & 0 \end{bmatrix} \sum_{m=0}^{\infty} \frac{(-1)^m (2\lambda L)^{2m}}{(2m+1)!} + O(B^2) \\ &= \mathbb{I} - \begin{bmatrix} 0 & B \sin(2\lambda L)/\lambda \\ B^* \sin(2\lambda L)/\lambda & 0 \end{bmatrix} + O(B^2). \end{aligned}$$

Therefore, in the limit $B \rightarrow 0$, the conjugate reflection coefficient takes the form

$$R(\lambda)^* = \frac{b(\lambda)^*}{a(\lambda)^*} = -\frac{S_{12}(\lambda)}{S_{11}(\lambda)} = \frac{B}{\lambda} \sin(2\lambda L) + O(B^2).$$

Noting the Fourier transform of the “barrier” initial condition $\psi(x)$:

$$\hat{\psi}(\lambda) := \int_{-\infty}^{\infty} \psi(x) e^{2i\lambda x} dx = B \int_{-L}^L e^{2i\lambda x} dx = \frac{B}{\lambda} \sin(2\lambda L),$$

we see that

$$R(\lambda)^* = \hat{\psi}(\lambda) + O(B^2), \quad B \rightarrow 0.$$

Therefore, just as in the small-amplitude limit the defocusing NLS equation (1) formally goes over into the linear Schrödinger equation, the nonlinear direct transform associating $R(\lambda)$ with $\psi(x)$ goes over into the familiar Fourier transform (up to scale factors and complex conjugation) in the same limit. This suggests that the reflection coefficient could be thought of as a nonlinear analogue of the Fourier transform of $\psi(x)$.

Many properties of the mapping $\psi \mapsto R$ have been carefully investigated. From the paper [2] one can deduce that if $\psi \in \mathcal{S}(\mathbb{R})$ (the Schwartz space of infinitely continuously differentiable functions for which the function and all derivatives vanish as $|x| \rightarrow \infty$ more rapidly than any negative power of $|x|$) then also $R \in \mathcal{S}(\mathbb{R})$ and R satisfies $|R(\lambda)| < 1$. See Proposition 1 below. Of course the Fourier transform also famously maps $\mathcal{S}(\mathbb{R})$ to itself.

Time evolution of the reflection coefficient. If the potential $\psi(x)$ is replaced by a time-dependent potential $\psi(x, t)$ that satisfies the defocusing NLS equation (1), then, as $\psi(x, t)$ evolves in time (within a suitable function space), the reflection coefficient $R(\lambda)$ becomes time dependent as well: $R(\lambda) = R(\lambda; t)$. The analogy with Fourier transforms is further strengthened at this point because it turns out that $R(\lambda; t)$ evolves in time in the same elementary way as does the (complex conjugate of the) Fourier transform under the linear Schrödinger equation.

To see this, we consider how the Jost matrices vary in time with $\psi(x, t)$; we now write them as $\mathbf{J}^{\pm}(x, t; \lambda)$.

Lemma 1. *Let $\psi(x, t)$ be a suitable solution of the defocusing NLS equation (1) that in particular decays to zero along with its x -derivative as $|x| \rightarrow \infty$, and let $\mathbf{J}^{\pm}(x, t; \lambda)$ be the corresponding Jost matrices defined for each t . Then, the matrices $\mathbf{W}^{\pm}(x, t; \lambda) := \mathbf{J}^{\pm}(x, t; \lambda) e^{-i\lambda^2 t \sigma_3}$ are simultaneous fundamental matrix solutions of the compatible linear problems (2) and (3) of the Lax pair.*

Proof. Obviously we have $\det(\mathbf{W}^{\pm}(x, t; \lambda)) = 1$ as a consequence of $\det(\mathbf{J}^{\pm}(x, t; \lambda)) = 1$. Since $e^{-i\lambda^2 t \sigma_3}$ is independent of x , it is equally obvious that the matrices $\mathbf{W}^{\pm}(x, t; \lambda)$ satisfy the “ x -part” of the Lax pair (2) as this is true by definition of $\mathbf{J}^{\pm}(x, t; \lambda)$. Since the zero-curvature (compatibility) condition holds, there exists a simultaneous fundamental solution matrix of (2) and the “ t -part” of the Lax pair (3) for every complex λ and in particular for $\lambda \in \mathbb{R}$. Every such matrix must have the form $\mathbf{W} = \mathbf{W}^+(x, t; \lambda) = \mathbf{J}^+(x, t; \lambda) \mathbf{C}^+(t; \lambda)$ or $\mathbf{W} = \mathbf{W}^-(x, t; \lambda) = \mathbf{J}^-(x, t; \lambda) \mathbf{C}^-(t; \lambda)$ because the Jost matrices $\mathbf{J}^{\pm}(x, t; \lambda)$ already are fundamental solution matrices of the “ x -part” (2). Substituting these forms $\mathbf{W}^{\pm}(x, t; \lambda)$ into the “ t -part” (3) one sees that the matrices $\mathbf{C}^{\pm}(t; \lambda)$ must satisfy the differential equations

$$(6) \quad \frac{\partial \mathbf{C}^{\pm}}{\partial t}(t; \lambda) = \mathbf{J}^{\pm}(x, t; \lambda)^{-1} \mathbf{V}(x, t; \lambda) \mathbf{J}^{\pm}(x, t; \lambda) \mathbf{C}^{\pm}(t; \lambda) - \mathbf{J}^{\pm}(x, t; \lambda)^{-1} \frac{\partial \mathbf{J}^{\pm}}{\partial t}(x, t; \lambda) \mathbf{C}^{\pm}(t; \lambda).$$

Since $\mathbf{C}^{\pm}(t; \lambda)$ are matrices independent of x , we can choose any value of x in this equation (they are all equivalent by compatibility). We will consider the limit $x \rightarrow \pm\infty$. First note that we can write

$$\mathbf{J}^{\pm}(x, t; \lambda) = e^{-i\lambda x \sigma_3} + \mathbf{E}^{\pm}(x, t; \lambda),$$

where $\mathbf{E}^{\pm}(x, t; \lambda)$ decays to zero as $x \rightarrow \pm\infty$. Provided that $\psi(x, t)$ is a suitable solution of (1) it will also be true that

$$\lim_{x \rightarrow \pm\infty} \frac{\partial \mathbf{E}^{\pm}}{\partial t}(x, t; \lambda) = \mathbf{0}.$$

Therefore for such $\psi(x, t)$,

$$(7) \quad \lim_{x \rightarrow \pm\infty} \frac{\partial \mathbf{J}^{\pm}}{\partial t}(x, t; \lambda) = \lim_{x \rightarrow \pm\infty} \frac{\partial \mathbf{E}^{\pm}}{\partial t}(x, t; \lambda) = \mathbf{0}$$

because $e^{-i\lambda\sigma_3 x}$ is obviously independent of t . We also observe that, as long as ψ and ψ_x decay as $|x| \rightarrow \infty$,

$$\lim_{x \rightarrow \pm\infty} \mathbf{V}(x, t, \lambda) = -i\lambda^2 \sigma_3.$$

So, taking the limit $x \rightarrow \pm\infty$ in (6) assuming that $\psi(x, t)$ is a suitable solution of (1) we see that the matrices $\mathbf{C}^\pm(t; \lambda)$ satisfy

$$\frac{\partial \mathbf{C}^\pm}{\partial t}(t; \lambda) = -i\lambda^2 \sigma_3 \mathbf{C}^\pm(t; \lambda).$$

A particular solution of the latter equation is obviously $\mathbf{C}^\pm(t; \lambda) = e^{-i\lambda^2 t \sigma_3}$, which completes the proof. \square

Since the matrices $\mathbf{W}^\pm(x, t; \lambda) = \mathbf{J}^\pm(x, t; \lambda) e^{-i\lambda^2 t \sigma_3}$ solve the “ t -part” of the Lax pair (3), by substitution it follows that the Jost matrices themselves satisfy

$$\frac{\partial \mathbf{J}^\pm}{\partial t} = i\lambda^2 \mathbf{J}^\pm \sigma_3 + \mathbf{V} \mathbf{J}^\pm.$$

This turns out to be enough information to obtain a differential equation in t for the scattering matrix $\mathbf{S}(\lambda; t)$. Solving (5) for $\mathbf{S}(\lambda) = \mathbf{S}(\lambda; t)$ and differentiating with respect to t gives

$$\begin{aligned} \frac{\partial \mathbf{S}}{\partial t} &= \frac{\partial}{\partial t} ((\mathbf{J}^-)^{-1} \mathbf{J}^+) = (\mathbf{J}^-)^{-1} \frac{\partial \mathbf{J}^+}{\partial t} - (\mathbf{J}^-)^{-1} \frac{\partial \mathbf{J}^-}{\partial t} (\mathbf{J}^-)^{-1} \mathbf{J}^+ \\ &= (\mathbf{J}^-)^{-1} (i\lambda^2 \mathbf{J}^+ \sigma_3 + \mathbf{V} \mathbf{J}^+) - (\mathbf{J}^-)^{-1} (i\lambda^2 \mathbf{J}^- \sigma_3 + \mathbf{V} \mathbf{J}^-) (\mathbf{J}^-)^{-1} \mathbf{J}^+ \\ &= i\lambda^2 \mathbf{S} \sigma_3 - i\lambda^2 \sigma_3 \mathbf{S} \\ &= i\lambda^2 [\mathbf{S}, \sigma_3], \end{aligned}$$

or, in terms of the functions $a = a(\lambda; t)$ and $b = b(\lambda; t)$,

$$\begin{bmatrix} a_t^* & -b_t^* \\ -b_t & a_t \end{bmatrix} = \begin{bmatrix} 0 & 2i\lambda^2 b^* \\ -2i\lambda^2 b & 0 \end{bmatrix} \Leftrightarrow a(\lambda; t) = a(\lambda; 0), \quad b(\lambda; t) = b(\lambda; 0) e^{2i\lambda^2 t}.$$

In particular, it follows from the definition $R(\lambda) := b(\lambda)/a(\lambda)$ that the reflection coefficient $R(\lambda; t)$ evolves explicitly in time.

Theorem 1 (Time evolution of the reflection coefficient). *Suppose that $\psi = \psi(x, t)$ is a suitable smooth solution³ of (1). Then the corresponding reflection coefficient $R(\lambda; t)$ satisfies*

$$R(\lambda; t) = R(\lambda; 0) e^{2i\lambda^2 t}, \quad \lambda \in \mathbb{R}.$$

Note the striking similarity to the Fourier formula $\hat{\psi}(\lambda, t)^* = \hat{\psi}(\lambda, 0)^* e^{2i\lambda^2 t}$ appropriate for the linear problem (4). In the appropriate “transform domain” adapted to the problem at hand, both the linear and the nonlinear dynamics are reduced to multiplication by exactly the same Fourier multiplier $e^{2i\lambda^2 t}$.

Another key point is that if $R(\cdot; 0) \in \mathcal{S}(\mathbb{R})$ and the condition $|R(\lambda; 0)| \leq 1$ holds, then clearly also $R(\cdot; t) \in \mathcal{S}(\mathbb{R})$ and $|R(\lambda; t)| \leq 1$ holds for all $t \in \mathbb{R}$.

The inverse transform for the defocusing NLS equation.

Integral equations for Jost solutions. While we’ve constructed the Jost solutions explicitly in a simple example, we have not yet understood how they can be found more generally. The Jost solutions are supposed to be particular vector solutions, for $\lambda \in \mathbb{R}$, of the linear differential equation (2) that also satisfy certain boundary conditions as $x \rightarrow \pm\infty$. How can we specify such solutions precisely? The central idea is the same as that which arises in the proof of existence of unique solutions for initial-value problems for differential equations: replace the differential equations by integral equations that build in the required auxiliary conditions.

Let us describe how to find $\mathbf{j}^{-,1}(x; \lambda)$ in some detail. Recall that the relevant boundary condition is in this case that

$$\mathbf{j}^{-,1}(x; \lambda) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \quad \text{as } x \rightarrow -\infty$$

³A smooth solution of (1) is “suitable” if the Jost matrices satisfy (7) and if ψ and ψ_x tend to zero as $|x| \rightarrow \infty$. Later it will be shown (Lemma 2 below) that the Jost matrices exist and satisfy (7) provided ψ and ψ_t (or equivalently $\frac{1}{2}\psi_{xx} - |\psi|^2\psi$) lie in the function space $L^1(\mathbb{R})$.

for each real λ . The key idea is to recall the Fundamental Theorem of Calculus

$$\int_a^b f'(x) dx = f(b) - f(a)$$

and use this to integrate both sides of the differential equation (2) satisfied by $\mathbf{w} = \mathbf{j}^{-,1}(x; \lambda)$ taking into account the boundary condition at $x = -\infty$. The problem is that $\mathbf{j}^{-,1}(x; \lambda)$ oscillates rapidly but has no limit as $x \rightarrow -\infty$, so the boundary terms will not be well-defined. However, if we write the components of $\mathbf{j}^{-,1}(x; \lambda)$ exactly in the form

$$\mathbf{j}^{-,1}(x; \lambda) = \begin{bmatrix} e^{-i\lambda x} u(x; \lambda) \\ e^{i\lambda x} v(x; \lambda) \end{bmatrix}$$

then from the differential equation (2) satisfied by $\mathbf{w} = \mathbf{j}^{-,1}(x; \lambda)$ we deduce the differential equations

$$(8) \quad \frac{\partial u}{\partial x}(x; \lambda) = e^{2i\lambda x} \psi(x) v(x; \lambda), \quad \frac{\partial v}{\partial x}(x; \lambda) = e^{-2i\lambda x} \psi(x)^* u(x; \lambda),$$

and now we have

$$\lim_{x \rightarrow -\infty} u(x; \lambda) = 1, \quad \lim_{x \rightarrow -\infty} v(x; \lambda) = 0,$$

for all real λ . Now we are in a position to apply the Fundamental Theorem of Calculus. Integrating from $-\infty$ to x we have

$$u(x; \lambda) = 1 + \int_{-\infty}^x e^{2i\lambda y} \psi(y) v(y; \lambda) dy, \quad v(x; \lambda) = \int_{-\infty}^x e^{-2i\lambda y} \psi(y)^* u(y; \lambda) dy.$$

Let's substitute the second equation into the first:

$$u(x; \lambda) = 1 + \int_{-\infty}^x e^{2i\lambda y} \psi(y) \int_{-\infty}^y e^{-2i\lambda z} \psi(z)^* u(z; \lambda) dz dy$$

This gives us a closed equation for $u(x; \lambda)$ that incorporates the boundary condition at $x = -\infty$. To analyze this equation, it is useful to exchange the order of integration using the formula

$$\int_{-\infty}^x \int_{-\infty}^y f(y, z) dz dy = \int_{-\infty}^x \int_z^x f(y, z) dy dz,$$

which gives a so-called *Volterra integral equation* for $u(x; \lambda)$:

$$(9) \quad u(x; \lambda) = 1 + \int_{-\infty}^x K(x, z; \lambda) u(z; \lambda) dz$$

where the kernel is

$$K(x, z; \lambda) := \psi(z)^* \int_z^x e^{2i\lambda(y-z)} \psi(y) dy, \quad x > z.$$

A general reference that includes properties of Volterra integral equations is the book by Tricomi [3]. One way to think about solving the Volterra equation (9) is to use iteration. That is, we think of the integral equation as a mapping turning a function $u_n(x; \lambda)$ into another function $u_{n+1}(x; \lambda)$:

$$u_{n+1}(x; \lambda) := 1 + \int_{-\infty}^x K(x, z; \lambda) u_n(z; \lambda) dz.$$

A reasonable way to start the iteration, in view of the boundary condition at $x = -\infty$, is to choose $u_0(x; \lambda) \equiv 1$. Then, it is easy to see that

$$u_1(x; \lambda) = 1 + \int_{-\infty}^x K(x, z_1; \lambda) dz_1, \\ u_2(x; \lambda) = 1 + \int_{-\infty}^x K(x, z_1; \lambda) dz_1 + \int_{-\infty}^x K(x, z_2; \lambda) \int_{-\infty}^{z_2} K(z_2, z_1; \lambda) dz_1 dz_2,$$

and so on. Based on these calculations, we may make an inductive hypothesis that

$$(10) \quad u_n(x; \lambda) = \sum_{k=0}^n I_k(x; \lambda),$$

where I_k denotes the k -fold integral

$$I_k(x; \lambda) := \int_{-\infty}^x K(x, z_k; \lambda) \int_{-\infty}^{z_k} K(z_k, z_{k-1}; \lambda) \cdots \int_{-\infty}^{z_2} K(z_2, z_1; \lambda) dz_1 \cdots dz_k.$$

By definition $I_0 = 1$. This certainly gives the correct iterates for $n = 1$ and $n = 2$. It is a direct matter to check that the recurrence step is consistent with the formula (10) for $u_n(x; \lambda)$, which completes an inductive proof that $u_n(x; \lambda)$ is indeed given by the claimed formula.

It is obvious from the formula (10) that the iterates $u_n(x; \lambda)$ form partial sums of an infinite series, which is called the *Neumann series* for the solution $u(x; \lambda)$ of (9). The question at hand is the convergence of this infinite series. Now, since $\lambda \in \mathbb{R}$,

$$\begin{aligned} |K(x, z; \lambda)| &= |\psi(z)| \left| \int_z^x e^{2i\lambda(y-z)} \psi(y) dy \right| \\ &\leq |\psi(z)| \int_z^x |e^{2i\lambda(y-z)}| |\psi(y)| dy \\ &= |\psi(z)| \int_z^x |\psi(y)| dy \\ &\leq |\psi(z)| \cdot \|\psi\|_1, \end{aligned} \tag{11}$$

where $\|\psi\|_1$ denotes the L^1 -norm:

$$\|\psi\|_1 := \int_{\mathbb{R}} |\psi(y)| dy.$$

For this upper bound on K to be useful, we have to introduce the first technical assumption we will make on $\psi(x)$: that it is an absolutely integrable function on $(-\infty, \infty)$, i.e., $\psi \in L^1(\mathbb{R})$. This is a restriction on the rate of decay we will require on $\psi(x)$ as $x \rightarrow \pm\infty$. This estimate gives us a corresponding estimate for the terms $I_k(x; \lambda)$: by the triangle inequality,

$$\begin{aligned} |I_k(x; \lambda)| &\leq \int_{-\infty}^x |K(x, z_k; \lambda)| \int_{-\infty}^{z_k} |K(z_k, z_{k-1}; \lambda)| \cdots \int_{-\infty}^{z_2} |K(z_2, z_1; \lambda)| dz_1 \cdots dz_k \\ &\leq \|\psi\|_1^k \int_{-\infty}^x |\psi(z_k)| \int_{-\infty}^{z_k} |\psi(z_{k-1})| \cdots \int_{-\infty}^{z_2} |\psi(z_1)| dz_1 \cdots dz_k. \end{aligned}$$

Now, the last line can be simplified. We claim that

$$\int_{-\infty}^x |\psi(z_k)| \int_{-\infty}^{z_k} |\psi(z_{k-1})| \cdots \int_{-\infty}^{z_2} |\psi(z_1)| dz_1 \cdots dz_k = \int_0^m \int_0^{m_k} \cdots \int_0^{m_2} dm_1 \cdots dm_k,$$

where

$$m := \int_{-\infty}^x |\psi(y)| dy.$$

This can be verified easily by making the change of variables

$$m_j := \int_{-\infty}^{z_j} |\psi(y)| dy, \quad j = 1, \dots, k$$

in each of the integrals on the right-hand side. Furthermore, a direct calculation shows that

$$\int_0^m \int_0^{m_k} \cdots \int_0^{m_2} dm_1 \cdots dm_k = \frac{m^k}{k!}.$$

Therefore, we have the estimate

$$|I_k(x; \lambda)| \leq \frac{1}{k!} \left(\|\psi\|_1 \int_{-\infty}^x |\psi(y)| dy \right)^k.$$

It follows that the infinite series

$$u(x; \lambda) = \sum_{k=0}^{\infty} I_k(x; \lambda) \tag{12}$$

converges absolutely by comparison with an exponential series; indeed we have

$$(13) \quad |u(x; \lambda)| \leq \sum_{k=0}^{\infty} |I_k(x; \lambda)| \leq \exp \left(\|\psi\|_1 \int_{-\infty}^x |\psi(y)| dy \right).$$

Note that by replacing x by $+\infty$ in the estimates for $|I_k(x; \lambda)|$ we can even show that the convergence is uniform for all $x \in \mathbb{R}$, and by doing so in the above upper bound for $|u(x; \lambda)|$ we see that the function $u(x; \lambda)$ is uniformly bounded as a function of $x \in \mathbb{R}$:

$$\|u\|_{\infty} := \sup_{x \in \mathbb{R}} |u(x; \lambda)| \leq \exp(\|\psi\|_1^2)$$

for each $\lambda \in \mathbb{R}$. It is easy to use the uniform convergence of the Neumann series to prove that $u(x; \lambda)$ is in fact a solution of the Volterra equation (9) whenever $\lambda \in \mathbb{R}$. This solution is the *unique* bounded solution of (9). Indeed suppose $\tilde{u}(x; \lambda)$ is another bounded solution of (9) for the same value of $\lambda \in \mathbb{R}$. Then the function $\Delta u(x; \lambda) := \tilde{u}(x; \lambda) - u(x; \lambda)$ must be a solution of the homogeneous equation

$$\Delta u(x; \lambda) = \int_{-\infty}^x K(x, z; \lambda) \Delta u(z; \lambda) dz.$$

Substituting this equation into itself $n - 1$ times yields

$$\Delta u(x; \lambda) = \int_{-\infty}^x K(x, z_n; \lambda) \int_{-\infty}^{z_n} K(z_n, z_{n-1}; \lambda) \cdots \int_{-\infty}^{z_2} K(z_2, z_1; \lambda) \Delta u(z_1; \lambda) dz_1 \cdots dz_n.$$

Since u and \tilde{u} are both bounded functions (by construction and by hypothesis, respectively), there is a positive constant $M > 0$ such that

$$\|\Delta u\|_{\infty} = \sup_{x \in \mathbb{R}} |\Delta u(x; \lambda)| = M.$$

Then, by the triangle inequality, the estimate of $K(x, z; \lambda)$, and similar arguments as above,

$$|\Delta u(x; \lambda)| \leq M \|\psi\|_1^n \int_{-\infty}^x |\psi(z_n)| \int_{-\infty}^{z_n} |\psi(z_{n-1})| \cdots \int_{-\infty}^{z_2} |\psi(z_1)| dz_1 \cdots dz_n \leq M \frac{\|\psi\|_1^{2n}}{n!}.$$

Now, since $\|\psi\|_1^{2n}/n!$ is the n^{th} term in the convergent Taylor series of $\exp(\|\psi\|_1^2)$, it tends to zero as $n \rightarrow \infty$. From this it easily follows that for every $\epsilon > 0$, independent of x , $|\Delta u(x; \lambda)| \leq \epsilon$. Therefore $\Delta u(x; \lambda) = 0$ for all $x \in \mathbb{R}$, i.e., $\tilde{u}(x; \lambda) = u(x; \lambda)$. See also [3, pgs. 14–15].

So, $u(x; \lambda)$ defined from the Neumann series (12) is, for each $\lambda \in \mathbb{R}$, the unique uniformly bounded solution of the Volterra integral equation (9). From the Lebesgue Dominated Convergence Theorem, it follows that $u(x; \lambda)$ has well-defined limiting values as $x \rightarrow \pm\infty$:

$$(14) \quad \lim_{x \rightarrow -\infty} u(x; \lambda) = 1, \quad \lim_{x \rightarrow +\infty} u(x; \lambda) = 1 + \int_{\mathbb{R}} K(+\infty, z; \lambda) u(z; \lambda) dz,$$

where

$$K(+\infty, z; \lambda) := \psi(z)^* \int_z^{+\infty} e^{2i\lambda(y-z)} \psi(y) dy.$$

Indeed, letting χ_S denote the characteristic function of a set $S \in \mathbb{R}$, i.e.,

$$\chi_S(x) := \begin{cases} 1, & x \in S \\ 0, & x \in \mathbb{R} \setminus S, \end{cases}$$

we may write

$$u(x; \lambda) = 1 + \int_{\mathbb{R}} K(x, z; \lambda) \chi_{(-\infty, x)}(z) u(z; \lambda) dz$$

and we note that for (Lebesgue) almost every $z \in \mathbb{R}$ the integrand converges to 0 as $x \rightarrow -\infty$ (in fact it is exactly zero for $x < z$) and to $K(+\infty, z; \lambda) u(z; \lambda)$ as $x \rightarrow +\infty$. Since $|K(x, z; \lambda) \chi_{(-\infty, x)}(z) u(z; \lambda)| \leq |\psi(z)| \cdot \|\psi\|_1 \cdot \|u\|_{\infty}$ which is integrable on \mathbb{R} and is independent of x , the Lebesgue Dominated Convergence Theorem allows us to take the limit $x \rightarrow \pm\infty$ under the integral sign, yielding the formulas (14).

It follows from the uniform boundedness of $u(x; \lambda)$ that $e^{-2i\lambda x} \psi(x)^* u(x; \lambda)$ is an absolutely integrable function. Therefore, the function $v(x; \lambda)$ defined as an antiderivative thereof:

$$(15) \quad v(x; \lambda) := \int_{-\infty}^x e^{-2i\lambda y} \psi(y)^* u(y; \lambda) dy$$

is an absolutely continuous (i.e., having an absolutely integrable derivative) and uniformly bounded function that by a Lebesgue Dominated Convergence Theorem argument has well-defined limiting values as $x \rightarrow \pm\infty$:

$$\lim_{x \rightarrow -\infty} v(x; \lambda) = 0, \quad \lim_{x \rightarrow +\infty} v(x; \lambda) = \int_{\mathbb{R}} e^{-2i\lambda y} \psi(y)^* u(y; \lambda) dy.$$

Putting back the exponential factors:

$$\mathbf{j}^{-,1}(x; \lambda) = \begin{bmatrix} e^{-i\lambda x} u(x; \lambda) \\ e^{i\lambda x} v(x; \lambda) \end{bmatrix}$$

we see that we have constructed, for each $\lambda \in \mathbb{R}$, the unique vector solution $\mathbf{w} = \mathbf{j}^{-,1}(x; \lambda)$ of the differential equation (2) satisfying

$$\mathbf{j}^{-,1}(x; \lambda) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \quad x \rightarrow -\infty,$$

and

$$\mathbf{j}^{-,1}(x; \lambda) = a(\lambda) \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + b(\lambda) \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \quad x \rightarrow +\infty,$$

where

$$a(\lambda) = 1 + \int_{\mathbb{R}} K(+\infty, z; \lambda) u(z; \lambda) dz, \quad b(\lambda) = \int_{\mathbb{R}} e^{-2i\lambda y} \psi(y)^* u(y; \lambda) dy.$$

With these formulae we can detail a Fourier-like mapping property of the direct transform $\psi \mapsto R$.

Proposition 1. *Suppose that $\psi \in \mathcal{S}(\mathbb{R})$. Then also $R \in \mathcal{S}(\mathbb{R})$ and $\sup_{\lambda \in \mathbb{R}} |R(\lambda)| < 1$.*

Proof. Since $|a(\lambda)| \geq 1$, to prove $R \in \mathcal{S}(\mathbb{R})$ it suffices to prove that $a(\lambda) - 1$ and $b(\lambda)$ are Schwartz functions. We merely indicate how this can be done using the above formulae for $a(\lambda)$ and $b(\lambda)$ in terms of ψ and u . By differentiating under the integral sign with respect to λ and repeated integration by parts one expresses $\lambda^P \partial_\lambda^Q (a(\lambda) - 1)$ and $\lambda^P \partial_\lambda^Q b(\lambda)$ explicitly in terms of integrals of mixed derivatives of u with respect to x and λ integrated against functions of x guaranteed to lie in $\mathcal{S}(\mathbb{R})$. Then one shows that the mixed derivatives of u are uniformly bounded for $(x, \lambda) \in \mathbb{R}^2$ by repeated differentiation of the Volterra equation (9). This gives uniform (over $\lambda \in \mathbb{R}$) bounds of $\lambda^P \partial_\lambda^Q$ acting on both $a(\lambda) - 1$ and $b(\lambda)$. Finally to show the strict inequality $|R(\lambda)| < 1$ one notes that $|T(\lambda)|^2 = 1/|a(\lambda)|^2$ cannot vanish so $|R(\lambda)|^2 = 1 - |T(\lambda)|^2 < 1$. We refer to [2] for further details. \square

Similar results hold for the other three columns of the Jost matrices: they are uniquely determined for each $\lambda \in \mathbb{R}$ by the differential equation (2) and associated boundary conditions at $x = \pm\infty$, provided only that $\psi \in L^1(\mathbb{R})$. If $\psi = \psi(x, t)$ is a solution of the defocusing nonlinear Schrödinger equation (1) lying in $L^1(\mathbb{R})$ for each $t \in \mathbb{R}$, we may ask for additional conditions under which (7) holds.

Lemma 2. *Suppose that ψ and ψ_t lie in $L^1(\mathbb{R})$ as functions of x for each $t \in \mathbb{R}$. Then (7) holds.*

Proof. We prove the result for $\mathbf{j}^{-,1}(x, t; \lambda)$, the first column of $\mathbf{J}^-(x, t; \lambda)$. Let $\lambda \in \mathbb{R}$ be fixed. Then the condition $\psi \in L^1(\mathbb{R})$ guarantees the existence of $\mathbf{j}^{-,1}(x, t; \lambda)$. Since

$$\frac{\partial \mathbf{j}^{-,1}}{\partial t}(x, t; \lambda) = \begin{bmatrix} e^{-i\lambda x} u_t(x, t; \lambda) \\ e^{i\lambda x} v_t(x, t; \lambda) \end{bmatrix},$$

because the factors $e^{\pm i\lambda x}$ are independent of t , the condition $\lambda \in \mathbb{R}$ implies that it is sufficient to prove that $u_t(x, t; \lambda)$ and $v_t(x, t; \lambda)$ tend to zero as $x \rightarrow -\infty$.

Differentiating (9) with respect to t gives a Volterra equation for $u_t(x, t; \lambda)$:

$$(16) \quad u_t(x, t; \lambda) = F(x, t; \lambda) + \int_{-\infty}^x K(x, z, t; \lambda) u_t(z, t; \lambda) dz, \quad F(x, t; \lambda) := \int_{-\infty}^x K_t(x, z, t; \lambda) u(z, t; \lambda) dz.$$

The forcing term $F(x, t; \lambda)$ is known once $u(x, t; \lambda)$ is determined for each $t \in \mathbb{R}$ from the Neumann series (12). In obtaining the terms in the latter series and in particular in constructing the kernel K , the potential $\psi(x)$ is replaced with $\psi(x, t)$. Therefore, the differentiated kernel $K_t(x, z, t; \lambda)$ is given by

$$K_t(x, z, t; \lambda) = \psi_t(z, t)^* \int_z^x e^{2i\lambda(y-z)} \psi(y, t) dy + \psi(z, t)^* \int_z^x e^{2i\lambda(y-z)} \psi_t(y, t) dy.$$

The analogue of (11) for K_t is (here we first use $\psi_t \in L^1(\mathbb{R})$)

$$(17) \quad |K_t(x, z, t; \lambda)| \leq |\psi_t(z, t)| \cdot \|\psi(\cdot, t)\|_1 + |\psi(z, t)| \cdot \|\psi_t(\cdot, t)\|_1.$$

The Neumann series corresponding to the new Volterra equation (16) is

$$u_t(x, t; \lambda) = \sum_{k=0}^{\infty} \tilde{I}_k(x, t; \lambda)$$

where $\tilde{I}_0(x, t; \lambda) := F(x, t; \lambda)$ and, for $k \geq 1$,

$$\tilde{I}_k(x, t; \lambda) := \int_{-\infty}^x K(x, z_k, t; \lambda) \int_{-\infty}^{z_k} K(z_k, z_{k-1}, t; \lambda) \cdots \int_{-\infty}^{z_2} K(z_2, z_1, t; \lambda) F(z_1, t; \lambda) dz_1 \cdots dz_k.$$

By similar estimates as for I_k ,

$$|\tilde{I}_k(x, t; \lambda)| \leq \frac{\|\psi(\cdot, t)\|_1^{2k}}{k!} \|F(\cdot, t; \lambda)\|_{\infty}.$$

Hence $u_t(x, t; \lambda)$ exists for a given $t \in \mathbb{R}$ by uniform convergence of its Neumann series provided $F(\cdot, t; \lambda)$ is uniformly bounded. But the latter holds because (using (13) and (17))

$$(18) \quad |F(x, t; \lambda)| \leq \|u(\cdot, t; \lambda)\|_{\infty} \int_{-\infty}^x |K_t(x, z, t; \lambda)| dz \leq 2 \exp(\|\psi(\cdot, t)\|_1^2) \|\psi(\cdot, t)\|_1 \|\psi_t(\cdot, t)\|_1.$$

Summing the upper bound on \tilde{I}_k , it then follows that an additional exponential factor is produced, so

$$(19) \quad \|u_t(\cdot, t; \lambda)\| \leq 2 \exp(2\|\psi(\cdot, t)\|_1^2) \|\psi(\cdot, t)\|_1 \|\psi_t(\cdot, t)\|_1.$$

The estimate (17) combined with the intermediate inequality in (18) shows also that $F(x, t; \lambda) \rightarrow 0$ as $x \rightarrow -\infty$. Using this fact and the estimates (11) and (19) in the integral equation (16) then proves that $u_t(x, t; \lambda) \rightarrow 0$ as $x \rightarrow -\infty$, as desired.

Differentiating the equation (15) with respect to t yields

$$v_t(x, t; \lambda) = \int_{-\infty}^x e^{-2i\lambda y} \psi_t(y, t)^* u(y, t; \lambda) dy + \int_{-\infty}^x e^{-2i\lambda y} \psi(y, t)^* u_t(y, t; \lambda) dy$$

and therefore for any $\lambda \in \mathbb{R}$,

$$|v_t(x, t; \lambda)| \leq \|u(\cdot, t; \lambda)\|_{\infty} \int_{-\infty}^x |\psi_t(y, t)| dy + \|u_t(\cdot, t; \lambda)\|_{\infty} \int_{-\infty}^x |\psi(y, t)| dy,$$

where $\|u\|_{\infty}$ and $\|u_t\|_{\infty}$ are finite from (13) and (19) respectively. Therefore the assumptions that $\psi \in L^1(\mathbb{R})$ and $\psi_t \in L^1(\mathbb{R})$ also guarantee that $v_t(x, t; \lambda) \rightarrow 0$ as $x \rightarrow -\infty$. \square

Analytic properties of Jost solutions. So far, we have been considering $\lambda \in \mathbb{R}$. But when we go back and look at what we have done, we can see that some of the steps are also valid for certain complex λ . Suppose that $\lambda = \alpha + i\beta$. Then our convergence analysis for the infinite series (12) representing $u(x; \lambda)$ begins as before with an estimate of the kernel K :

$$\begin{aligned} |K(x, z; \lambda)| &= |\psi(z)| \left| \int_z^x e^{2i\lambda(y-z)} \psi(y) dy \right| \\ &\leq |\psi(z)| \int_z^x \left| e^{2i\lambda(y-z)} \right| |\psi(y)| dy \\ &= |\psi(z)| \int_z^x e^{-2\beta(y-z)} |\psi(y)| dy. \end{aligned}$$

Now, in the integrand $y \geq z$, so if $\beta \geq 0$, we will have again the estimate (cf., (11))

$$|K(x, z; \lambda)| \leq |\psi(z)| \cdot \|\psi\|_1.$$

Therefore, the existence of $u(x; \lambda)$ as a uniformly bounded function of x satisfying (cf., (13))

$$\|u\|_\infty \leq \exp(\|\psi\|_1^2)$$

is guaranteed as long as $\text{Im}(\lambda) = \beta \geq 0$ by the same arguments as before. Moreover, the formulae (14) for the limiting values of u as $x \rightarrow \pm\infty$ remain valid. The formula (15) for $v(x; \lambda)$ in terms of $u(x; \lambda)$ also remains valid for $\text{Im}(\lambda) \geq 0$ since the exponential factor in the integrand is decaying in the direction of $x \rightarrow -\infty$. The statement that $v(x; \lambda) \rightarrow 0$ as $x \rightarrow -\infty$ therefore also remains valid for $\text{Im}(\lambda) \geq 0$. In fact, we even get an exponential rate of decay for v :

$$\begin{aligned} |v(x; \lambda)| &\leq \int_{-\infty}^x e^{2\beta y} |\psi(y)| |u(y; \lambda)| dy \\ &\leq e^{2\beta x} \int_{-\infty}^x |\psi(y)| |u(y; \lambda)| dy \\ &\leq e^{2\beta x} \|\psi u\|_1 \\ &\leq e^{2\beta x} \|u\|_\infty \|\psi\|_1 \\ &\leq e^{2\beta x} \|\psi\|_1 \exp(\|\psi\|_1^2). \end{aligned}$$

This is $O(e^{2\beta x})$ as $x \rightarrow -\infty$. However, if $\text{Im}(\lambda) > 0$ we can no longer deduce a limiting value for $v(x; \lambda)$ in the limit $x \rightarrow +\infty$. (Why not?)

The Jost solution $\mathbf{j}^{-,1}(x; \lambda)$ therefore exists for all λ in the upper half-plane as well as for real λ . From the conditions that $u(x; \lambda) \rightarrow 1$ as $x \rightarrow -\infty$ and $v(x; \lambda) = O(e^{2\beta x})$ that hold true for $\text{Im}(\lambda) > 0$, we see by putting back in the exponential factors $e^{\pm i\lambda x}$ that

$$\mathbf{j}^{-,1}(x; \lambda) = O(e^{\beta x}), \quad x \rightarrow -\infty,$$

and therefore represents a vector solution of (2) that, for $\text{Im}(\lambda) > 0$, decays exponentially to zero as $x \rightarrow -\infty$.

Now other important features of $\mathbf{j}^{-,1}(x; \lambda)$ arise from thinking of the dependence of $u(x; \lambda)$ on λ with $\text{Im}(\lambda) \geq 0$ for fixed $x \in \mathbb{R}$. We begin by determining two key properties of the kernel $K(x, z; \lambda)$.

Lemma 3. *Let $\psi \in L^1(\mathbb{R})$. Then for all $x \in \mathbb{R}$ and (Lebesgue) almost every $z \leq x$, $K(x, z; \lambda)$ is a continuous function of $\lambda \in \mathbb{C}$.*

Proof. Pick $\lambda_0 \in \mathbb{C}$. Obviously $e^{2i\lambda(y-z)}\psi(y) \rightarrow e^{2i\lambda_0(y-z)}\psi(y)$ as $\lambda \rightarrow \lambda_0$ for almost every $y \in [z, x]$. Letting D_0 denote the closed unit disk centered at λ_0 , we also observe that $e^{2i\lambda(y-z)}$ is continuous on the compact set $(\lambda, y) \in D_0 \times [z, x]$ so there is a constant C such that $|e^{2i\lambda(y-z)}| \leq C$ for all $(\lambda, y) \in D_0 \times [z, x]$. Therefore $\lambda \in D_0$ implies $|e^{2i\lambda(y-z)}\psi(y)| \leq C|\psi(y)|$ for $y \in [z, x]$, an upper bound that is independent of $\lambda \in D_0$ and integrable. Therefore by the Lebesgue Dominated Convergence Theorem,

$$\lim_{\lambda \rightarrow \lambda_0} \int_z^x e^{2i\lambda(y-z)}\psi(y) dy = \int_z^x e^{2i\lambda_0(y-z)}\psi(y) dy$$

for all real $z \leq x$. Putting back the factor $\psi(z)^*$ which is defined for almost all z completes the proof. \square

Lemma 4. *Let $x \geq z$ be fixed real numbers and suppose that $\psi \in L^1(\mathbb{R})$. Then $K(x, z; \lambda)$ is an entire function of $\lambda \in \mathbb{C}$ (i.e., analytic in the whole complex plane).*

Proof. Let Γ be any piecewise-smooth closed curve in the complex plane. Then

$$\oint_{\Gamma} K(x, z; \lambda) d\lambda = \psi(z)^* \oint_{\Gamma} \int_z^x e^{2i\lambda(y-z)} \psi(y) dy d\lambda.$$

Now $|e^{2i\lambda(y-z)}\psi(y)| \leq e^{-2\text{Im}(\lambda)(y-z)}|\psi(y)|$ which is integrable on the product $(y, \lambda) \in [z, x] \times \Gamma$ (here we are using the facts that $[z, x]$ is a bounded interval and $\psi \in L^1(\mathbb{R})$). Therefore Fubini's Theorem allows us to exchange the order of integration:

$$\oint_{\Gamma} K(x, z; \lambda) d\lambda = \psi(z)^* \int_z^x \psi(y) \oint_{\Gamma} e^{2i\lambda(y-z)} d\lambda dy.$$

But by Cauchy's Integral Theorem, for each $y, z \in \mathbb{R}$,

$$\oint_{\Gamma} e^{2i\lambda(y-z)} d\lambda = 0$$

because $e^{2i\lambda(y-z)}$ is entire in λ . So, $K(x, z; \lambda)$ is a continuous function of $\lambda \in \mathbb{C}$ by Lemma 3 that satisfies

$$\oint_{\Gamma} K(x, z; \lambda) d\lambda = 0$$

for every nice closed curve Γ in the complex plane. Hence the analyticity of $K(x, z; \lambda)$ for all $\lambda \in \mathbb{C}$ follows by Morera's Theorem. \square

With these results in hand, we can establish corresponding properties of the summand $I_k(x; \lambda)$ of the Neumann series (12). Note that these functions satisfy the recursion relation

$$I_k(x; \lambda) = \int_{-\infty}^x K(x, z; \lambda) I_{k-1}(z; \lambda) dz, \quad k \geq 1$$

with the "initial condition" $I_0(x; \lambda) \equiv 1$.

Lemma 5. *Let $x \in \mathbb{R}$ be fixed and suppose that $\psi \in L^1(\mathbb{R})$. Then for each $k \geq 0$, $I_k(x; \lambda)$ is a continuous function of λ for $\text{Im}(\lambda) \geq 0$.*

Proof. The result is obvious for $k = 0$. Suppose now that for some $k \geq 1$ that $I_{k-1}(x; \lambda)$ is continuous on the set $\text{Im}(\lambda) \geq 0$. Suppose that $\text{Im}(\lambda_0) \geq 0$. For almost every $z \leq x$, we have $K(x, z; \lambda) I_{k-1}(z; \lambda) \rightarrow K(x, z; \lambda_0) I_{k-1}(z; \lambda_0)$ as $\lambda \rightarrow \lambda_0$ with $\text{Im}(\lambda_0) \geq 0$, according to Lemma 3 and the inductive hypothesis. Also, whenever $\text{Im}(\lambda) \geq 0$, we have

$$|K(x, z; \lambda) I_{k-1}(z; \lambda)| \leq |\psi(z)| \cdot \|\psi\|_1 \cdot \frac{\|\psi\|_1^{2(k-1)}}{(k-1)!}$$

an upper bound that is integrable on $(-\infty, x)$ and independent of λ . Therefore by the Lebesgue Dominated Convergence Theorem again,

$$\lim_{\substack{\lambda \rightarrow \lambda_0 \\ \lambda, \lambda_0 \in \overline{\mathbb{C}_+}}} \int_{-\infty}^x K(x, z; \lambda) I_{k-1}(z; \lambda) dz = \int_{-\infty}^x K(x, z; \lambda_0) I_{k-1}(z; \lambda_0) dz.$$

This proves the desired continuity of $I_k(x; \lambda)$. \square

Lemma 6. *Let $x \in \mathbb{R}$ be fixed and suppose that $\psi \in L^1(\mathbb{R})$. Then for each $k \geq 0$, $I_k(x; \lambda)$ is analytic on the open upper half-plane $\text{Im}(\lambda) > 0$.*

Proof. Again the result is obvious for $k = 0$. Suppose that $I_{k-1}(x; \lambda)$ is analytic for $\text{Im}(\lambda) > 0$. Let Γ be a piecewise-smooth closed curve contained in the open upper half-plane. Then

$$\oint_{\Gamma} I_k(x; \lambda) d\lambda = \oint_{\Gamma} \int_{-\infty}^x K(x, z; \lambda) I_{k-1}(z; \lambda) dz d\lambda.$$

By the estimate

$$|K(x, z; \lambda) I_{k-1}(z; \lambda)| \leq |\psi(z)| \cdot \|\psi\|_1 \frac{\|\psi\|_1^{2(k-1)}}{(k-1)!},$$

the integrand lies in $L^1((-\infty, x) \times \Gamma)$ so by Fubini's Theorem,

$$\oint_{\Gamma} I_k(x; \lambda) d\lambda = \int_{-\infty}^x \oint_{\Gamma} K(x, z; \lambda) I_{k-1}(z; \lambda) d\lambda dz.$$

Since $K(x, z; \lambda)$ is entire in λ by Lemma 4 and since $I_{k-1}(z; \lambda)$ is analytic for $\text{Im}(\lambda) > 0$ by the inductive hypothesis, it follows from Cauchy's Integral Theorem that

$$\oint_{\Gamma} K(x, z; \lambda) I_{k-1}(z; \lambda) d\lambda = 0.$$

Therefore $I_k(x; \lambda)$ is a continuous function of λ in the upper half-plane (Lemma 5) that satisfies

$$\oint_{\Gamma} I_k(x; \lambda) d\lambda = 0$$

for every piecewise-smooth closed curve Γ in the open upper half-plane. By Morera's Theorem, $I_k(x; \lambda)$ is therefore analytic in the same domain. \square

The latter two results can be combined to yield the following.

Lemma 7. *Assume $\psi \in L^1(\mathbb{R})$, let $u(x; \lambda)$ be defined for $\text{Im}(\lambda) \geq 0$ by the convergent Neumann series (12), and let $v(x; \lambda)$ be defined in terms of u by (15). Then for each $x \in \mathbb{R}$, $u(x; \lambda)$ and $v(x; \lambda)$ are continuous functions of λ in the closed upper half-plane and analytic functions of λ in the open upper half-plane.*

Proof. From the continuity and analyticity properties of the summands $I_k(x; \lambda)$ formulated in Lemma 5 and Lemma 6 respectively, the uniform convergence of the Neumann series on $\text{Im}(\lambda) \geq 0$ implies the continuity of $u(x; \lambda)$ for $\text{Im}(\lambda) \geq 0$ and its analyticity for $\text{Im}(\lambda) > 0$. With these properties of $u(x; \lambda)$ in hand, one can analyze the integral formula (15) to deduce the corresponding properties of $v(x; \lambda)$. Indeed, starting from

$$(20) \quad e^{2i\lambda x} v(x; \lambda) = \int_{-\infty}^x e^{2i\lambda(x-y)} \psi(y)^* u(y; \lambda) dy$$

we see that if $\text{Im}(\lambda_0) \geq 0$ then the integrand converges pointwise to $e^{2i\lambda_0(x-y)} \psi(y)^* u(y; \lambda_0)$ as $\lambda \rightarrow \lambda_0$ with $\text{Im}(\lambda) \geq 0$ for almost every $y \leq x$ by continuity of $u(x; \lambda)$. Also, since $x \geq y$ and $\text{Im}(\lambda) \geq 0$, we have $|e^{2i\lambda(x-y)} \psi(y)^* u(y; \lambda)| \leq |\psi(y)| \exp(\|\psi\|_1^2)$ which is integrable on $(-\infty, x)$ and independent of λ , so continuity of $e^{2i\lambda x} v(x; \lambda)$ and hence of $v(x; \lambda)$ follows from the Lebesgue Dominated Convergence Theorem. To prove analyticity of $v(x; \lambda)$ for $\text{Im}(\lambda) > 0$ given the corresponding property of $u(x; \lambda)$ is then another application of Morera's Theorem. \square

We conclude that, if $\psi \in L^1(\mathbb{R})$, both components of the Jost solution vector $\mathbf{j}^{-1}(x; \lambda)$ are analytic functions of λ in the upper half-plane for each fixed $x \in \mathbb{R}$, and they extend continuously to the real axis $\lambda \in \mathbb{R}$.

Next we ask: if $x \in \mathbb{R}$ is fixed, how does $\mathbf{j}^{-1}(x; \lambda)$ behave as $\lambda \rightarrow \infty$ with $\text{Im}(\lambda) > 0$? To answer this question, we can start with the uniform upper bound for $u(x; \lambda)$:

$$\sup_{x \in \mathbb{R}, \text{Im}(\lambda) \geq 0} |u(x; \lambda)| \leq \exp(\|\psi\|_1^2).$$

Then, from (20) we get the estimate

$$|e^{2i\lambda x} v(x; \lambda)| \leq \exp(\|\psi\|_1^2) \int_{-\infty}^x e^{2\beta(y-x)} |\psi(y)| dy.$$

The integrand tends to zero as $\beta \rightarrow +\infty$ for almost every $y \leq x$, and since $\beta \geq 0$ guarantees that

$$e^{2\beta(y-x)} |\psi(y)| \leq |\psi(y)|, \quad y \leq x,$$

with the upper bound being independent of β and absolutely integrable, the Lebesgue Dominated Convergence Theorem says that

$$\lim_{\text{Im}(\lambda) \rightarrow +\infty} e^{2i\lambda x} v(x; \lambda) = 0$$

for each $x \in \mathbb{R}$. Note also that if we make the further assumption that $\psi \in L^\infty(\mathbb{R})$, that is, $\psi(x)$ is a uniformly bounded function, then we can also see that

$$|e^{2i\lambda x} v(x; \lambda)| \leq \exp(\|\psi\|_1^2) \int_{-\infty}^x e^{2\beta(y-x)} |\psi(y)| dy \leq \|\psi\|_\infty \exp(\|\psi\|_1^2) \int_{-\infty}^x e^{2\beta(y-x)} dy = \frac{1}{2\beta} \|\psi\|_\infty \exp(\|\psi\|_1^2),$$

so we get a *rate* of decay of $O(1/\beta)$. Under some additional conditions on ψ it is also true that $e^{2i\lambda x} v(x; \lambda)$ decays like $1/\lambda$ as $\lambda \rightarrow \infty$ with $\text{Im}(\lambda) \geq 0$, even in directions parallel to the real axis. To see this, assume

that $\psi(\cdot)$ is a continuously differentiable function of x with compact support (which also implies $\psi' \in L^1(\mathbb{R})$). Then integrate by parts:

$$\begin{aligned} e^{2i\lambda x} v(x; \lambda) &= -\frac{1}{2i\lambda} \int_{-\infty}^x \frac{d}{dy} \left(e^{2i\lambda(x-y)} \right) \psi(y)^* u(y; \lambda) dy \\ &= -\frac{1}{2i\lambda} \left[\psi(x)^* u(x; \lambda) - \int_{-\infty}^x e^{2i\lambda(x-y)} \psi'(y)^* u(y; \lambda) dy - \int_{-\infty}^x e^{2i\lambda(x-y)} \psi(y)^* u'(y; \lambda) dy \right] \end{aligned}$$

where we used compact support of ψ to eliminate the boundary term at $y = -\infty$. Now substitute for $u'(y; \lambda)$ from (8):

$$e^{2i\lambda x} v(x; \lambda) = -\frac{1}{2i\lambda} \left[\psi(x)^* u(x; \lambda) - \int_{-\infty}^x e^{2i\lambda(x-y)} \psi'(y)^* u(y; \lambda) dy - \int_{-\infty}^x e^{2i\lambda(x-y)} |\psi(y)|^2 e^{2i\lambda y} v(y; \lambda) dy \right].$$

Now using $|e^{2i\lambda(x-y)}| \leq 1$ for $y \leq x$ and $\text{Im}(\lambda) \geq 0$, it is easy to get the estimate

$$\begin{aligned} |e^{2i\lambda x} v(x; \lambda)| &\leq \frac{1}{2|\lambda|} \left[|\psi(x)| |u(x; \lambda)| + \int_{-\infty}^x |\psi'(y)| |u(y; \lambda)| dy + \int_{-\infty}^x |\psi(y)|^2 |e^{2i\lambda y} v(y; \lambda)| dy \right] \\ &\leq \frac{1}{2|\lambda|} \left[\|\psi\|_\infty \|u\|_\infty + \|\psi'\|_1 \|u\|_\infty + \|\psi\|_2^2 \sup_{x \in \mathbb{R}} |e^{2i\lambda x} v(x; \lambda)| \right]. \end{aligned}$$

Since the upper bound on the right doesn't depend on x , we can take a supremum over $x \in \mathbb{R}$ and arrive at

$$\left(1 - \frac{\|\psi\|_2^2}{2|\lambda|} \right) \sup_{x \in \mathbb{R}} |e^{2i\lambda x} v(x; \lambda)| \leq \frac{\|u\|_\infty}{2|\lambda|} (\|\psi\|_\infty + \|\psi'\|_1), \quad \text{Im}(\lambda) \geq 0.$$

If $|\lambda| \geq \|\psi\|_2^2$, then the multiplier on the left-hand side is bounded below by $1/2$, so

$$\sup_{x \in \mathbb{R}} |e^{2i\lambda x} v(x; \lambda)| \leq \frac{\|u\|_\infty}{|\lambda|} (\|\psi\|_\infty + \|\psi'\|_1), \quad \text{Im}(\lambda) \geq 0, \quad |\lambda| \geq \|\psi\|_2^2.$$

Note that for ψ with compact support and $\psi' \in L^1$, $\|\psi\|_1$ is certainly finite, and as before $\|u\|_\infty \leq \exp(\|\psi\|_1^2)$, and also

$$\|\psi\|_\infty = \sup_{x \in \mathbb{R}} \left| \int_{-\infty}^x \psi'(y) dy \right| \leq \|\psi'\|_1 \quad \text{and} \quad \|\psi\|_2^2 = \int_{\mathbb{R}} |\psi(y)|^2 dy \leq \|\psi\|_\infty \|\psi\|_1 \leq \|\psi\|_1 \|\psi'\|_1.$$

These ideas can be used to prove the following.

Lemma 8. *Suppose that $\psi \in L^1(\mathbb{R})$. Then $u(x; \lambda) \rightarrow 1$ and $e^{2i\lambda x} v(x; \lambda) \rightarrow 0$ uniformly over $x \in \mathbb{R}$ as $\lambda \rightarrow \infty$ with $\text{Im}(\lambda) \geq 0$.*

Proof. Let $\epsilon > 0$ be given, arbitrarily small, and let $\tilde{\psi}_\epsilon(x)$ be a differentiable function of compact support that approximates $\psi(x)$ in the L^1 -sense:

$$\|\psi - \tilde{\psi}_\epsilon\|_1 = \int_{\mathbb{R}} |\psi(x) - \tilde{\psi}_\epsilon(x)| dx < \frac{\epsilon}{2} \exp(-\|\psi\|_1^2).$$

Then, since $\psi(x) = (\psi(x) - \tilde{\psi}_\epsilon(x)) + \tilde{\psi}_\epsilon(x)$,

$$e^{2i\lambda x} v(x; \lambda) = \int_{-\infty}^x e^{2i\lambda(x-y)} (\psi(y) - \tilde{\psi}_\epsilon(y))^* u(y; \lambda) dy + \int_{-\infty}^x e^{2i\lambda(x-y)} \tilde{\psi}_\epsilon(y)^* u(y; \lambda) dy.$$

For the first term on the right-hand side, we simply use $|e^{2i\lambda(x-y)}| \leq 1$ and $|u(y; \lambda)| \leq \|u\|_\infty \leq \exp(\|\psi\|_1^2)$ to get

$$\left| \int_{-\infty}^x e^{2i\lambda(x-y)} (\psi(y) - \tilde{\psi}_\epsilon(y))^* u(y; \lambda) dy \right| \leq \int_{-\infty}^x |e^{2i\lambda(x-y)}| |\psi(y) - \tilde{\psi}_\epsilon(y)| |u(y; \lambda)| dy \leq \|u\|_\infty \|\psi - \tilde{\psi}_\epsilon\|_1 \leq \frac{\epsilon}{2}.$$

For the second term, we use the fact that $\tilde{\psi}_\epsilon$ has compact support and $\tilde{\psi}'_\epsilon \in L^1(\mathbb{R})$ and apply the integration-by-parts argument to obtain that

$$\left| \int_{-\infty}^x e^{2i\lambda(x-y)} \tilde{\psi}_\epsilon(y)^* u(y; \lambda) dy \right| \leq \frac{\|u\|_\infty}{|\lambda|} (\|\tilde{\psi}_\epsilon\|_\infty + \|\tilde{\psi}'_\epsilon\|_1) \leq 2 \frac{\|\tilde{\psi}'_\epsilon\|_1 \exp(\|\psi\|_1^2)}{|\lambda|}$$

holds for $\text{Im}(\lambda) \geq 0$ with $|\lambda| \geq \|\tilde{\psi}_\epsilon\|_1 \|\tilde{\psi}'_\epsilon\|_1 \geq \|\tilde{\psi}_\epsilon\|_2^2$. Therefore, provided that $\text{Im}(\lambda) \geq 0$ and

$$|\lambda| > \max \left\{ \|\tilde{\psi}_\epsilon\|_1 \|\tilde{\psi}'_\epsilon\|_1, \frac{4}{\epsilon} \|\tilde{\psi}'_\epsilon\|_1 \exp(\|\psi\|_1^2) \right\},$$

we will have $\sup_{x \in \mathbb{R}} |e^{2i\lambda x} v(x; \lambda)| < \epsilon$, proving the desired uniform convergence to zero.

Next, consider the difference $u(x; \lambda) - 1$ as given in terms of $v(x; \lambda)$ (this is just the original integral equation for u before v was eliminated to give a closed equation):

$$u(x; \lambda) - 1 = \int_{-\infty}^x e^{2i\lambda y} \psi(y) v(y; \lambda) dy.$$

It follows easily that

$$\sup_{x \in \mathbb{R}} |u(x; \lambda) - 1| \leq \int_{\mathbb{R}} |\psi(y)| |e^{2i\lambda y} v(y; \lambda)| dy \leq \|\psi\|_1 \sup_{x \in \mathbb{R}} |e^{2i\lambda x} v(x; \lambda)|.$$

Since the right-hand side tends to zero as $\lambda \rightarrow \infty$ with $\text{Im}(\lambda) \geq 0$, we have shown the desired uniform convergence of $u(x; \lambda)$ to 1. \square

Therefore, it follows under the hypothesis that $\psi \in L^1(\mathbb{R})$ that

$$\lim_{\substack{\lambda \rightarrow \infty \\ \text{Im}(\lambda) \geq 0}} e^{i\lambda x} \mathbf{j}^{-,1}(x; \lambda) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

in the sense of uniform convergence over $x \in \mathbb{R}$. If ψ satisfies additional conditions, the rate of decay in the second row is $O(1/\lambda)$.

Other Jost solutions. Analyticity properties of the scattering matrix elements. Parallel analysis applies to the other Jost solutions: $\mathbf{j}^{-,2}(x; \lambda)$, $\mathbf{j}^{+,1}(x; \lambda)$, and $\mathbf{j}^{+,2}(x; \lambda)$. Namely, under the same hypotheses on ψ as above:

- $\mathbf{j}^{-,2}(x; \lambda)$ is continuous for $\text{Im}(\lambda) \leq 0$ and analytic for $\text{Im}(\lambda) < 0$, and satisfies

$$\mathbf{j}^{-,2}(x; \lambda) = O(e^{-\text{Im}(\lambda)x}), \quad \text{as } x \rightarrow -\infty$$

for $\text{Im}(\lambda) < 0$ (exponential decay for negative x) and

$$\lim_{\substack{\lambda \rightarrow \infty \\ \text{Im}(\lambda) \leq 0}} e^{-i\lambda x} \mathbf{j}^{-,2}(x; \lambda) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

in the sense of uniform convergence over $x \in \mathbb{R}$.

- $\mathbf{j}^{+,1}(x; \lambda)$ is continuous for $\text{Im}(\lambda) \leq 0$ and analytic for $\text{Im}(\lambda) < 0$, and satisfies

$$\mathbf{j}^{+,1}(x; \lambda) = O(e^{\text{Im}(\lambda)x}), \quad \text{as } x \rightarrow +\infty$$

for $\text{Im}(\lambda) < 0$ (exponential decay for positive x) and

$$\lim_{\substack{\lambda \rightarrow \infty \\ \text{Im}(\lambda) \leq 0}} e^{i\lambda x} \mathbf{j}^{+,1}(x; \lambda) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

in the sense of uniform convergence over $x \in \mathbb{R}$.

- $\mathbf{j}^{+,2}(x; \lambda)$ is continuous for $\text{Im}(\lambda) \geq 0$ and analytic for $\text{Im}(\lambda) > 0$, and satisfies

$$\mathbf{j}^{+,2}(x; \lambda) = O(e^{-\text{Im}(\lambda)x}), \quad \text{as } x \rightarrow +\infty$$

for $\text{Im}(\lambda) > 0$ (exponential decay for positive x) and

$$\lim_{\substack{\lambda \rightarrow \infty \\ \text{Im}(\lambda) \geq 0}} e^{-i\lambda x} \mathbf{j}^{+,2}(x; \lambda) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

in the sense of uniform convergence over $x \in \mathbb{R}$.

Now consider the first column of the scattering relation $\mathbf{J}^-(x; \lambda) = \mathbf{J}^+(x; \lambda)\mathbf{S}(\lambda)^{-1}$:

$$\mathbf{j}^{-,1}(x; \lambda) = a(\lambda)\mathbf{j}^{+,1}(x; \lambda) + b(\lambda)\mathbf{j}^{+,2}(x; \lambda).$$

Let's make both sides of this equation into 2×2 matrices by adjoining as a second column to both sides the same vector, $\mathbf{j}^{+,2}(x; \lambda)$:

$$[\mathbf{j}^{-,1}(x; \lambda), \mathbf{j}^{+,2}(x; \lambda)] = [a(\lambda)\mathbf{j}^{+,1}(x; \lambda) + b(\lambda)\mathbf{j}^{+,2}(x; \lambda), \mathbf{j}^{+,2}(x; \lambda)].$$

Now take determinants of both sides using linearity of the determinant in the first column to simplify the right-hand side:

$$\det [\mathbf{j}^{-,1}(x; \lambda), \mathbf{j}^{+,2}(x; \lambda)] = a(\lambda) \det [\mathbf{j}^{+,1}(x; \lambda), \mathbf{j}^{+,2}(x; \lambda)] + b(\lambda) \det [\mathbf{j}^{+,2}(x; \lambda), \mathbf{j}^{+,2}(x; \lambda)] = a(\lambda)$$

with the final result obtained by noting that the coefficient of $b(\lambda)$ is obviously the determinant of a singular matrix and that the coefficient of $a(\lambda)$ is simply $\det(\mathbf{J}^+(x; \lambda))$, which we previously evaluated by taking a limit as $x \rightarrow +\infty$. This gives a formula for the scattering coefficient $a(\lambda)$ in terms of the Jost solutions $\mathbf{j}^{-,1}(x; \lambda)$ and $\mathbf{j}^{+,2}(x; \lambda)$. Since both of these have analytic continuations into the upper half-plane, we learn that $a(\lambda)$ is an analytic function in the upper half-plane as well. Moreover, from the asymptotic behavior of the relevant Jost solutions for large λ , we have

$$\lim_{\substack{\lambda \rightarrow \infty \\ \text{Im}(\lambda) \geq 0}} a(\lambda) = 1.$$

A formula for $b(\lambda)$ may also be obtained by taking determinants, this time after adjoining $\mathbf{j}^{+,1}(x; \lambda)$ as a first column to both sides of the scattering relation. This gives

$$b(\lambda) = \det [\mathbf{j}^{+,1}(x; \lambda), \mathbf{j}^{-,1}(x; \lambda)].$$

Now as one of the relevant Jost functions is analytic in the upper half-plane while the other is analytic in the lower half-plane, we learn that, generally speaking, the scattering coefficient $b(\lambda)$ while defined and continuous for $\lambda \in \mathbb{R}$, generally does not have an analytic continuation into either of the half-planes. We do learn, however, that $b(\lambda) \rightarrow 0$ as $\lambda \rightarrow \pm\infty$ on the real axis.

We now have the following important result.

Lemma 9. *Suppose that $\psi \in L^1(\mathbb{R})$, and let $a(\lambda) = S_{22}(\lambda)$ be the corresponding element of the scattering matrix. Then $a(\lambda) \neq 0$ for $\text{Im}(\lambda) \geq 0$.*

Proof. For real λ , from the identity $\mathbf{S}(\lambda)^* = \sigma_1 \mathbf{S}(\lambda) \sigma_1$ and the fact that $\det(\mathbf{S}(\lambda)) = 1$ one easily obtains the inequality $|a(\lambda)|^2 \geq 1$.

Now suppose that $a(\lambda_0) = 0$ for some λ_0 with $\text{Im}(\lambda_0) > 0$. From the Wronskian formula for $a(\lambda)$ it follows that the Jost solutions $\mathbf{j}^{-,1}(x; \lambda_0)$ and $\mathbf{j}^{+,2}(x; \lambda_0)$ are proportional to each other. But because $\text{Im}(\lambda_0) > 0$, $\mathbf{j}^{-,1}(x; \lambda_0)$ is exponentially decaying as $x \rightarrow -\infty$ while $\mathbf{j}^{+,2}(x; \lambda_0)$ is exponentially decaying as $x \rightarrow +\infty$. This implies the existence of a nonzero solution \mathbf{w} of (2) that decays exponentially as $|x| \rightarrow \infty$. Now the linear system (2) can be written in the form of an eigenvalue problem:

$$L\mathbf{w} = \lambda\mathbf{w}, \quad L := i\sigma_3 \frac{d}{dx} + \begin{bmatrix} 0 & -i\psi(x) \\ i\psi(x)^* & 0 \end{bmatrix}$$

and a key point is that the matrix differential operator L is *self-adjoint* on the space of square-integrable vector functions $\mathbf{w}(x)$ equipped with the Euclidean inner product

$$\langle \mathbf{w}, \mathbf{v} \rangle := \int_{\mathbb{R}} [w_1(x)v_1(x)^* + w_2(x)v_2(x)^*] dx.$$

However, $\lambda = \lambda_0$ is a complex eigenvalue of L , which contradicts selfadjointness. □

That $b(\lambda)$ is continuous and tends to zero for large real λ together with Lemma 9 implies that under the assumption that $\psi \in L^1(\mathbb{R})$, the reflection coefficient $R(\lambda) := b(\lambda)/a(\lambda)$ is a continuous function of $\lambda \in \mathbb{R}$ that decays to zero as $\lambda \rightarrow \pm\infty$, a fact that is a *nonlinear analogue of the Riemann-Lebesgue Lemma* from Fourier transform theory.

Riemann-Hilbert problem. Let's recombine the columns of the Jost matrices in order to obtain matrices whose columns extend together into the same half-plane:

$$(21) \quad \mathbf{M}(\lambda; x) := \begin{cases} \left[\frac{e^{i\lambda x}}{a(\lambda)} \mathbf{j}^{-,1}(x; \lambda), e^{-i\lambda x} \mathbf{j}^{+,2}(x; \lambda) \right], & \text{Im}(\lambda) > 0, \\ \left[e^{i\lambda x} \mathbf{j}^{+,1}(x; \lambda), \frac{e^{-i\lambda x}}{a(\lambda)^*} \mathbf{j}^{-,2}(x; \lambda) \right], & \text{Im}(\lambda) < 0. \end{cases}$$

The notation we are using is meant to suggest that we are now changing our point of view and viewing λ as the basic (complex) variable, and x as the parameter. From the relations

$$\mathbf{j}^{\pm,2}(x; \lambda)^* = \sigma_1 \mathbf{j}^{\pm,1}(x; \lambda), \quad \lambda \in \mathbb{R},$$

we get by analytic continuation that

$$\mathbf{j}^{-,2}(x; \lambda^*)^* = \sigma_1 \mathbf{j}^{-,1}(x; \lambda), \quad \text{Im}(\lambda) \geq 0,$$

and

$$\mathbf{j}^{+,2}(x; \lambda^*)^* = \sigma_1 \mathbf{j}^{+,1}(x; \lambda), \quad \text{Im}(\lambda) \leq 0.$$

From these it follows easily that (by definition, really)

$$\mathbf{M}(\lambda; x) = \sigma_1 \mathbf{M}(\lambda^*; x)^* \sigma_1.$$

From the definition (21) it follows that $\det(\mathbf{M}(\lambda; x)) = 1$ at each point where \mathbf{M} is defined.

Lemma 10 (Analyticity of $\mathbf{M}(\lambda; x)$). *For every $x \in \mathbb{R}$, the elements of the matrix $\mathbf{M}(\lambda; x)$ are all analytic functions of λ for $\text{Im}(\lambda) \neq 0$.*

Proof. This follows from the analyticity properties of the Jost solutions and of the function $a(\lambda)$, because $a(\lambda) \neq 0$ for $\text{Im}(\lambda) > 0$ according to Lemma 9. \square

Note that $\mathbf{M}(x; \lambda)$ has not actually been defined for $\lambda \in \mathbb{R}$. This is not an omission, but rather reflects the reality of the situation; the matrix $\mathbf{M}(\lambda; x)$ generally does not extend to the real λ -axis as an analytic (or even continuous) function. Indeed, there is a mismatch in the boundary values taken by $\mathbf{M}(\lambda; x)$ as λ approaches the real axis from the upper and lower half-planes. Let

$$\mathbf{M}_{\pm}(\lambda; x) := \lim_{\epsilon \downarrow 0} \mathbf{M}(\lambda \pm i\epsilon; x), \quad \lambda \in \mathbb{R},$$

and let $\mathbf{m}_{\pm}^1(\lambda; x)$ and $\mathbf{m}_{\pm}^2(\lambda; x)$ denote the corresponding columns. (The subscript notation on \mathbf{M} and its columns differs in meaning from the superscript notation on \mathbf{J} and its columns.) Thus,

$$\mathbf{m}_{+}^1(\lambda; x) = \frac{e^{i\lambda x}}{a(\lambda)} \mathbf{j}^{-,1}(x; \lambda), \quad \mathbf{m}_{+}^2(\lambda; x) = e^{-i\lambda x} \mathbf{j}^{+,2}(x; \lambda),$$

and

$$\mathbf{m}_{-}^1(\lambda; x) = e^{i\lambda x} \mathbf{j}^{+,1}(x; \lambda), \quad \mathbf{m}_{-}^2(\lambda; x) = \frac{e^{-i\lambda x}}{a(\lambda)^*} \mathbf{j}^{-,2}(x; \lambda),$$

all for $\lambda \in \mathbb{R}$. From the scattering relation

$$\mathbf{J}^{-}(x; \lambda) = \mathbf{J}^{+}(x; \lambda) \begin{bmatrix} a(\lambda) & b(\lambda)^* \\ b(\lambda) & a(\lambda)^* \end{bmatrix},$$

it easily follows that

$$\mathbf{m}_{+}^1(\lambda; x) = \frac{e^{i\lambda x}}{a(\lambda)} (a(\lambda) \mathbf{j}^{+,1}(x; \lambda) + b(\lambda) \mathbf{j}^{+,2}(x; \lambda)) = \mathbf{m}_{-}^1(\lambda; x) + e^{2i\lambda x} \frac{b(\lambda)}{a(\lambda)} \mathbf{m}_{+}^2(\lambda; x),$$

and

$$\mathbf{m}_{-}^2(\lambda; x) = \frac{e^{-i\lambda x}}{a(\lambda)^*} (b(\lambda)^* \mathbf{j}^{+,1}(x; \lambda) + a(\lambda)^* \mathbf{j}^{+,2}(x; \lambda)) = e^{-2i\lambda x} \frac{b(\lambda)^*}{a(\lambda)^*} \mathbf{m}_{-}^1(\lambda; x) + \mathbf{m}_{+}^2(\lambda; x).$$

Recalling the definition of the reflection coefficient $R(\lambda) := b(\lambda)/a(\lambda)$, these can be written respectively as

$$\mathbf{M}_{+}(\lambda; x) \begin{bmatrix} 1 \\ -e^{2i\lambda x} R(\lambda) \end{bmatrix} = \mathbf{M}_{-}(\lambda; x) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{+}(\lambda; x) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{M}_{-}(\lambda; x) \begin{bmatrix} -e^{-2i\lambda x} R(\lambda)^* \\ 1 \end{bmatrix},$$

which can be combined into a single matrix equation as

$$\mathbf{M}_+(\lambda; x) \begin{bmatrix} 1 & 0 \\ -e^{2i\lambda x} R(\lambda) & 1 \end{bmatrix} = \mathbf{M}_-(\lambda; x) \begin{bmatrix} 1 & -e^{-2i\lambda x} R(\lambda)^* \\ 0 & 1 \end{bmatrix}.$$

We therefore arrive at the following result:

Lemma 11 (Jump condition for $\mathbf{M}(\lambda; x)$). *$\mathbf{M}(\lambda; x)$ takes continuous boundary values $\mathbf{M}_\pm(\lambda; x)$ on the real line $\lambda \in \mathbb{R}$ that are related by the following jump condition:*

$$\mathbf{M}_+(\lambda; x) = \mathbf{M}_-(\lambda; x) \mathbf{D}(\lambda; x), \quad \lambda \in \mathbb{R},$$

where the so-called jump matrix is

$$\mathbf{D}(\lambda; x) = \begin{bmatrix} 1 - |R(\lambda)|^2 & -e^{-2i\lambda x} R(\lambda)^* \\ e^{2i\lambda x} R(\lambda) & 1 \end{bmatrix}, \quad \lambda \in \mathbb{R}.$$

Finally, we have the following property of $\mathbf{M}(\lambda; x)$.

Lemma 12 (Normalization of $\mathbf{M}(\lambda; x)$). *For every $x \in \mathbb{R}$, the matrix $\mathbf{M}(\lambda; x)$ satisfies*

$$\lim_{\lambda \rightarrow \infty} \mathbf{M}(\lambda; x) = \mathbb{I}.$$

Here the limit can be taken in any direction in the complex plane, including along one or the other side of the real line.

Proof. This follows from the definition of $\mathbf{M}(\lambda; x)$ with the use of the asymptotic conditions satisfied by the Jost solutions and the function $a(\lambda)$ as $\lambda \rightarrow \infty$. \square

In Lemma 10, Lemma 11, and Lemma 12, we have isolated three properties of the matrix $\mathbf{M}(\lambda; x)$ that we have deduced from scattering theory. The remarkable thing is that these properties are sufficient (in most cases) to determine the matrix $\mathbf{M}(\lambda; x)$ from the reflection coefficient $R(\lambda)$, and from $\mathbf{M}(\lambda; x)$ it is easy to extract the potential $\psi(x)$ whose reflection coefficient is $R(\lambda)$. This is the process of *inverse scattering*. Indeed, if the matrix $\mathbf{M}(\lambda; x)$ can be recovered for each x , then from the differential equation (2) satisfied by $\mathbf{j}^{+,2}(x; \lambda)$ we can easily deduce a differential equation satisfied by $\mathbf{m}^2(\lambda; x)$ for $\text{Im}(\lambda) > 0$:

$$\frac{\partial}{\partial x} \mathbf{m}^2(\lambda, x) = \begin{bmatrix} -2i\lambda & \psi(x) \\ \psi(x)^* & 0 \end{bmatrix} \mathbf{m}^2(\lambda, x).$$

Recalling that $\mathbf{m}^2(\lambda; x) \rightarrow [0, 1]^\top$ as $\lambda \rightarrow \infty$, we see from the first row that

$$o(1) = -2i\lambda M_{12}(\lambda; x) + \psi(x)(1 + o(1)), \quad \lambda \rightarrow \infty.$$

Taking the limit $\lambda \rightarrow \infty$ gives us a formula for $\psi(x)$ in terms of $\mathbf{M}(\lambda; x)$:

$$\psi(x) = 2i \lim_{\lambda \rightarrow \infty} \lambda M_{12}(\lambda; x).$$

The problem of finding the matrix $\mathbf{M}(\lambda; x)$ satisfying the three properties listed in Lemmas 10–12 given a suitable reflection coefficient $R(\lambda)$ is called a *Riemann-Hilbert problem*.

Summary: the inverse-scattering transform solution algorithm for the initial-value problem for the defocusing NLS equation.

Theorem 2. *Let $\psi_0(x)$ be a suitable complex-valued function of x decaying to zero as $x \rightarrow \pm\infty$. Let $R_0(\lambda)$ denote the corresponding reflection coefficient under the direct transform: $\psi_0 \mapsto R_0$. Then, the solution of the initial value problem for the defocusing NLS equation*

$$i\psi_t + \frac{1}{2}\psi_{xx} - |\psi|^2\psi = 0, \quad \lim_{x \rightarrow \pm\infty} \psi(x, t) = 0, \quad \psi(x, 0) = \psi_0(x)$$

is given by the formula

$$\psi(x, t) = 2i \lim_{\lambda \rightarrow \infty} \lambda M_{12}(\lambda; x, t),$$

where $\mathbf{M}(\lambda; x, t)$ is the solution of the following Riemann-Hilbert problem: find a 2×2 matrix $\mathbf{M}(\lambda; x, t)$ with the following properties:

- **Analyticity:** $\mathbf{M}(\lambda; x, t)$ is an analytic function of λ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

- **Jump Condition:** The matrix $\mathbf{M}(\lambda; x, t)$ takes continuous boundary values $\mathbf{M}_{\pm}(\lambda; x, t)$ on the real axis from \mathbb{C}_{\pm} , and they are related by the condition

$$\mathbf{M}_{+}(\lambda; x, t) = \mathbf{M}_{-}(\lambda; x, t)\mathbf{D}(\lambda; x, t), \quad \lambda \in \mathbb{R},$$

where

$$\mathbf{D}(\lambda; x, t) := \begin{bmatrix} 1 - |R_0(\lambda)|^2 & -e^{-2i(\lambda x + \lambda^2 t)} R_0(\lambda)^* \\ e^{2i(\lambda x + \lambda^2 t)} R_0(\lambda) & 1 \end{bmatrix}.$$

- **Normalization:** As $\lambda \rightarrow \infty$, $\mathbf{M}(\lambda; x, t) \rightarrow \mathbb{I}$.

Later, after we develop the theory of Riemann-Hilbert problems, we will see that this problem has a unique solution for all $(x, t) \in \mathbb{R}^2$ if R_0 is a suitable function of $\lambda \in \mathbb{R}$.

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